Latin American Journal of Solids and Structures
www.lajss.org

A complete set of equations for piezo-magneto-elastic analysis of a functionally graded thick shell of revolution

Abstract

Tensor analysis and an orthogonal curvilinear coordinate system have been used to derive a complete set of equations for piezo-magneto-elastic analysis of a functionally graded (FG) thick shell of revolution with variable thickness and curvature. The mentioned structure can be subjected to mechanical, electrical and magnetic fields. It was assumed that all material properties (mechanical, electrical and magnetic properties) change functionally throughout the three axis of employed coordinate system. Kinetic and potential energies of the system have been evaluated in order to constitute the functional of the system. Final partial differential equations of the system can be derived by using minimization of the energy functional with respect to five employed functions of the system. For validation, the obtained differential equations have been reduced to two previously studied problems i.e. functionally graded piezoelectric materials and functionally graded piezo-magnetic cylinders. Furthermore, numerical results are evaluated for a case study.

Keywords

piezo-magneto-elastic analysis, functionally graded material, tensor analysis, curvilinear.

Mohammad Arefi [a]

[a] Department of Solid Mechanics, Faculty of Mechanical Engineering, University of Kashan, Kashan, I.R. Iran, 87317-51167.
E-mail: arefi63@gmail.com and arefi@kashanu.ac.ir

1 INTRODUCTION

Since scientific introduction of piezoelectric materials as an important element of intelligent and smart systems, numerous researches have been devoted to analysis of these materials in different configurations and geometries. Application of piezoelectric materials in various environments with opposite conditions has forced researchers to introduce new group of materials with variable properties along the directions of coordinate system. These materials were known as functionally graded materials (FGMs). The first idea for producing this group of materials was their applica-

tion in environments with high temperature gradient and their forming ability. FGMs are made of a mixture with arbitrary composition of two different materials, and volume fraction of each material changes continuously and gradually at the entire volume of the material. A brief review on the literature indicates that cylinder, plate and sphere have been mostly used for mentioned analyses.

Woo and Meguid (2001) investigated the nonlinear analysis of FG plates and shallow shells under transverse mechanical loads and temperature field. Von Karman theory for large deflection analysis has been employed for solution. They proposed an analytical solution for the coupled large deflection analysis of mentioned structures. A spherical shell made of homogenous materials subjected to mechanical and electrical loads has been studied by Chen *et al.* (2002). Liu *et al.* (2002) developed an analytical model for free vibration analysis of a cylindrical shell under mechanical and electrical loads using Mindlin's theory. Distribution of electric potential has been modeled by a sinusoidal function along the thickness direction. Piezoelectric analysis of a cylindrical shell was performed by Peng-Fei and Andrew (2004). They used the theory of elasticity for evaluating the behavior of piezoelectric structure. Exact analysis of a cylinder made of functionally graded piezoelectric materials (FGPMs) under bending was studied by Lee and Lu (2005). It was supposed that the material properties changes continuously and gradually along the radial coordinate system based on a power function distribution.

Dai *et al.* (2007) analyzed electro-magneto-elastic behavior of functionally graded piezoelectric (FGP) cylindrical and spherical pressure vessels. All mechanical, electrical and magnetic properties were assumed variable along the radial coordinate system based on a power function.

Electro-magneto-elastic analysis of an orthotropic cylinder has been performed by Babaei and Chen (2008). The cylinder was assumed infinitely long and consequently, an one dimensional analysis has been presented. The cylinder has been polarized and magnetized radially. The effect of centrifugal load also was studied on the electro-magneto-elastic responses of the problem. Two dimensional theory of elasticity has been employed in order to investigate the thermo-elastic behavior of a FG cylinder under thermal and mechanical loads by Jabbari *et al.* (2009). Sarfaraz Khabbaz *et al.* (2009) investigated the nonlinear analysis of FG plates under pressure based on the higher-order shear deformation theory. The effect of the thickness and non-homogeneous index was investigated on the distribution of the displacements and stresses. It was supposed that cylinder has been constrained with simply supported boundary conditions.

Khoshgoftar *et al.* (2009) presented the thermo-elastic analysis of a FGP cylindrical pressure vessel. It was supposed that all thermal, mechanical and electrical properties obey a power function along the radial direction of the cylinder. This analysis was performed using the plane elasticity theory.

Electro-magneto-thermo-elastic behaviors of a FGP hollow cylinder under a uniform magnetic field and subjected to thermo-electro-mechanical loads was investigated by Dai *et al.* (2010). All mechanical, electric, magnetic and thermal properties were considered variable as a power function along the thickness direction. They presented benefits of material nonhomogeneity for design optimization of electro mechanical structures and systems. Sheng and Wang (2010) presented thermo-elastic vibration and buckling characteristics of a FGP cylindrical shell. Description of deformation has been performed by first order shear deformation theory and a quadratic function

has been used for distribution of electric potential along the thickness. The Hamilton's principle and Maxwell's equation were considered for solving the problem. The critical values of axial load, temperature and voltage were investigated for different boundary conditions. Arefi and Rahimi (2011) studied thermo-piezo-magneto-elastic analysis of a functionally graded piezomagnetic cylinder.

As mentioned in literature review, most researches about piezoelectric materials have been limited to three famous structures (cylindrical, spherical and plate). The author thinks that it is necessary to develope a complete formulation to exactly predict the piezo-magneto-elastic behavior of an arbitrary structure made of functionally graded material (FGM) under mechanical loads and electric and magnetic fields.

Although some researches have been presented to study the behavior of a shell of revolution, this study tries to present a complete set of partial differential equations for piezo-magneto-elastic analysis of a functionally graded piezo-magnetic thick shell of revolution with variable thickness and curvature. Some incompleteness and incomprehensive relations in previous researches can be resolved in this study. Some of these works can be presented.

Three dimensional formulation of elasticity for elastic analysis of a thick shell of revolution with arbitrary thickness and curvature made of isotropic and homogenous materials has been studied by Kang (2007). This work was performed using the tensor formulation in curvilinear coordinate system. The assumed limitations in the mentioned work can be improved for more general applications in the present study.

Arciniega and Reddy (2007) suggested the finite element formulation for nonlinear analysis of a shell structure based on the tensor analysis. Curvilinear coordinate system with higher order elements were employed for this purpose. The thickness strain was considered and therefore, a three dimensional analysis was employed. It was assumed that shear stresses and strains did not have any effect on the normal strains and stresses, respectively. Some appropriate and useful relations about tensor analysis and piezoelectric structures were presented in literature (Arciniega and Reddy 2007, Carrera and Brischetto 2007, Ray and Reddy 2005, Santos *et al.*, 2008).

It will be understood that there are not considered any limitation on the constitutive equations nor on the direction of functionalities of the used material in the present research. This work is performed using tensor analysis and an orthogonal curvilinear coordinate system for a completely general thick shell of revolution (variable thickness and curvature). Using energy method and minimization of the functional of the system using Euler equation, governing partial differential equations of the system can be derived. The outcome of this study can be used for a complete anisotropic structure. The present formulation can be validated with results of cylinders made of functionally graded piezoelectric or functionally graded piezomagnetic materials.

2 FORMULATION

2.1 Fundamental geometric relations of used curvilinear coordinate system

For analysis of an arbitrary thick shell of revolution, a curvilinear coordinate system is employed. Figure 1 shows this curvilinear coordinate system. Two curvilinear and Cartesian coordinate systems are presented in this figure. For derivation of fundamental geometric relations, let us to

define the appropriate relation between curvilinear and Cartesian coordinate systems. The components of curvilinear coordinate system include $(\psi, \mathrm{z}, \theta)$, which are meridian, normal and circumferential components, respectively.

The appropriate transformation from $(\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3) = (\psi, \mathrm{z}, \theta)$ to $(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)$, is expressed as follows (Kang, 2007):

$$X = \rho_1 \sin\psi \cos\theta \vec{e}_1 + \rho_1 \sin\psi \sin\theta \vec{e}_2 + (y - z\cos\psi)\vec{e}_3 \tag{1}$$

where, $\vec{e}_i$ is unit vectors along the axis of Cartesian coordinate system, $\rho_1 = \rho_\psi + \mathrm{z}$ and ρ_ψ is the normal distance of mid-plane from the vertical axis. The mentioned parameters of the shell can be observed in Figure 1.

Using Eq. (1), the first important base vector (covariant base vector, g_i) can be defined. Covariant base vector (g_i) along every component of curvilinear coordinate system can be obtained by using partial derivative of the position vector with respect to the mentioned component of curvilinear coordinate system using Eq. (2) (Kang 2007, Arciniega and Reddy 2007):

$$\begin{aligned}
g_1 &= X_{,q_1} = \frac{\partial X}{\partial \psi} = (\rho_1 \sin\psi)_{,\psi} \cos\theta \vec{e}_1 + (\rho_1 \sin\psi)_{,\psi} \sin\theta \vec{e}_2 + (y - z\cos\psi)_{,\psi} \vec{e}_3 \\
g_2 &= X_{,q_2} = \frac{\partial X}{\partial z} = \rho_{1,z} \sin\psi \cos\theta \vec{e}_1 + \rho_{1,z} \sin\psi \sin\theta \vec{e}_2 - \cos\psi \vec{e}_3 \\
g_3 &= X_{,q_3} = \frac{\partial X}{\partial \theta} = -\rho_1 \sin\psi \sin\theta \vec{e}_1 - \rho_1 \sin\psi \cos\theta \vec{e}_2
\end{aligned} \tag{2}$$

For completion of derivation of covariant base vector, three undefined expressions including $(\rho_1 \sin\psi)_{,\psi}, (y - z\cos\psi)_{,\psi}$ and $\rho_{1,z}$ appeared in Eq. (2) must be evaluated. For this case, we have following geometric relations:

$$\begin{aligned}
&\mathrm{d}\bar{\mathrm{r}} = \mathrm{ds}\cos\psi, \mathrm{dy} = \mathrm{ds}\sin\psi, \mathrm{ds} = \rho_\theta \mathrm{d}\psi, \mathrm{r} = \bar{\mathrm{r}} + \mathrm{z}\sin\psi = (\rho_\psi + \mathrm{z})\sin\psi = \rho_1 \sin\psi \\
&\frac{\mathrm{d}\bar{\mathrm{r}}}{\mathrm{d}\psi} = \frac{\mathrm{ds}}{\mathrm{d}\psi}\cos\psi = \rho_\theta \cos\psi, \frac{\mathrm{dy}}{\mathrm{d}\psi} = \frac{\mathrm{ds}}{\mathrm{d}\psi}\sin\psi = \rho_\theta \sin\psi
\end{aligned} \tag{3}$$

where, ρ_θ is radius of curvature, $\bar{\mathrm{r}}$ is the radius of mid-plane of curve and r is the radius of an arbitrary point with orthogonal distance (z) from the mid-plane.

Now, the undefined expressions can be evaluated as follows:

$$\begin{aligned}
&(\rho_1 \sin\psi)_{,\psi} = \frac{dr}{d\psi} = \rho_\theta \cos\psi + z\cos\psi = \rho_2 \cos\psi, \rho_{1,z} = 1 \\
&(y - z\cos\psi)_{,\psi} = \rho_\theta \sin\psi + z\sin\psi = \rho_2 \sin\psi
\end{aligned} \tag{4}$$

where $\rho_2 = \rho_\theta + z$. The above geometric relations simplify the evaluation of the covariant base vector as follows:

$$\begin{aligned} g_1 &= X_{,1} = \frac{\partial X}{\partial \psi} = \rho_2 \cos\psi\cos\theta\vec{e}_1 + \rho_2 \cos\psi\sin\theta\vec{e}_2 + \rho_2 \sin\psi\vec{e}_3 \\ g_2 &= X_{,2} = \frac{\partial X}{\partial z} = \sin\psi\cos\theta\vec{e}_1 + \sin\psi\sin\theta\vec{e}_2 - \cos\psi\vec{e}_3 \\ g_3 &= X_{,3} = \frac{\partial X}{\partial \theta} = -r\sin\theta\vec{e}_1 + r\cos\theta\vec{e}_2 \end{aligned} \tag{5}$$

After evaluation of covariant base vector, other important base vector (Contra variant base vector, g^{i}) can be defined.

$$g^i = g^{ij} g_j \tag{6}$$

where, g^{ij} are metric tensors. The covariant and contra variant metric tensors g_{ij}, g^{ij} and the relation between them are expressed as (Arciniega and Reddy 2007):

$$g_{ij} = g_i \cdot g_j, \left\| g_i \right\| = \sqrt{g_{ii}}, \; g^{ij} = g^i \cdot g^j, \left\| g^i \right\| = \sqrt{g^{ii}}, g^{ij} = \frac{1}{g_{ij}}, g^i = g^{ij} g_j \tag{7}$$

which, gives Contra variant base vector g^{i} as follows:

$$\begin{aligned} g^1 &= \frac{1}{\rho_2}[\cos\psi\cos\theta\vec{e}_1 + \cos\psi\sin\theta\vec{e}_2 + \sin\psi\vec{e}_3] \\ g^2 &= \sin\psi\cos\theta\vec{e}_1 + \sin\psi\sin\theta\vec{e}_2 - \cos\psi\vec{e}_3 \\ g^3 &= \frac{1}{r}[-\sin\theta\vec{e}_1 + \cos\theta\vec{e}_2] \end{aligned} \tag{8}$$

Substitution of Eq. (5) into Eq. (7) gives three components of covariant and contra-variant metric tensors as follows (Kang 2007, Arciniega and Reddy 2007):

$$g_{11} = g_1 \cdot g_1 = \rho_2^2, \; g_{22} = g_2 \cdot g_2 = 1 \;, g_{33} = g_3 \cdot g_3 = r^2, g^{11} = \frac{1}{\rho_2^2}, \; g_{22} = 1 \;, g_{33} = \frac{1}{r^2} \tag{9}$$

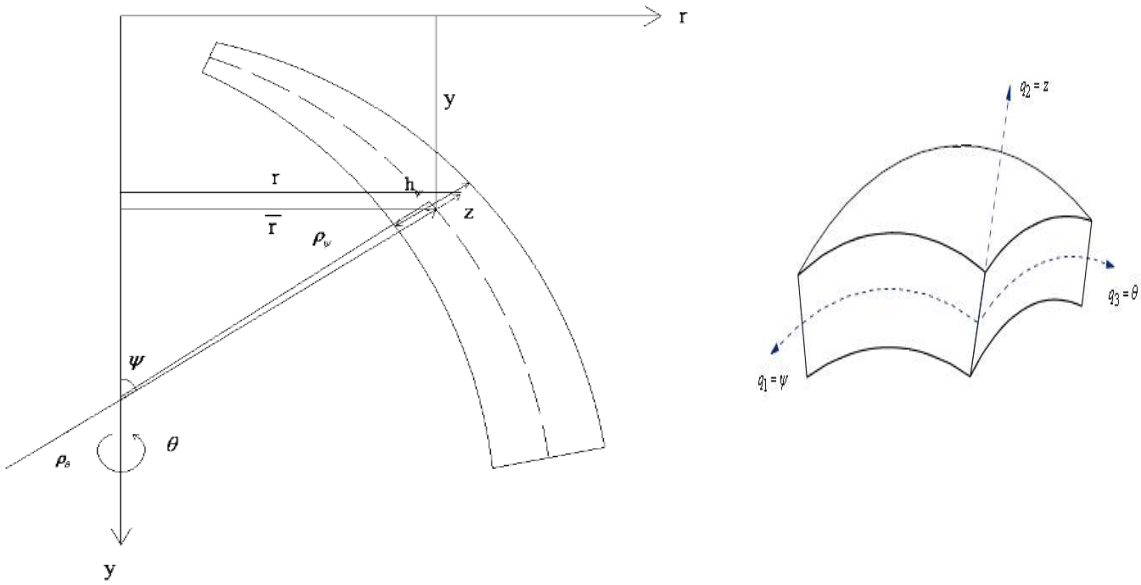


Figure 1: the schematic figure of a thick shell of revolution with assumed curvilinear coordinate system.

The other important step in derivation of basic geometric relations is evaluation of derivatives of a vector with respect to components of coordinate system. The covariant and contra-variant derivatives of a vector (v_j and v^j) can be presented by defining the Christoffel symbols of first and second kind (Γ^f_{ij} and Γ_{fij}) (Kang 2007):

$$v_j\big|_i = v_{j,i} - v_f \Gamma^f_{ij},\ v^j\big|_i = v^j_{,i} - v^f \Gamma_{fij} \tag{10}$$

The Christoffel symbols are derived as:

$$\begin{aligned} &\Gamma^k_{ij} = g^k . g_{i,j} \\ &\Gamma^1_{11} = \frac{\rho_{\psi,\psi}}{\rho_2}, \Gamma^1_{12} = \frac{1}{\rho_2}, \Gamma^1_{33} = -\frac{r}{\rho_2}\cos\psi \\ &\Gamma^2_{11} = -\rho_2, \Gamma^2_{33} = -r\sin\psi, \Gamma^3_{13} = \frac{\rho_2}{r}\cos\psi, \Gamma^3_{23} = \frac{\rho_{\psi,\psi}}{\rho_2} \end{aligned} \tag{11}$$

In this step, the strain-displacement relation can be expressed as (Kang 2007):

$$\gamma_{ij} = \frac{1}{2}[v_i\big|_j + v_j\big|_i] \tag{12}$$

where, γ_{ij} are tensor components of strain. Using the appropriate transformation relation between the physical (ε_{ij}) and tensor (γ_{ij}) components ($\gamma_{ij} = \sqrt{g^{ii}}\sqrt{g^{jj}}\varepsilon_{ij}$), we have the physical strain components as (Kang 2007):

$$\varepsilon_{\psi\psi}=\frac{(u_{\psi,\psi}+u_z)}{\rho_2},\varepsilon_{\theta\theta}=\frac{(u_{\theta,\theta}+u_z\sin\psi+u_\psi\cos\psi)}{r},\varepsilon_{zz}=u_{z,z}$$
$$\varepsilon_{\psi z}=\frac{1}{2}[u_{\psi,z}-\frac{(u_\psi-u_{z,\psi})}{\rho_2}],\varepsilon_{\psi\theta}=\frac{1}{2}[\frac{(u_{\psi,\theta}-u_\theta\cos\psi)}{r}-\frac{u_{\theta,\psi}}{\rho_2}] \quad (13)$$
$$\varepsilon_{z\theta}=\frac{1}{2}[\frac{(u_{z,\theta}-u_\theta\sin\psi)}{r}+u_{\theta,z}]$$

The final step in derivation of basic equations is evaluation of divergence of the scalar fields. If ϕ is electric potential and Φ is magnetic potential, therefore, $\vec{E},\vec{H}$ electric and magnetic fields can be obtained as follows (Khoshgoftar *et al.* 2009, Santos *et al.* 2008):

$$\vec{E}=-\vec{\nabla}\phi,\quad \vec{H}=-\vec{\nabla}\Phi \quad (14)$$

Divergence of a scalar filed such as electric potential (ϕ) may be evaluated by using the metric tensor for obtaining electric field (E) as follows (Santos *et al.* 2008):

$$E_i=-\vec{\nabla}\phi=-g^i\frac{\partial\phi}{\partial q^i}=-\sqrt{g^{ii}}\frac{\partial\phi}{\partial q^i}$$
$$E_\psi=-\sqrt{g^{11}}\frac{\partial\phi}{\partial\psi}=-\frac{\partial\phi}{\rho_2\partial\psi},\ E_z=-\sqrt{g^{22}}\frac{\partial\phi}{\partial z}=-\frac{\partial\phi}{\partial z},\ E_\theta=-\sqrt{g^{33}}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\phi}{r\partial\theta} \quad (15)$$

The same equations can be obtained for derivation of magnetic field (H) as follows:

$$H_i=-\vec{\nabla}\Phi\rightarrow H_\psi=-\frac{\partial\Phi}{\rho_2\partial\psi},\ H_z=-\frac{\partial\Phi}{\partial z},\ H_3=-\frac{\partial\Phi}{r\partial\theta} \quad (16)$$

2.2 Constitutive equations for piezo-magneto-elastic formulation

The piezo-magneto-elastic behavior of a thick shell of revolution made of functionally graded piezomagnetic materials can be presented in this section. Three constitutive equations that are necessary for identification of the behavior of the piezomagnetic structures are expressed in vector form (Khoshgoftar *et al.* 2009, Santos *et al.* 2008):

$$\sigma=C\varepsilon-eE-dH$$
$$D=e\varepsilon+\eta E+gH \quad (17)$$
$$B=d\varepsilon+gE+\mu H$$

where, σ and ε are stress and strain tensors of order two, E and D are electric field and electric displacement tensors of order one, H and B are magnetic field and magnetic displacement components of order one and C,e and η are elastic stiffness, piezoelectric and dielectric tensors

of order four, three and two, respectively. d, g and μ are piezo magnetic, electromagnetic and magnetic coefficient tensors, respectively. The constitutive and piezoelectric relations are defined in terms of physical components (Santos *et al.* 2008):

$$\begin{aligned} \sigma_{ij} &= \mathrm{C}_{ijkl}\varepsilon_{kl} - \mathrm{e}_{ijk}\mathrm{E}_k - \mathrm{d}_{ijk}\mathrm{H}_k \\ \mathrm{D}_i &= \mathrm{e}_{ijk}\varepsilon_{jk} + \eta_{ik}\mathrm{E}_k + \mathrm{g}_{ik}\mathrm{H}_k \\ \mathrm{B}_i &= \mathrm{d}_{ijk}\varepsilon_{jk} + \mathrm{g}_{ik}\mathrm{E}_k + \mu_{ik}\mathrm{H}_k \end{aligned} \tag{18}$$

Imposing the symmetric condition for stress and strain tensors $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$, reduces the relation in the extended form as follows (Arciniega and Reddy, 2007):

$$
\begin{cases}
\sigma_{\psi\psi} = C_{1111}\varepsilon_{\psi\psi} + C_{1122}\varepsilon_{zz} + C_{1133}\varepsilon_{\theta\theta} + 2C_{1112}\varepsilon_{\psi z} + 2C_{1113}\varepsilon_{\psi\theta} + 2C_{1123}\varepsilon_{z\theta} - e_{111}E_{\psi} - e_{112}E_{z} \\
-e_{113}E_{\theta} - d_{111}H_{\psi} - d_{112}H_{z} - d_{113}H_{\theta} \\
\sigma_{zz} = C_{2211}\varepsilon_{\psi\psi} + C_{2222}\varepsilon_{zz} + C_{2233}\varepsilon_{\theta\theta} + 2C_{2212}\varepsilon_{\psi z} + 2C_{2213}\varepsilon_{\psi\theta} + 2C_{2223}\varepsilon_{z\theta} - e_{221}E_{\psi} - e_{222}E_{z} \\
-e_{223}E_{\theta} - d_{221}H_{\psi} - d_{222}H_{z} - d_{223}H_{\theta} \\
\sigma_{\theta\theta} = C_{3311}\varepsilon_{\psi\psi} + C_{3322}\varepsilon_{zz} + C_{3333}\varepsilon_{\theta\theta} + 2C_{3312}\varepsilon_{\psi z} + 2C_{3313}\varepsilon_{\psi\theta} + 2C_{3323}\varepsilon_{z\theta} - e_{331}E_{\psi} - e_{332}E_{z} \\
-e_{333}E_{\theta} - d_{331}H_{\psi} - d_{332}H_{z} - d_{333}H_{\theta} \\
\sigma_{\psi z} = C_{1211}\varepsilon_{\psi\psi} + C_{1222}\varepsilon_{zz} + C_{1233}\varepsilon_{\theta\theta} + 2C_{1212}\varepsilon_{\psi z} + 2C_{1213}\varepsilon_{\psi\theta} + 2C_{1223}\varepsilon_{z\theta} - e_{121}E_{\psi} - e_{122}E_{z} \\
-e_{123}E_{\theta} - d_{121}H_{\psi} - d_{122}H_{z} - d_{123}H_{\theta} \\
\sigma_{\psi\theta} = C_{1311}\varepsilon_{\psi\psi} + C_{1322}\varepsilon_{zz} + C_{1333}\varepsilon_{\theta\theta} + 2C_{1312}\varepsilon_{\psi z} + 2C_{1313}\varepsilon_{\psi\theta} + 2C_{1323}\varepsilon_{z\theta} - e_{131}E_{\psi} - e_{132}E_{z} \\
-e_{133}E_{\theta} - d_{131}H_{\psi} - d_{132}H_{z} - d_{133}H_{\theta} \\
\sigma_{z\theta} = C_{2311}\varepsilon_{\psi\psi} + C_{2322}\varepsilon_{zz} + C_{2333}\varepsilon_{\theta\theta} + 2C_{2312}\varepsilon_{\psi z} + 2C_{2313}\varepsilon_{\psi\theta} + 2C_{2323}\varepsilon_{z\theta} - e_{231}E_{\psi} - e_{232}E_{z} \\
-e_{233}E_{\theta} - d_{231}H_{\psi} - d_{232}H_{z} - d_{233}H_{\theta} \\
D_{\psi} = e_{111}\varepsilon_{\psi\psi} + e_{122}\varepsilon_{zz} + e_{133}\varepsilon_{\theta\theta} + 2e_{112}\varepsilon_{\psi z} + 2e_{113}\varepsilon_{\psi\theta} + 2e_{123}\varepsilon_{z\theta} + \eta_{11}E_{\psi} + \eta_{12}E_{z} + \eta_{13}E_{\theta} \\
+g_{11}H_{\psi} + g_{12}H_{z} + g_{13}H_{\theta} \\
D_{z} = e_{211}\varepsilon_{\psi\psi} + e_{222}\varepsilon_{zz} + e_{233}\varepsilon_{\theta\theta} + 2e_{212}\varepsilon_{\psi z} + 2e_{213}\varepsilon_{\psi\theta} + 2e_{223}\varepsilon_{z\theta} + \eta_{21}E_{\psi} + \eta_{22}E_{z} + \eta_{23}E_{\theta} \\
+g_{21}H_{\psi} + g_{22}H_{z} + g_{23}H_{\theta} \\
D_{\theta} = e_{311}\varepsilon_{\psi\psi} + e_{322}\varepsilon_{zz} + e_{333}\varepsilon_{\theta\theta} + 2e_{312}\varepsilon_{\psi z} + 2e_{313}\varepsilon_{\psi\theta} + 2e_{323}\varepsilon_{z\theta} + \eta_{31}E_{\psi} + \eta_{32}E_{z} + \eta_{33}E_{\theta} \\
+g_{31}H_{\psi} + g_{32}H_{z} + g_{33}H_{\theta} \\
B_{\psi} = d_{111}\varepsilon_{\psi\psi} + d_{122}\varepsilon_{zz} + d_{133}\varepsilon_{\theta\theta} + 2d_{112}\varepsilon_{\psi z} + 2d_{113}\varepsilon_{\psi\theta} + 2d_{123}\varepsilon_{z\theta} + g_{11}E_{\psi} + g_{12}E_{z} + g_{13}E_{\theta} \\
+\mu_{11}H_{\psi} + \mu_{12}H_{z} + \mu_{13}H_{\theta} \\
B_{z} = d_{211}\varepsilon_{\psi\psi} + d_{222}\varepsilon_{zz} + d_{233}\varepsilon_{\theta\theta} + 2d\varepsilon_{\psi z} + 2d_{213}\varepsilon_{\psi\theta} + 2d_{223}\varepsilon_{z\theta} + g_{21}E_{\psi} + g_{22}E_{z} + g_{23}E_{\theta} \\
+\mu_{21}H_{\psi} + \mu_{22}H_{z} + \mu_{23}H_{\theta} \\
B_{\theta} = d_{311}\varepsilon_{\psi\psi} + d_{322}\varepsilon_{zz} + d_{333}\varepsilon_{\theta\theta} + 2d_{312}\varepsilon_{\psi z} + 2d_{313}\varepsilon_{\psi\theta} + 2d_{323}\varepsilon_{z\theta} + g_{31}E_{\psi} + g_{32}E_{z} + g_{33}E_{\theta} \\
+\mu_{31}H_{\psi} + \mu_{32}H_{z} + \mu_{33}H_{\theta}
\end{cases}
\tag{19}
$$

The energy method can be employed in order to obtain functional of the system and derivation of governing differential equations. Total energy of the system includes potential and kinetic energies. Potential energy is decomposed into mechanical, electrical and magnetic energies that can be defined per unit volume of the structure as follows (Ray and Reddy, 2005):

$$
\bar{u}_{p} = \frac{1}{2}\left[\sigma_{ij}\varepsilon_{ij} - D_{i}E_{i} - B_{i}H_{i}\right] - f_{i}u_{i} \tag{20}
$$

where, $f_i(i=\psi,z,\theta)$ are the component of body force per unit volume. The kinetic energy is defined as (Ray and Reddy, 2005):

$$\bar{u}_k = \frac{1}{2}\lambda(\psi,z,\theta)[\dot{u}_\psi^2+\dot{u}_z^2+\dot{u}_\theta^2] \tag{21}$$

where, $\lambda(\psi,z,\theta)$ is density of material. By substituting the total non-zero components of σ_{ij}, ε_{ij}, D_i and E_i, we arrive to the potential energy per unit volume of the structure as:

$$\begin{aligned}\bar{u}_p = \frac{1}{2}[&\sigma_{\psi\psi}\varepsilon_{\psi\psi}+\sigma_{zz}\varepsilon_{zz}+\sigma_{\theta\theta}\varepsilon_{\theta\theta}+2\sigma_{\psi z}\varepsilon_{\psi z}+2\sigma_{\psi\theta}\varepsilon_{\psi\theta}+2\sigma_{z\theta}\varepsilon_{z\theta}-D_\psi E_\psi-D_z E_z-D_\theta\\&-B_\psi H_\psi-B_z H_z-B_\theta H_\theta]-f_\psi u_\psi-f_z u_z-f_\theta u_\theta\end{aligned} \tag{22}$$

Total energy of the system is obtained by subtraction of the potential energy from the kinetic energy (Ray and Reddy, 2005):

$$\bar{u} = \bar{u}_k - \bar{u}_p \tag{23}$$

By definition of unit volume of the structure $dV=(\rho_2 d\psi)(r d\theta)dz=\rho_2 r d\psi d\theta dz$ and considering Eqs. (21, 22 and 23), the total energy and the energy functional of the system can be constituted as follows:

$$U=\int_\theta\int_z\int_\psi \rho_2 r\bar{u}\,d\psi dz d\theta=\int_\theta\int_z\int_\psi F(u_\psi,u_z,u_\theta,\phi,t)\,d\psi dz d\theta \rightarrow F(u_\psi,u_z,u_\theta,\phi,t)=\rho_2 r\bar{u} \tag{24}$$

The obtained functional of the system (Eqs. (24)) can be substituted in the Euler's equations in order to derive the final governing differential equations of the system as follows:

$$\begin{cases}\dfrac{\partial F}{\partial u_\psi}-\dfrac{\partial}{\partial t}\left(\dfrac{\partial F}{\partial u_{\psi,t}}\right)-\dfrac{\partial}{\partial\psi}\left(\dfrac{\partial F}{\partial u_{\psi,\psi}}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{\partial F}{\partial u_{\psi,z}}\right)-\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial F}{\partial u_{\psi,\theta}}\right)=0\\ \dfrac{\partial F}{\partial u_z}-\dfrac{\partial}{\partial t}\left(\dfrac{\partial F}{\partial u_{z,t}}\right)-\dfrac{\partial}{\partial\psi}\left(\dfrac{\partial F}{\partial u_{z,\psi}}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{\partial F}{\partial u_{z,z}}\right)-\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial F}{\partial u_{z,\theta}}\right)=0\\ \dfrac{\partial F}{\partial u_\theta}-\dfrac{\partial}{\partial t}\left(\dfrac{\partial F}{\partial u_{\theta,t}}\right)-\dfrac{\partial}{\partial\psi}\left(\dfrac{\partial F}{\partial u_{\theta,\psi}}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{\partial F}{\partial u_{\theta,z}}\right)-\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial F}{\partial u_{\theta,\theta}}\right)=0\\ \dfrac{\partial F}{\partial\phi}-\dfrac{\partial}{\partial t}\left(\dfrac{\partial F}{\partial\phi_{,t}}\right)-\dfrac{\partial}{\partial\psi}\left(\dfrac{\partial F}{\partial\phi_{,\psi}}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{\partial F}{\partial\phi_{,z}}\right)-\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial F}{\partial\phi_{,\theta}}\right)=0\\ \dfrac{\partial F}{\partial\Phi}-\dfrac{\partial}{\partial t}\left(\dfrac{\partial F}{\partial\Phi_{,t}}\right)-\dfrac{\partial}{\partial\psi}\left(\dfrac{\partial F}{\partial\Phi_{,\psi}}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{\partial F}{\partial\Phi_{,z}}\right)-\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial F}{\partial\Phi_{,\theta}}\right)=0\end{cases} \tag{25}$$

where, $F(u_\psi,u_z,u_\theta,\phi,\Phi,t)=\rho_2 r\bar{u}$ is defined as the functional of the system. By performing the appropriate operations and simplifications, we will have the final energy functional as:

$$
\begin{aligned}
&F(u_\psi, u_z, u_\theta, \phi, t) = \rho_2 r\bar{u}\\
&= \tfrac{1}{2}\lambda\rho_2 r[\dot{u}_\psi^{\,2} + \dot{u}_z^{\,2} + \dot{u}_\theta^{\,2}] - \frac{1}{2}\rho_2 r[C_{1111}\varepsilon_{\psi\psi}^{\,2} + C_{2222}\varepsilon_{zz}^{\,2} + C_{3333}\varepsilon_{\theta\theta}^{\,2} + 4C_{1212}\varepsilon_{\psi z}^{\,2} + 4C_{1313}\varepsilon_{\psi\theta}^{\,2}\\
&+4C_{2323}\varepsilon_{z\theta}^{\,2} + 2C_{1122}\varepsilon_{zz}\varepsilon_{\psi\psi} + 2C_{1133}\varepsilon_{\theta\theta}\varepsilon_{\psi\psi} + 2C_{2233}\varepsilon_{\theta\theta}\varepsilon_{zz} + 4\varepsilon_{\psi\psi}(C_{1112}\varepsilon_{\psi z} + C_{1113}\varepsilon_{\psi\theta} + C_{1123}\varepsilon_{z\theta})\\
&+4\varepsilon_{zz}(C_{2212}\varepsilon_{\psi z} + C_{2213}\varepsilon_{\psi\theta} + C_{2223}\varepsilon_{z\theta}) + 4\varepsilon_{\theta\theta}(C_{3312}\varepsilon_{\psi z} + C_{3313}\varepsilon_{\psi\theta} + C_{3323}\varepsilon_{z\theta})\\
&+8C_{1213}\varepsilon_{\psi\theta}\varepsilon_{\psi z} + 8C_{1223}\varepsilon_{z\theta}\varepsilon_{\psi z} + 8C_{1323}\varepsilon_{z\theta}\varepsilon_{\psi\theta} - 2e_{111}E_\psi\varepsilon_{\psi\psi} - 2e_{222}\varepsilon_{zz}E_z - 2e_{333}\varepsilon_{\theta\theta}E_\theta - 2e_{112}E_z\varepsilon_{\psi\psi}\\
&-2e_{113}E_\theta\varepsilon_{\psi\psi} - 2e_{221}E_\psi\varepsilon_{zz} - 2e_{223}E_\theta\varepsilon_{zz} - 2e_{331}E_\psi\varepsilon_{\theta\theta} - 2e_{332}E_z\varepsilon_{\theta\theta} - 4E_\psi(e_{112}\varepsilon_{\psi z} + e_{113}\varepsilon_{\psi\theta} + e_{123}\varepsilon_{z\theta})\\
&-4E_z(e_{212}\varepsilon_{\psi z} + e_{213}\varepsilon_{\psi\theta} + e_{223}\varepsilon_{z\theta}) - 4E_\theta(e_{312}\varepsilon_{\psi z} + e_{313}\varepsilon_{\psi\theta} + e_{323}\varepsilon_{z\theta}) - \eta_{11}E_\psi^{\,2} - \eta_{22}E_z^{\,2} - \eta_{33}E_\theta^{\,2}\\
&-2\eta_{12}E_zE_\psi - 2\eta_{13}E_\theta E_\psi - 2\eta_{23}E_zE_\theta\\
&-2d_{111}H_\psi\varepsilon_{\psi\psi} - 2d_{222}H_z\varepsilon_{zz} - 2d_{333}H_\theta\varepsilon_{\theta\theta} - 2H_\psi(d_{221}\varepsilon_{zz} + d_{331}\varepsilon_{\theta\theta} + 2d_{121}\varepsilon_{\psi z} + 2d_{131}\varepsilon_{\psi\theta} + 2d_{231}\varepsilon_{z\theta})\\
&-2H_z(d_{112}\varepsilon_{\psi\psi} + d_{332}\varepsilon_{\theta\theta} + 2d_{122}\varepsilon_{\psi z} + 2d_{132}\varepsilon_{\psi\theta} + 2d_{232}\varepsilon_{z\theta})\\
&-2H_\theta(d_{113}\varepsilon_{\psi\psi} + d_{223}\varepsilon_{zz} + 2d_{123}\varepsilon_{\psi z} + 2d_{133}\varepsilon_{\psi\theta} + 2d_{233}\varepsilon_{z\theta})\\
&-2g_{11}H_\psi E_\psi - 2g_{22}H_zE_z - 2g_{33}H_\theta E_\theta\\
&-2H_\psi(g_{21}E_z + g_{31}E_\theta) - 2H_z(g_{12}E_\psi + g_{32}E_\theta) - 2H_\theta(g_{13}E_\psi + g_{23}E_z)\\
&-\mu_{11}H_\psi^{\,2} - \mu_{22}H_z^{\,2} - \mu_{33}H_\theta^{\,2} - 2\mu_{12}H_zH_\psi - 2\mu_{13}H_\theta H_\psi - 2\mu_{23}H_\theta H_z] + \rho_2 r(f_\psi u_\psi + f_z u_z + f_\theta u_\theta)
\end{aligned}
\tag{26}
$$

Substituting Eq. (26) into Eq. (25) gives five field equations of an arbitrary thick shell of revolution for a general nonhomogenous piezomagnetic material as follows:

$$
\begin{aligned}
&A_1u_{\psi,\psi\psi} + A_2u_{z,zz} + A_3u_{\theta,\theta\theta} + A_4\phi_{,\psi\psi} + A_5\phi_{,zz} + A_6\phi_{,\theta\theta} + A_7\Phi_{,\psi\psi} + A_8\Phi_{,zz} + A_9\Phi_{,\theta\theta} + A_{10}u_{z,\psi\psi}\\
&+A_{11}u_{\theta,\psi\psi} + A_{12}u_{\psi,zz} + A_{13}u_{\theta,zz} + A_{14}u_{\psi,\theta\theta} + A_{15}u_{z,\theta\theta} + A_{16}u_{z,z\psi} + A_{17}u_{\theta,\theta\psi} + A_{18}u_{\psi,z\psi} + A_{19}u_{\psi,\theta\psi}\\
&+A_{20}u_{z,\theta\psi} + A_{21}u_{\theta,z\psi} + A_{22}u_{\theta,\theta z} + A_{23}u_{\psi,\theta z} + A_{24}u_{z,\theta z} + A_{25}\phi_{,z\psi} + A_{26}\phi_{,\theta\psi} + A_{27}\phi_{,\theta z} + A_{28}\Phi_{,z\psi}\\
&+A_{29}\Phi_{,\theta\psi} + A_{30}\Phi_{,\theta z} + A_{31}u_{\psi,\psi} + A_{32}u_{z,z} + A_{33}u_{\theta,\theta} + A_{34}u_{\psi,z} + A_{35}u_{z,\psi} + A_{36}u_{\psi,\theta} + A_{37}u_{\theta,\psi}\\
&+A_{38}u_{z,\theta} + A_{39}u_{\theta,z} + A_{40}\phi_{,\psi} + A_{41}\phi_{,z} + A_{42}\phi_{,\theta} + A_{43}\phi_{,\psi} + A_{44}\phi_{,z} + A_{45}\phi_{,\theta} + A_{46}u_\psi + A_{47}u_z\\
&+A_{48}u_\theta + \rho_2 rf_\psi = A_{49}\ddot{u}_\psi
\end{aligned}
\tag{27-1}
$$

$$
\begin{aligned}
&B_1u_{\psi,\psi\psi} + B_2u_{z,zz} + B_3u_{\theta,\theta\theta} + B_4\phi_{,\psi\psi} + B_5\phi_{,zz} + B_6\phi_{,\theta\theta} + B_7\Phi_{,\psi\psi} + B_8\Phi_{,zz} + B_9\Phi_{,\theta\theta} + B_{10}u_{z,\psi\psi} + B_{11}u_{\theta,\psi\psi}\\
&+B_{12}u_{\psi,zz} + B_{13}u_{\theta,zz} + B_{14}u_{z,\theta\theta} + B_{15}u_{\psi,\theta\theta} + B_{16}u_{\psi,z\psi} + B_{17}u_{z,z\psi} + B_{18}u_{\theta,\theta\psi} + B_{19}u_{\psi,\theta\psi} + B_{20}u_{z,\theta\psi} + B_{21}u_{\theta,z\psi}\\
&+B_{22}u_{\theta,\theta z} + B_{23}u_{\psi,\theta z} + B_{24}u_{z,\theta z} + B_{25}\phi_{,z\psi} + B_{26}\phi_{,\theta\psi} + B_{27}\phi_{,\theta z} + B_{28}\Phi_{,z\psi} + B_{29}\Phi_{,\theta\psi} + B_{30}\Phi_{,\theta z} + B_{31}u_{\psi,\psi}\\
&+B_{32}u_{z,z} + B_{33}u_{\theta,\theta} + B_{34}u_{\psi,z} + B_{35}u_{z,\psi} + B_{36}u_{\psi,\theta} + B_{37}u_{\theta,\psi} + B_{38}u_{z,\theta} + B_{39}u_{\theta,z} + B_{40}\phi_{,\psi} + B_{41}\phi_{,z}\\
&+B_{42}\phi_{,\theta} + B_{43}\Phi_{,\psi} + B_{44}\Phi_{,z} + B_{45}\Phi_{,\theta} + B_{46}u_\psi + B_{47}u_z + B_{48}u_\theta + \rho_2 rf_z = B_{49}\ddot{u}_z
\end{aligned}
\tag{27-2}
$$

$$
\begin{aligned}
&C_1u_{\psi,\psi\psi}+C_2u_{z,zz}+C_3u_{\theta,\theta\theta}+C_4\phi_{,\psi\psi}+C_5\phi_{,zz}+C_6\phi_{,\theta\theta}+C_7\Phi_{,\psi\psi}+C_8\Phi_{,zz}+C_9\Phi_{,\theta\theta}+C_{10}u_{\theta,\psi\psi}+C_{11}u_{z,\psi\psi}\\
&+C_{12}u_{\theta,zz}+C_{13}u_{\psi,zz}+C_{14}u_{\psi,\theta\theta}+C_{15}u_{z,\theta\theta}+C_{16}u_{\psi,\theta\psi}+C_{17}u_{z,z\psi}+C_{18}u_{\theta,\theta\psi}+C_{19}u_{\psi,z\psi}+C_{20}u_{z,\theta\psi}+C_{21}u_{\theta,z\psi}\\
&+C_{22}u_{z,\theta z}+C_{23}u_{\theta,\theta z}+C_{24}u_{\psi,\theta z}+C_{25}\phi_{,z\psi}+C_{26}\phi_{,\theta\psi}+C_{27}\phi_{,\theta z}+C_{28}\Phi_{,z\psi}+C_{29}\Phi_{,\theta\psi}+C_{30}\Phi_{,\theta z}+C_{31}u_{\psi,\psi}\\
&+C_{32}u_{z,z}+C_{33}u_{\theta,\theta}+C_{34}u_{\psi,\theta}+C_{35}u_{\theta,\psi}+C_{36}u_{z,\theta}+C_{37}u_{\theta,z}+C_{38}u_{\psi,z}+C_{39}u_{z,\psi}+C_{40}\phi_{,\psi}+C_{41}\phi_{,z}+C_{42}\phi_{,\theta}\\
&+C_{43}\Phi_{,\psi}+C_{44}\Phi_{,z}+C_{45}\Phi_{,\theta}+C_{46}u_{\psi}+C_{47}u_{z}+C_{48}u_{\theta}+\rho_2 rf_{\theta}=C_{49}\ddot{u}_{\theta}
\end{aligned}
\tag{27-3}
$$

$$
\begin{aligned}
&D_1u_{\psi,\psi\psi}+D_2u_{z,zz}+D_3u_{\theta,\theta\theta}+D_4\phi_{,\psi\psi}+D_5\phi_{,zz}+D_6\phi_{,\theta\theta}+D_7\Phi_{,\psi\psi}+D_8\Phi_{,zz}+D_9\Phi_{,\theta\theta}+D_{10}u_{z,\psi\psi}\\
&+D_{11}u_{\theta,\psi\psi}+D_{12}u_{\psi,zz}+D_{13}u_{\theta,zz}+D_{14}u_{\psi,\theta\theta}+D_{15}u_{z,\theta\theta}+D_{16}u_{z,z\psi}+D_{17}u_{\theta,\theta\psi}+D_{18}u_{\psi,z\psi}+D_{19}u_{\psi,\theta\psi}\\
&+D_{20}u_{z,\theta\psi}+D_{21}u_{\theta,z\psi}+D_{22}u_{\theta,\theta z}+D_{23}u_{\psi,\theta z}+D_{24}u_{z,\theta z}+D_{25}\phi_{,z\psi}+D_{26}\phi_{,\theta\psi}+D_{27}\phi_{,\theta z}+D_{28}\Phi_{,z\psi}\\
&+D_{29}\Phi_{,\theta\psi}+D_{30}\Phi_{,\theta z}+D_{31}u_{\psi,\psi}+D_{32}u_{z,z}+D_{33}u_{\theta,\theta}+D_{34}u_{z,\psi}+D_{35}u_{\psi,z}+D_{36}u_{\psi,\theta}+D_{37}u_{\theta,\psi}\\
&+D_{38}u_{z,\theta}+D_{39}u_{\theta,z}+D_{40}\phi_{,\psi}+D_{41}\phi_{,z}+D_{42}\phi_{,\theta}+D_{43}\Phi_{,\psi}+D_{44}\Phi_{,z}+D_{45}\Phi_{,\theta}+D_{46}u_{\psi}+D_{47}u_{z}+D_{48}u_{\theta}=0
\end{aligned}
\tag{27-4}
$$

$$
\begin{aligned}
&E_1u_{\psi,\psi\psi}+E_2u_{z,zz}+E_3u_{\theta,\theta\theta}+E_4\phi_{,\psi\psi}+E_5\phi_{,zz}+E_6\phi_{,\theta\theta}+E_7\Phi_{,\psi\psi}+E_8\Phi_{,zz}+E_9\Phi_{,\theta\theta}+E_{10}u_{z,\psi\psi}+E_{11}u_{\theta,\psi\psi}\\
&+E_{12}u_{\psi,zz}+E_{13}u_{\theta,zz}+E_{14}u_{\psi,\theta\theta}+E_{15}u_{z,\theta\theta}+E_{16}u_{z,z\psi}+E_{17}u_{\theta,\theta\psi}+E_{18}u_{\psi,z\psi}+E_{19}u_{\psi,\theta\psi}+E_{20}u_{z,\theta\psi}\\
&+E_{21}u_{\theta,z\psi}+E_{22}u_{\theta,\theta z}+E_{23}u_{\psi,\theta z}+E_{24}u_{z,\theta z}+E_{25}\phi_{,z\psi}+E_{26}\phi_{,\theta\psi}+E_{27}\phi_{,\theta z}+E_{28}\Phi_{,z\psi}+E_{29}\Phi_{,\theta\psi}+E_{30}\Phi_{,\theta z}\\
&+E_{31}u_{\psi,\psi}+E_{32}u_{z,z}+E_{33}u_{\theta,\theta}+E_{34}u_{z,\psi}+E_{35}u_{\psi,z}+E_{36}u_{\psi,\theta}+E_{37}u_{\theta,\psi}+E_{38}u_{z,\theta}+E_{39}u_{\theta,z}+E_{40}\phi_{,\psi}\\
&+E_{41}\phi_{,z}+E_{42}\phi_{,\theta}+E_{43}\Phi_{,\psi}+E_{44}\Phi_{,z}+E_{45}\Phi_{,\theta}+E_{46}u_{\psi}+E_{47}u_{z}+E_{48}u_{\theta}=0
\end{aligned}
\tag{27-5}
$$

where, A_i, B_i, C_i, D_i, E_i are variable coefficients presented in the Appendices 1-5. Eqs.27-1, 27-2, 27-3, 27-4 and 27-5 are the complete set of partial differential equations, which describes the time dependent piezo-magneto-elastic behavior of a functionally graded piezomagnetic thick shell of revolution.

3 RESULTS AND DISCUSSION

This section presents two applied problems of functionally graded materials. These problems are solved using the obtained differential equations (Eqs. (27-1), (27-2), (27-3), (27-4) and (27-5)) with some assumptions.

3.1. Reduction of obtained equations for a functionally graded piezoelectric cylindrical shell

For validation of the obtained differential equations and applying them for well-known applied problems, it is convenient to investigate the relation between mechanical and electrical components in a rotationally symmetric functionally graded piezoelectric cylinder with stable conditions (Khoshgoftar *et al.*, 2009).

Due to symmetric assumption $(\frac{\partial}{\partial\theta}=0)$ and disregarding of any axial mechanical and electrical loads $(\frac{\partial}{\partial\psi}=0)$, only the nonzero components of displacement is radial displacement

$u_r = u_r(r) \neq 0$, which is only a function of radial coordinate. Based on the above assumptions, the electric potential is a function of radial coordinate $\phi = \phi(r)$.

By applying the above simplifications, within the numerous coefficients, only six coefficients remain non-zero through the analysis procedure. They include $C_{2222}, C_{3333}, C_{2233}, e_{222}, e_{332}$ and η_{22}. Considering the above assumptions and then the geometric relations for a cylindrical shell, ($\psi = \pi/2, \rho_2, \rho_\theta \to \infty, \rho_\psi + \mathrm{z} = \mathrm{r}, \rho_\psi = \mathrm{const}, \mathrm{f_z} = \lambda \mathrm{r}\omega^2, \partial/\partial \mathrm{r} = \partial/\partial \mathrm{z}$), we will have strain-displacement and electric potential-electric field relations as follows:

$$\varepsilon_{zz} = \varepsilon_{\mathrm{rr}} = \mathrm{u_{r,r}} = \frac{\partial \mathrm{u_r}}{\partial \mathrm{r}}, \varepsilon_{\theta\theta} = \frac{\mathrm{u_r}}{\mathrm{r}}, \mathrm{E_r} = -\frac{\partial \phi}{\partial \mathrm{r}} \tag{28}$$

The corresponding components of stress and electric displacement may be obtained using the above relation and the variable distribution of material properties as follows:

$$\begin{cases} \sigma_{\mathrm{rr}} = \mathrm{C_{2222}(r)}\varepsilon_{\mathrm{rr}} + \mathrm{C_{2233}(r)}\varepsilon_{\theta\theta} - \mathrm{e_{222}(r)E_r} \\ \sigma_{\theta\theta} = \mathrm{C_{2233}(r)}\varepsilon_{\mathrm{rr}} + \mathrm{C_{3333}(r)}\varepsilon_{\theta\theta} - \mathrm{e_{332}(r)E_r} \\ \mathrm{D_r} = \mathrm{e_{222}(r)}\varepsilon_{\mathrm{rr}} + \mathrm{e_{332}(r)}\varepsilon_{\theta\theta} + \eta_{22}\mathrm{(r)E_r} \end{cases} \tag{29}$$

The variable material properties can be considered as a power function along the thickness direction of the cylinder (Khoshgoftar *et al.*, 2009; Rahimi *et al.*, 2011):

$$\begin{aligned} &\mathrm{C_{2222}(r) = C_{2222\,0}r^l, C_{2233}(r) = C_{2233\,0}r^l, C_{3333}(r) = C_{3333\,0}r^l} \\ &\mathrm{e_{222}(r) = e_{222\,0}r^l, e_{332}(r) = e_{332\,0}r^l}, \eta_{22}\mathrm{(r)} = \eta_{22\,0}\mathrm{r^l}, \lambda\mathrm{(r)} = \lambda_0\mathrm{r^l} \end{aligned} \tag{30}$$

By applying the mentioned assumptions in this section, only Eqs. (27-2) and (27-4) remain non-vanished. The coefficients of two non-vanishing differential equations are evaluated as follows:

$$B_2 = (C_{2222}\rho_2 r) = C_{2222\,0} r^{l+1}\rho_2,$$

$$B_5 = e_{222}\rho_2 r = e_{222\,0} r^{l+1}\rho_2$$

$$B_{26} = (-rC_{1122} - \rho_2 C_{2233}\sin\psi + (rC_{2212})_{,\psi} + (C_{2222}\rho_2 r)_{,z} + C_{1122}r + C_{2233}\rho_2\sin\psi + (\rho_2 C_{2223})_{,\theta})$$

$$= (C_{2222}\rho_2 r)_{,r} = (C_{2222\,0} r^l \rho_2 r)_{,r} = (l+1)\rho_2 r^l C_{2222\,0}$$

$$B_{35} = (-\rho_2 \sin\psi e_{332} + (e_{222}\rho_2 r)_{,z}) = (-\rho_2 e_{332\,0} r^l + (e_{222\,0} r^l \rho_2 r)_{,z}) = r^l \rho_2((l+1)e_{222\,0} - e_{332\,0})$$

$$B_{38} = (-\frac{\rho_2 C_{3333}}{r}\sin^2\psi - C_{1133}\sin\psi - \frac{rC_{1111}}{\rho_2} + (\frac{rC_{1112}}{\rho_2})_{,\psi} + C_{3312,\psi}\sin\psi + C_{3312}\cos\psi + (C_{1122}r)_{,z} + (C_{2233}\rho_2)_{,z}\sin\psi$$

$$+C_{1123,\theta} + (\frac{\rho_2 C_{3323}}{r})_{,\theta}\sin\psi) = (-\frac{\rho_2 C_{3333\,0} r^l}{r}\sin^2\psi + (C_{2233\,0} r^l \rho_2)_{,r}\sin\psi) = r^{l-1}\rho_2(-C_{3333\,0} + lC_{2233\,0})$$

In order to attain to appropriate results and better comparison with the literature, it is better to divide all the sentences by $\rho_2 r^{l-1}$. Therefore, the first differential equation can be composed as:

$$r^2 C_{2222\,0} u_{r,rr} + e_{222\,0} r^2 \phi_{,rr} + (l+1)rC_{2222\,0} u_{r,r} + r((l+1)e_{222\,0} - e_{332\,0})\phi_{,r} + (-C_{3333\,0} + lC_{2233\,0})u_r + \lambda_0 r^3 \omega^2 = 0 \tag{31}$$

The second equation must be evaluated using Eq. (27-4). The appropriate coefficients can be then calculated as:

$$D_2 = \rho_2 r e_{222} = \rho_2 e_{2220} r^{l+1}$$

$$D_5 = -\rho_2 r \eta_{22} = -\rho_2 r^{l+1} \eta_{220}$$

$$D_{26} = ((\rho_2 r e_{222})_{,z} + \rho_2 e_{332} \sin\psi) = ((\rho_2 r e_{2220} r^l)_{,r} + \rho_2 e_{3320} r^l) = \rho_2 r^l ((l+1) e_{2220} + e_{3320})$$

$$D_{35} = -((\rho_2 r \eta_{22})_{,z}) = -((\rho_2 r \eta_{220} r^l)_{,r}) = -(l+1)\eta_{220} r^l, D_{38} = (\rho_2 e_{332})_{,z} \sin\psi = (\rho_2 e_{3320} r^l)_{,r} = l\rho_2 r^{l-1} e_{3320}$$

Therefore, second equation can be composed as follows:

$$e_{2220} r^2 u_{r,rr} - \eta_{220} r^2 \phi_{,rr} + r((l+1) e_{2220} + e_{3320}) u_{r,r} - (l+1)\eta_{220} r \phi_{,r} + l e_{3320} u_r = 0 \tag{32}$$

It is observed that the present general multi-field formulation can predict exactly the behavior of a functionally graded cylinder. The obtained simplified equations indicate that the present formulation using the curvilinear coordinate system and Hamilton's principle for arbitrary nonhomogeneity and variable thickness and curvature is identical with that of the literature that has been derived using the plane elasticity theory [10]. This case study justifies the capability and necessity of the present study in derivation of the governing differential equations of the curvilinear coordinate system for piezoelectric structures with arbitrary geometry and material properties.

3.2. Reduction of obtained equations for a functionally graded piezomagnetic cylindrical shell

The relation between mechanical and magnetic components in a symmetric functionally graded piezomagnetic cylinder with stable conditions can be derived using the tensor analysis and compared with reference (Arefi *et al.*, 2011).

After applying previously mentioned assumptions in Sec 3.1, and considering magnetic potential instead of electric potential $\Phi = \Phi(r)$, within the numerous coefficients, only six coefficients remain non-zero through the analysis procedure. They include $C_{2222}, C_{3333}, C_{2233}, d_{222}, d_{332}$ and μ_{22}. The corresponding components of stress and magnetic displacement may be obtained using:

$$\begin{cases} \sigma_{rr} = C_{2222}(r)\varepsilon_{rr} + C_{2233}(r)\varepsilon_{\theta\theta} - d_{222}(r)H_r \\ \sigma_{\theta\theta} = C_{2233}(r)\varepsilon_{rr} + C_{3333}(r)\varepsilon_{\theta\theta} - d_{332}(r)H_r \\ B_r = d_{222}(r)\varepsilon_{rr} + d_{332}(r)\varepsilon_{\theta\theta} + \mu_{22}(r)H_r \end{cases} \tag{33}$$

where $\sigma_{ij}, \varepsilon_{kl}, H_k$ are components of stress, strains and magnetic field, respectively. $C_{ijkl}, d_{ijk}, \mu_{ij}$ are stiffness and piezomagnetic and magnetic coefficients, respectively.

The variable material properties can be considered as a power function along the thickness direction of the cylinder (Arefi *et al.*, 2011):

$$\begin{aligned} &C_{2222}(r) = C_{2222\,0} r^{l}, C_{2233}(r) = C_{2233\,0} r^{l}, C_{3333}(r) = C_{3333\,0} r^{l} \\ &d_{222}(r) = d_{222\,0} r^{l}, d_{332}(r) = d_{332\,0} r^{l}, \mu_{22}(r) = \mu_{22\,0} r^{l} \end{aligned} \tag{34}$$

By applying the mentioned assumptions in this section, only Eqs. (27-2) and (27-5) remain non-vanished. The coefficients of two non-vanishing differential equations are evaluated as follows:

$$\begin{aligned} &B_2 u_{r,rr} + B_8 \Phi_{,rr} + B_{32} u_{r,r} + B_{44} \Phi_{,r} = 0 \\ &B_2 = (C_{2222}\rho_2 r), B_8 = d_{222}\rho_2 r, B_{32} = ((rC_{2212})_{,\psi} + (C_{2222}\rho_2 r)_{,z} + (\rho_2 C_{2223})_{,\theta}) \\ &B_{44} = (-rd_{112} - \rho_2 \sin\psi d_{332} + (rd_{212})_{,\psi} + (d_{222}\rho_2 r)_{,z} + (\rho_2 d_{223})_{,\theta}) \end{aligned}$$

After some simplifications the first differential equation can be composed as:

$$r^2 C_{2222\,0} u_{r,rr} + d_{222\,0} r^2 \Phi_{,rr} + (l+1) r C_{2222\,0} u_{r,r} + r((l+1) d_{222\,0} - d_{332\,0}) \Phi_{,r} + (-C_{3333\,0} + l C_{2233\,0}) u_r = 0 \tag{35}$$

The second equation must be evaluated using Eq. (27-5). The appropriate coefficients can be then calculated as:

$$\begin{aligned} &E_2 u_{r,rr} + E_8 \Phi_{,rr} + E_{32} u_{r,r} + E_{44} \Phi_{,r} = 0 \\ &E_2 = \rho_2 r d_{222}, E_8 = \rho_2 r \mu_{22}, E_{32} = ((r d_{221})_{,\psi} + (\rho_2 r d_{222})_{,z}, E_{44} = -((r\mu_{12})_{,\psi} + (\rho_2 r \mu_{22})_{,z} + (\mu_{23}\rho_2)_{,\theta}) \end{aligned}$$

Therefore, second equation can be composed as follows:

$$d_{222\,0} r^2 u_{r,rr} - \mu_{22\,0} r^2 \Phi_{,rr} + r((l+1) d_{222\,0} + d_{332\,0}) u_{r,r} - (l+1)\mu_{22\,0} r \Phi_{,r} + l d_{332\,0} u_r = 0 \tag{36}$$

It can be indicated that the obtained formulation in this paper based on the tensor analysis and in an orthogonal curvilinear coordinate system is identical with those relations derived in previous work (Arefi *et al.*, 2011).

4 NUMERICAL RESULTS

In this section, for validation of the obtained differential equations for the simpler geometries and boundary conditions, the responses of a stationary functionally graded piezoelectric cylinder under 80MPa inner pressure is presented. The inner and outer radii of cylinder is considered 0.6, 1. All material properties are considered to varying along the radial direction by a power function (Eq.30). In order to evaluate the responses of the system, Eqs. (31) and (32) must be concurrently solved. For convenience, these equations are reconsidered by setting $\omega = 0$.

$$\begin{aligned} &r^2 C_{2222\,0} u_{r,rr} + e_{222\,0} r^2 \phi_{,rr} + (l+1) r C_{2222\,0} u_{r,r} + r((l+1) e_{222\,0} - e_{332\,0}) \phi_{,r} + (-C_{3333\,0} + l C_{2233\,0}) u_r = 0 \\ &e_{222\,0} r^2 u_{r,rr} - \eta_{22\,0} r^2 \phi_{,rr} + r((l+1) e_{222\,0} + e_{332\,0}) u_{r,r} - (l+1)\eta_{22\,0} r \phi_{,r} + l e_{332\,0} u_r = 0 \end{aligned} \tag{37}$$

One can detect that these equations are a system of differential equations of order two. Unknown functions are radial displacement u and electric potential function ϕ. These equations are known as Cauchy- Euler equations. The solution of the system can be defined by changing the variable from r to s as $r = e^S$. This change yields two differential equations as follows:

$$C_{2222\,0}\bar{u}_{r,ss}+e_{222\,0}\bar{\phi}_{,ss}+lC_{2222\,0}\bar{u}_{r,s}+(le_{222\,0}-e_{332\,0})\bar{\phi}_{,s}+(-C_{3333\,0}+lC_{2233\,0})\bar{u}_r=0$$
$$e_{222\,0}\bar{u}_{r,ss}-\eta_{22\,0}\bar{\phi}_{,ss}+(le_{222\,0}+e_{332\,0})\bar{u}_{r,s}-l\eta_{22\,0}\phi_{,s}+le_{332\,0}\bar{u}_r=0 \tag{38}$$

where, $\bar{u}_r=u_r(r=e^s)$ and $\bar{\phi}_r=\phi_r(r=e^s)$. Solution of Eq. (38) can be expressed as follows:

$$\bar{u}_r=\sum_{i=j}^{4}u_{ci}r^{\lambda_i}$$
$$\bar{\varphi}=\sum_{i=j}^{4}M_i u_{ci}r^{\lambda_i} \tag{39}$$
$$M_i=-\frac{C_{2222\,0}\lambda_i^2+C_{2222\,0}l\lambda_i+C_{2233\,0}l-C_{3333\,0}}{e_{222\,0}\lambda_i^2+(e_{222\,0}l-e_{332\,0})\lambda_i}$$

where, λ_i are the roots of characteristic equation and u_{ci} are constants of integration. Characteristic equation is determianat of coefficient of Eq. (38) as defined as follows:

$$[D^2C_{2222\,0}+DlC_{2222\,0}+D(-C_{3333\,0}+lC_{2233\,0})]\bar{u}_r+[D^2e_{222\,0}+D(le_{222\,0}-e_{332\,0})]\bar{\phi}=0$$
$$[D^2e_{222\,0}+D(le_{222\,0}+e_{332\,0})+le_{332\,0}]\bar{u}_r-[D^2\eta_{22\,0}+Dl\eta_{22\,0}]\bar{\phi}=0 \tag{40}$$

where, D is defined as the derivative with respect to s: $(D=\frac{d}{ds})$. By setting $D=\lambda$, the characteristic equation can be defined as follows:

$$\begin{vmatrix}\lambda^2C_{2222\,0}+\lambda lC_{2222\,0}+(-C_{3333\,0}+lC_{2233\,0}) & \lambda^2e_{222\,0}+\lambda(le_{222\,0}-e_{332\,0})\\ \lambda^2e_{222\,0}+\lambda(le_{222\,0}+e_{332\,0})+le_{332\,0} & -(\lambda^2\eta_{22\,0}+\lambda l\eta_{22\,0})\end{vmatrix}=0 \tag{41}$$

Solution of characteristic equation defined in Eq. (41) presents four roots of the problem. Solution procedure can be completed by employing required boundary condition.

In this section the appropriate electrical and mechanical boundary conditions are determined. The four boundary conditions can be written as:

$$\sigma_{rr}(r=a)=-80MPa \qquad \sigma_{rr}(r=b)=0$$
$$\varphi(r=a)=0 \qquad \varphi(r=b)=0 \tag{42}$$

After employing boundary conditions of the system, numerical results can be presented. Figures 2, 3 show the radial distribution of radial displacement and electric potential along the thickness direction for different values of non homogenous index.

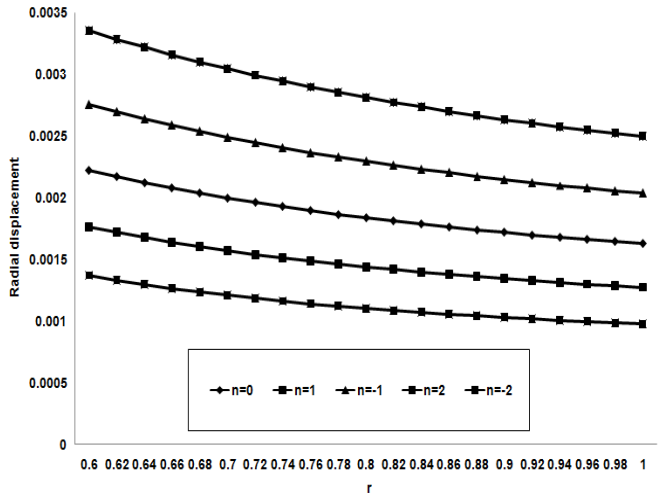


Figure 2: The radial distribution of radial displacement for a functionally graded piezoelectric cylinder.

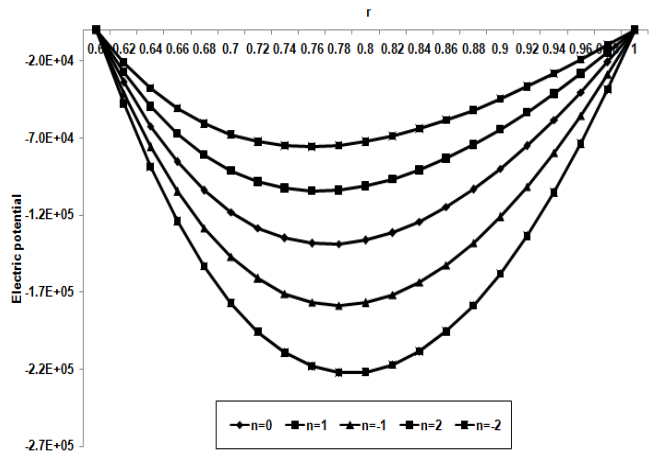


Figure 3: The radial distribution of electric potential for a functionally graded piezoelectric cylinder.

5 CONCLUSION

1. Based on the tensor analysis and using an orthogonal curvilinear coordinate system, a complete set of linear partial differential equations for piezo-magneto-elastic behavior of a functionally graded piezomagnetic shell of revolution with variable thickness and curvature has been derived. These formulations can be applied for studying a general shell and can be reduced for analysis of simpler geometries such as cylinders and spheres.

2. The governing differential equations have been derived with considering no limitation on the constitutive equations. It was assumed that all strain, electric and magnetic fields have direct and explicit effect on the all stresses, electric and magnetic displacements. This is a comprehensive model for studying an arbitrary structure made of completely anisotropic functionally graded piezomagnetic materials.

3. The derived governing differential equations have been reduced for studying the behaviors of functionally graded piezoelectric and piezomagnetic cylinders. These reductions have been validated with literatures (Khoshgoftar *et al.*, 2009, Arefi *et al.*, 2011).

Nomenclature

$A_i, B_i, C_i (i = 1..49), D_i, E_i (i = 1..48)$	The coefficients of partial time dependent differential equation of system
C_{ijkl}	Elastic stiffness coefficient
D_i	Physical components of electric displacement
d_i	Piezomagnetic coefficient
$\vec{E}, E_i$	Vector and components of electric field, respectively.
e_{ijk}	Piezoelectric coefficient
$F(u_\psi, u_z, u_\theta, \phi, \Phi, t)$	Functional of the system
g_i	Covariant base vector
g^i	Contra-variant base vector
g_{ij}	Covariant metric vector
g^{ij}	Contra-variant metric vector
$\vec{H}, H_i$	Vector and components of magnetic field, respectively.
l	Nonhomogenous index
r	Radius of revolution for any arbitrary point
$\bar{r}$	Radius of revolution of mid plane
ds	Differential distance in meridian direction
S_i, S_{ij}	A symbolic tensor of order one and two, respectively.
u_ψ, u_z, u_θ	Orthogonal components of displacement in orthogonal coordinate system
$\bar{u}$	Total energy per unit volume of the structure
$\bar{u}_p$	Potential energy per unit volume of the structure
$\bar{u}_k$	Kinetic energy per unit volume of the structure

$\dot{u}_{\psi}, \dot{u}_{z}, \dot{u}_{\theta}$	Velocity components in curvilinear coordinate system
dV	Unit volume of the structure
X	Position vector of an arbitrary point
q_i, X_i	Component of position vector in Cartesian coordinate system
y	Vertical distance of mid-plane
z	Second component in curvilinear coordinate system that describes normal distance of an arbitrary point from mid-plane
v	Poisson ratio
ψ	First component in curvilinear coordinate system that describes the angle between normal to mid-plane and vertical axis (axis of revolution)
θ	Third component in curvilinear coordinate system that describes circumferential direction and it's angle
ρ_{ψ}	Distance between mid-plane and axis of revolution along normal to mid-plane.
ρ_{θ}	Meridian radius of curvature
ρ_1	$= \rho_{\psi} + z$
ρ_2	$= \rho_{\theta} + z$
$\Gamma_{ijk}, \Gamma_{ij}^{\ k}$	Christoffel symbols of first and second kind
γ_{ij}	Tensor component of strain
ε_{ij}	Physical component of strain
ϕ	Electric potential
Φ	Magnetic potential
σ_{ij}	Physical components of stress tensor
η_{ij}	Dielectric coefficient
μ_{ij}	Magnetic coefficients
$\vec{\nabla}$	Del operator
$\| \ \|$	Describes the magnitude of a component

References

Arciniega, R. A., Reddy, J. N. (2007). Tensor-based finite element formulation for geometrically nonlinear analysis of shell structures. Com. Met. Appl. Mech. Engrg. 196: 1048–1073.

Arciniega, R. A, Reddy, J. N. (2007). Large deformation analysis of functionally graded shells. Int. J. Solids. Struct. 44: 2036–2052.

Arefi M., Rahimi G. H. and Khoshgoftar M. J., (2011). Optimized design of a cylinder under mechanical, magnetic and thermal loads as a sensor or actuator using a functionally graded piezomagnetic material, Int. J. Phy. Sci., 6(27), 6315-6322.

Brischetto, S., Carrera, E., (2009). Refined 2D models for the analysis of functionally graded piezoelectric plates. J. Intell. Mat. Syst. Str. 20(15): 1783-1797.

Babaei M. H., Chen Z. T., (2008). Exact solutions for radially polarized and magnetized magnetoelectroelastic rotating cylinders . Smart. Mater. Struct. 17: 025035 (11pp).

Chen W. Q., Lu Y., Ye J. R., Cai J. B., (2002). 3D electroelastic fields in a functionally graded piezoceramic hollow sphere under mechanical and electric loading . Arch. Appl. Mech. 72: 39-51.

Carrera, E., Brischetto, S., (2007). Reissner mixed theorem applied to static analysis of piezoelectric shells. J. Intell. Mat. Syst. Str. 18(10): 1083-1107.

Carrera, E., Brischetto, S., (2007). Piezoelectric shell theories with a priori continuous transverse electromechanical variables. J. Mech. Mater. Struct. 2(2): 377-399.

Dai H. L., Fu Y. M., Yang J. H., (2007). Electromagnetoelastic behaviors of functionally graded piezoelectric solid cylinder and sphere . Acta. Mech. Solida. Sin. 23: 55-63.

Dai, H. L., Hong, L., Fu, Y. M., Xiao, X., (2010). Analytical solution for electromagnetothermoelastic behaviors of a functionally graded piezoelectric hollow cylinder . Appl. Math. Modelling. 34: 343–357.

Jabbari M., Bahtui A., Eslami M. R., (2009). Axisymmetric Mechanical and Thermal Stresses in Thick Short Length FGM Cylinders . Int. J. Pres. Ves. Pip. 86: 296–306.

Kang, J. H. (2007). Field equations, equations of motion, and energy functionals for thick shells of revolution with arbitrary curvature and variable thickness from a three-dimensional theory. Acta. Mech. 188: 21–37.

Khoshgoftar, M. J., G Arani, A., Arefi, M., (2009). Thermoelastic analysis of a thick walled cylinder made of functionally graded piezoelectric material . Smart. Mater. Struct. 18: 115007 (8 pp).

Liu X., Wang Q., Quek S. T., (2002). Analytical solution for free vibration of piezoelectric coupled moderately thick circular plates . Int. J. Solids. Struct. 39: 2129–2151.

Lu P., Lee H. P., Lu C., (2005). An exact solution for functionally graded piezoelectric laminated in cylindrical bending . Int. J. Mech. Sci. 47: 437-458.

Peng-Fei H., Andrew Y. T., (2004). The transient responses of magneto-electro-elastic hollow cylinders , Smart. Mater. Struct. 13: 762–776.

Ray, M. C., Reddy, J. N., (2005). Active control of laminated cylindrical shells using piezoelectric fiber reinforced composites. Composites. Sci. Tech. 65: 1226–1236.

Rahimi, G. H., Arefi, M., Khoshgoftar, M. J., (2011). Application and analysis of functionally graded piezoelectrical rotating cylinder as mechanical sensor subjected to pressure and thermal loads. Appl. Math. Mech. (Engl. Ed) 32(8): 997-1008.

S Khabbaz, R. D Manshadi, B. and Abedian, A. (2009). Non-linear analysis of FGM plates under pressure loads using the higher-order shear deformation theories, Compos. Struct., 89, 333–344

Sheng G. G., Wang X., (2010). Thermoelastic vibration and buckling analysis of functionally graded piezoelectric cylindrical shells. Appl. Math. Modelling. 34: 2630–2643.

Santos H., Mota, Soares C. M. Mota., Soares, C.A., Reddy, J. N., (2008). A finite element model for the analysis of 3D axisymmetric laminated shells with piezoelectric sensors and actuators: Bending and free vibrations. Comput. Struct. 86: 940–947.

Woo, J., Meguid, S.A., (2001). Nonlinear analysis of functionally graded plates and shallow shells. Int. J. Solids. Struct., 38: 7409-7421.

Acknowledgements

The authors would like to gratefully acknowledge the financial support by University of Kashan (Grant Number: 263475/10).

Appendix 1

$$A_1=\frac{rC_{1111}}{\rho_2},A_2=C_{2212}\rho_2 r,A_3=\frac{C_{3313}\rho_2}{r},A_4=\frac{re_{111}}{\rho_2},A_5=e_{212}\rho_2 r,A_6=\frac{\rho_2 e_{313}}{r},A_7=\frac{rd_{111}}{\rho_2},A_8=d_{212}\rho_2 r,$$

$$A_9=\frac{\rho_2 d_{313}}{r},A_{10}=\frac{rC_{1112}}{\rho_2},A_{11}=(\frac{rC_{1113}}{\rho_2}),A_{12}=C_{1212}\rho_2 r,A_{13}=C_{1223}\rho_2 ru,A_{14}=\frac{C_{1313}\rho_2}{r},A_{15}=2\frac{C_{1323}\rho_2}{r},$$

$$A_{16}=r(C_{1122}+C_{1212}),A_{17}=(C_{1133}+C_{1313}),A_{18}=r(C_{1112}+C_{1112}),A_{19}=(C_{1113}+C_{1113}),A_{20}=(C_{1123}+2C_{1213}),$$

$$A_{21}=r(C_{1123}+C_{1213}),A_{22}=\rho_2(C_{3312}+2C_{1323}),A_{23}=\rho_2(C_{1213}+2C_{1213}),A_{24}=\rho_2(C_{1223}+C_{2213}),A_{25}=r(e_{112}+e_{112}),$$

$$A_{26}=(e_{113}+e_{113}),A_{27}=\rho_2(e_{312}+e_{213}),A_{28}=r(d_{112}+d_{112}),A_{29}=(d_{113}+d_{113}),A_{30}=\rho_2(d_{312}+d_{213}),$$

$$A_{31}=(+\frac{rC_{1112}}{\rho_2}+(\frac{rC_{1111}}{\rho_2})_{,\psi}-\frac{rC_{1112}}{\rho_2}+(C_{1112}r)_{,z}+C_{1113,\theta}),A_{32}=(-\rho_2\cos\psi C_{2233}+rC_{2212}+(rC_{1122})_{,\psi}+C_{1112}r$$

$$+(C_{2212}\rho_2 r)_{,z}+C_{3312}\rho_2\sin\psi+(\rho_2 C_{2213})_{,\theta}),A_{33}=(-\frac{\rho_2 C_{3333}}{r}\cos\psi+C_{3312}+C_{1133,\psi}+(C_{3312}\rho_2)_{,z}-\frac{C_{1313}\rho_2}{r}\cos\psi$$

$$+(\frac{C_{3313}\rho_2}{r})_{,\theta}-2\frac{C_{1323}\rho_2}{r}\sin\psi),A_{34}=((rC_{1112})_{,\psi}+(C_{1212}\rho_2 r)_{,z}+2(\rho_2 C_{1213})_{,\theta}),A_{35}=(\frac{rC_{1212}}{\rho_2}-\cos\psi C_{3312}+\frac{rC_{1111}}{\rho_2}$$

$$+C_{1133}\sin\psi-(\frac{rC_{1112}}{\rho_2})_{,\psi}+(C_{1212}r)_{,z}+2C_{1213,\theta}),A_{36}=(+C_{1213}+C_{1113,\psi}+(C_{1213}\rho_2)_{,z}+(\frac{C_{1313}\rho_2}{r})_{,\theta}-2C_{1213})$$

$$A_{37}=(-\cos\psi C_{3313}+\frac{rC_{1213}}{\rho_2}-C_{1113}\cos\psi+(\frac{rC_{1113}}{\rho_2})_{,\psi}-C_{1123}\sin\psi+(C_{1213}r)_{,z}+C_{1313,\theta})$$

$$A_{38}=(-\frac{\rho_2\cos\psi C_{3323}}{r}+C_{1223}+C_{1123,\psi}+(C_{1223}\rho_2)_{,z}+C_{1113}+\frac{C_{3313}\rho_2}{r}\sin\psi+2(\frac{C_{1323}\rho_2}{r})_{,\theta})$$

$$A_{39}=(-\rho_2\cos\psi C_{3323}+rC_{1223}+(rC_{1123})_{,\psi}-C_{1213}\rho_2\cos\psi-C_{1223}\rho_2\sin\psi+(C_{1223}\rho_2 r)_{,z}+2(C_{1323}\rho_2)_{,\theta})$$

$$A_{40}=((\frac{re_{111}}{\rho_2})_{,\psi}+(e_{112}r)_{,z}+(e_{113})_{,\theta}-\cos\psi e_{331}-\frac{re_{112}}{\rho_2}),A_{41}=((e_{112}r)_{,\psi}+(e_{212}\rho_2 r)_{,z}+(\rho_2 e_{213})_{,\theta}$$

$$-\rho_2\cos\psi e_{332}-re_{212}),A_{42}=(e_{113,\psi}+(e_{312}\rho_2)_{,z}+(\frac{\rho_2 e_{313}}{r})_{,\theta}-\frac{\rho_2\cos\psi e_{333}}{r}-e_{312}),A_{43}=((\frac{rd_{111}}{\rho_2})_{,\psi}$$

$$+(d_{112}r)_{,z}+(d_{113})_{,\theta}-\cos\psi d_{331}-\frac{rd_{112}}{\rho_2}),A_{44}=((d_{112}r)_{,\psi}+(d_{212}\rho_2 r)_{,z}+(\rho_2 d_{213})_{,\theta}-\rho_2\cos\psi d_{332}-rd_{212})$$

$$A_{45}=(d_{113,\psi}+(d_{312}\rho_2)_{,z}+(\frac{\rho_2 d_{313}}{r})_{,\theta}-\frac{\rho_2\cos\psi d_{333}}{r}-d_{312}),A_{46}=(-\frac{\rho_2 C_{3333}\cos^2\psi}{r}-\frac{rC_{1212}}{\rho_2}+2\cos\psi C_{3312}$$

$$+C_{1133,\psi}\cos\psi-C_{1133}\sin\psi-(\frac{rC_{1112}}{\rho_2})_{,\psi}-(C_{1212}r)_{,z}+(C_{3312}\rho_2)_{,z}\cos\psi+(\frac{C_{3313}\rho_2}{r})_{,\theta}\cos\psi-2C_{1213,\theta})$$

$$A_{47}=(-\frac{\rho_2 C_{3333}\cos\psi\sin\psi}{r}-\cos\psi C_{1133}+\frac{rC_{1112}}{\rho_2}+C_{3312}\sin\psi+(\frac{rC_{1111}}{\rho_2})_{,\psi}+C_{1133,\psi}\sin\psi+C_{1133}\cos\psi$$

$$+(C_{1112}r)_{,z}+(C_{3312}\rho_2)_{,z}\sin\psi+C_{1113,\theta}+(\frac{C_{3313}\rho_2}{r})_{,\theta}\sin\psi),A_{48}=(\frac{\rho_2 C_{3313}\cos^2\psi}{r}+\frac{\rho_2\cos\psi C_{3323}}{r}$$

$$-C_{1213}\cos\psi-C_{1223}\sin\psi-C_{1113,\psi}\cos\psi+C_{1113}\sin\psi-C_{1123,\psi}\sin\psi-C_{1123}\cos\psi-(C_{1213}\rho_2)_{,z}\cos\psi$$

$$-(C_{1223}\rho_2)_{,z}\sin\psi-(\frac{C_{1313}\rho_2}{r})_{,\theta}\cos\psi-2(\frac{C_{1323}\rho_2}{r})_{,\theta}\sin\psi),A_{49}=\lambda(\psi,z,\theta)\rho_2 r$$

Appendix 2

$$B_1=\frac{rC_{1112}}{\rho_2},B_2=(C_{2222}\rho_2 r),B_3=\frac{\rho_2 C_{3323}}{r},B_4=\frac{re_{112}}{\rho_2},B_5=e_{222}\rho_2 r,B_6=\frac{\rho_2 e_{323}}{r},B_7=\frac{rd_{112}}{\rho_2},B_8=d_{222}\rho_2 r,$$

$$B_9=\frac{\rho_2 d_{323}}{r},B_{10}=\frac{rC_{1212}}{\rho_2},B_{11}=\frac{rC_{1213}}{\rho_2},B_{12}=(C_{2212}\rho_2 r),B_{13}=C_{2223}\rho_2 r,B_{14}=\frac{\rho_2 C_{2323}}{r},B_{15}=\frac{\rho_2 C_{1323}}{r},$$

$$B_{16}=r(C_{1212}+C_{1122}),B_{17}=r(C_{2212}+C_{2212}),B_{18}=(C_{3312}+C_{1323}),B_{19}=(C_{1213}+C_{1123}),B_{20}=(C_{1223}+C_{1223}),$$

$$B_{21}=r(C_{1223}+C_{2213}),B_{22}=\rho_2(C_{2233}+C_{2323}),B_{23}=\rho_2(C_{2213}+C_{1223}),B_{24}=\rho_2(C_{2223}+C_{2223}),B_{25}=r(e_{212}+e_{221})$$

$$B_{26}=(e_{312}+e_{123}),B_{27}=\rho_2(e_{223}+e_{223}),B_{28}=r(d_{212}+d_{221}),B_{29}=(d_{312}+d_{123}),B_{30}=\rho_2(d_{223}+d_{223}),$$

$$B_{31}=(-\frac{rC_{1111}}{\rho_2}-\frac{rC_{1212}}{\rho_2}+(\frac{rC_{1112}}{\rho_2})_{,\psi}+(C_{1122}r)_{,z}+C_{1123,\theta}+C_{3312}\cos\psi),B_{32}=((rC_{2212})_{,\psi}+(C_{2222}\rho_2 r)_{,z}+(\rho_2 C_{2223})_{,\theta}),$$

$$B_{33}=(-\frac{\rho_2 C_{3333}}{r}\sin\psi-C_{1133}+C_{3312,\psi}+(C_{2233}\rho_2)_{,z}-\frac{\rho_2 C_{2323}}{r}\sin\psi+(\frac{\rho_2 C_{3323}}{r})_{,\theta}-\frac{\rho_2 C_{1323}}{r}\cos\psi)$$

$$B_{34}=(-rC_{1112}-\rho_2\sin\psi C_{3312}+(rC_{1212})_{,\psi}+(C_{2212}\rho_2 r)_{,z}-C_{2212}r+(\rho_2 C_{1223})_{,\theta}+C_{2233}\rho_2\cos\psi)$$

$$B_{35}=((\frac{rC_{1212}}{\rho_2})_{,\psi}+(C_{2212}r)_{,z}+C_{1223,\theta}),B_{36}=(-C_{1113}-\frac{\rho_2\sin\psi C_{3313}}{r}+C_{1213,\psi}+(C_{2213}\rho_2)_{,z}+\frac{\rho_2 C_{3323}}{r}\cos\psi-C_{1223}$$

$$+(\frac{\rho_2 C_{1323}}{r})_{,\theta}),B_{37}=(-\frac{rC_{1113}}{\rho_2}-\sin\psi C_{3313}-C_{1213}\cos\psi+(\frac{rC_{1213}}{\rho_2})_{,\psi}-C_{1223}\sin\psi+(C_{2213}r)_{,z}+C_{1323,\theta})$$

$$B_{38}=(-C_{1123}+C_{1223,\psi}+(C_{2223}\rho_2)_{,z}+(\frac{\rho_2 C_{2323}}{r})_{,\theta}+(\rho_2 C_{2323})_{,\theta}+C_{1123}),B_{39}=(-rC_{1123}-\rho_2\sin\psi C_{3323}$$

$$+(rC_{1223})_{,\psi}-C_{2213}\rho_2\cos\psi-C_{2223}\rho_2\sin\psi+(C_{2223}\rho_2 r)_{,z}),B_{40}=(-\frac{re_{111}}{\rho_2}-\sin\psi e_{331}+(\frac{re_{112}}{\rho_2})_{,\psi}+(e_{221}r)_{,z}$$

$$+e_{123,\theta}),B_{41}=(-re_{112}-\rho_2\sin\psi e_{332}+(re_{212})_{,\psi}+(e_{222}\rho_2 r)_{,z}+(\rho_2 e_{223})_{,\theta}),B_{42}=(-e_{113}-\frac{\rho_2\sin\psi e_{333}}{r}+e_{312,\psi}$$

$$+(e_{223}\rho_2)_{,z}+(\frac{\rho_2 e_{323}}{r})_{,\theta}),B_{43}=(-\frac{rd_{111}}{\rho_2}-\sin\psi d_{331}+(\frac{rd_{112}}{\rho_2})_{,\psi}+(d_{221}r)_{,z}+e_{123,\theta}),B_{44}=(-rd_{112}-\rho_2\sin\psi d_{332}$$

$$+(rd_{212})_{,\psi}+(d_{222}\rho_2 r)_{,z}+(\rho_2 d_{223})_{,\theta}),B_{45}=(-d_{113}-\frac{\rho_2\sin\psi d_{333}}{r}+d_{312,\psi}+(d_{223}\rho_2)_{,z}+(\frac{\rho_2 d_{323}}{r})_{,\theta})$$

$$B_{46}=(-\frac{\rho_2 C_{3333}}{r}\sin\psi\cos\psi-C_{1133}\cos\psi+\frac{rC_{1112}}{\rho_2}-(\frac{rC_{1212}}{\rho_2})_{,\psi}+C_{3312,\psi}\cos\psi+(C_{2233}\rho_2)_{,z}\cos\psi$$

$$-(C_{2212}r)_{,z}+(\frac{\rho_2 C_{3323}}{r})_{,\theta}\cos\psi-C_{1223,\theta})$$

$$B_{47}=(-\frac{\rho_2 C_{3333}}{r}\sin^2\psi-C_{1133}\sin\psi-\frac{rC_{1111}}{\rho_2}+(\frac{rC_{1112}}{\rho_2})_{,\psi}+C_{3312,\psi}\sin\psi+C_{3312}\cos\psi+(C_{1122}r)_{,z}$$

$$+(C_{2233}\rho_2)_{,z}\sin\psi+C_{1123,\theta}+(\frac{\rho_2 C_{3323}}{r})_{,\theta}\sin\psi)$$

$$B_{48}=(C_{1113}\cos\psi+C_{1123}\sin\psi+\frac{\rho_2\sin\psi\cos\psi C_{3313}}{r}+\frac{\rho_2\sin^2\psi C_{3323}}{r}-C_{1213,\psi}\cos\psi+C_{1213}\sin\psi-C_{1223,\psi}\sin\psi$$

$$-C_{1223}\cos\psi-(C_{2213}\rho_2)_{,z}\cos\psi-(C_{2223}\rho_2)_{,z}\sin\psi-(\frac{\rho_2 C_{2323}}{r})_{,\theta}\sin\psi-(\frac{\rho_2 C_{1323}}{r})_{,\theta}\sin\psi),B_{49}=\lambda(\psi,z,\theta)\rho_2 r$$

Appendix 3

$$C_1=\frac{rC_{1113}}{\rho_2},C_2=\rho_2 rC_{2223},C_3=\frac{C_{3333}\rho_2}{r},C_4=\frac{re_{113}}{\rho_2},C_5=\rho_2 re_{223},C_6=\frac{\rho_2 e_{333}}{r},C_7=\frac{rd_{113}}{\rho_2},C_8=\rho_2 rd_{223},$$

$$C_9=\frac{\rho_2 d_{333}}{r},C_{10}=\frac{rC_{1313}}{\rho_2},C_{11}=\frac{rC_{1213}}{\rho_2},C_{12}=C_{2323}\rho_2 r,C_{13}=C_{1223}\rho_2 r,C_{14}=\frac{\rho_2 C_{3313}}{r},C_{15}=\frac{\rho_2 C_{3323}}{r},$$

$$C_{16}=(C_{1313}+C_{1133}),C_{17}=r(C_{2213}+C_{1223}),C_{18}=(C_{3313}+C_{3313})C_{19}=r(C_{1213}+C_{1123}),C_{20}=(C_{1323}+C_{3312}),$$

$$C_{21}=r(C_{1323}+C_{1323}),C_{22}=\rho_2(C_{2323}+C_{2233}),C_{23}=\rho_2(C_{3323}+C_{3323})C_{24}=\rho_2(C_{1323}+C_{3312}),$$

$$C_{25}=r(e_{213}+e_{123}),C_{26}=(e_{313}+e_{331}),C_{27}=\rho_2(e_{323}+e_{332}),C_{28}=r(d_{213}+d_{123}),C_{29}=(d_{313}+d_{331}),$$

$$C_{30}=\rho_2(d_{323}+d_{332}),C_{31}=(C_{1113}\cos\psi+\sin\psi C_{1123}+(\frac{rC_{1113}}{\rho_2})_{,\psi}+C_{3313}\cos\psi-\frac{rC_{1213}}{\rho_2}+(rC_{1123})_{,z}+C_{1133,\theta}),$$

$$C_{32}=(\rho_2\cos\psi C_{2213}+\rho_2\sin\psi C_{2223}+(rC_{2213})_{,\psi}+rC_{1123}+(\rho_2 rC_{2223})_{,z}+\rho_2 C_{3323}\sin\psi+(C_{2233}\rho_2)_{,\theta}),$$

$$C_{33}=(+\frac{\rho_2\sin\psi C_{3323}}{r}+C_{3313,\psi}+(\rho_2 C_{3323})_{,z}+(\frac{C_{3333}\rho_2}{r})_{,\theta}-\frac{\rho_2 C_{3323}}{r}\sin\psi),C_{34}=(\frac{C_{1313}\rho_2\cos\psi}{r}$$

$$+\frac{C_{1323}\rho_2\sin\psi}{r}+C_{1313,\psi}+(C_{1323}\rho_2)_{,z}+\frac{C_{3333}\rho_2}{r}\cos\psi-C_{3312}+(\frac{\rho_2 C_{3313}}{r})_{,\theta}),C_{35}=((\frac{rC_{1313}}{\rho_2})_{,\psi}+(C_{1323}r)_{,z}$$

$$+C_{3313,\theta}),C_{36}=(\frac{C_{2323}\rho_2\sin\psi}{r}-\frac{C_{1323}\rho_2\cos\psi}{r}+C_{1323,\psi}+(C_{2323}\rho_2)_{,z}+\frac{C_{3333}\rho_2}{r}\sin\psi+C_{1133}+(\frac{\rho_2 C_{3323}}{r})_{,\theta}),$$

$$C_{37}=((rC_{1323})_{,\psi}+(C_{2323}\rho_2 r)_{,z}+(\rho_2 C_{3323})_{,\theta}),C_{38}=(-\rho_2\cos\psi C_{1213}-\rho_2\sin\psi C_{1223}+(rC_{1213})_{,\psi}$$

$$+\rho_2 C_{3323}\cos\psi+(C_{1223}\rho_2 r)_{,z}-C_{1223}r+(\rho_2 C_{3312})_{,\theta}),C_{39}=(-\cos\psi C_{1213}-\sin\psi C_{1223}+\frac{rC_{1113}}{\rho_2}+C_{3313}\sin\psi$$

$$+(\frac{rC_{1213}}{\rho_2})_{,\psi}+(C_{1223}r)_{,z}+C_{3312,\theta}),C_{40}=(-\cos\psi e_{113}-\sin\psi e_{123}+(\frac{re_{113}}{\rho_2})_{,\psi}+(re_{123})_{,z}+e_{331,\theta}),$$

$$C_{41}=(-\rho_2\cos\psi e_{213}-\rho_2\sin\psi e_{223}+(re_{213})_{,\psi}+(\rho_2 re_{223})_{,z}+(\rho_2 e_{332})_{,\theta}),C_{42}=(-\frac{\rho_2(\cos\psi e_{313}+\sin\psi e_{323})}{r}$$

$$+e_{313,\psi}+(\rho_2 e_{323})_{,z}+(\frac{\rho_2 e_{333}}{r})_{,\theta}),C_{43}=(-\cos\psi d_{113}-\sin\psi d_{123}+(\frac{rd_{113}}{\rho_2})_{,\psi}+(rd_{123})_{,z}+d_{331,\theta}),$$

$$C_{44}=(-\rho_2\cos\psi d_{213}-\rho_2\sin\psi d_{223}+(rd_{213})_{,\psi}+(\rho_2 rd_{223})_{,z}+(\rho_2 d_{332})_{,\theta}),C_{45}=(-\frac{\rho_2(\cos\psi d_{313}+\sin\psi d_{323})}{r}$$

$$+d_{313,\psi}+(\rho_2 d_{323})_{,z}+(\frac{\rho_2 d_{333}}{r})_{,\theta}),C_{46}=(-\frac{\rho_2\cos^2\psi C_{3313}}{r}-\frac{\rho_2\sin\psi\cos\psi C_{3323}}{r}+\cos\psi C_{1213}+\sin\psi C_{1223}$$

$$+C_{3313,\psi}\cos\psi-C_{3313}\sin\psi-(\frac{rC_{1213}}{\rho_2})_{,\psi}+(\rho_2 C_{3323})_{,z}\cos\psi-(C_{1223}r)_{,z}+(\frac{C_{3333}\rho_2}{r})_{,\theta}\cos\psi-C_{3312,\theta})$$

$$C_{47}=(-C_{1113}\cos\psi-\frac{\rho_2\cos\psi\sin\psi C_{3313}}{r}-\sin\psi C_{1123}-\frac{\rho_2\sin^2\psi C_{3323}}{r}+(\frac{rC_{1113}}{\rho_2})_{,\psi}+C_{3313,\psi}\sin\psi$$

$$+C_{3313}\cos\psi+(rC_{1123})_{,z}+(\rho_2 C_{3323})_{,z}\sin\psi+(\frac{C_{3333}\rho_2}{r})_{,\theta}\sin\psi+C_{1133,\theta}),C_{48}=(-\frac{C_{1313}\rho_2\cos^2\psi}{r}$$

$$-\frac{C_{2323}\rho_2\sin^2\psi}{r}+\frac{\rho_2\sin\psi\cos\psi C_{1323}}{r}-\frac{\rho_2\sin\psi\cos\psi C_{1323}}{r}-C_{1313,\psi}\cos\psi+C_{1313}\sin\psi-C_{1323,\psi}\sin\psi$$

$$-C_{1323}\cos\psi-(C_{2323}\rho_2)_{,z}\sin\psi-(C_{1323}\rho_2)_{,z}\cos\psi-(\frac{\rho_2 C_{3313}}{r})_{,\theta}\cos\psi-(\frac{\rho_2 C_{3323}}{r})_{,\theta}\sin\psi),$$

$$C_{49}=\lambda(\psi,z,\theta)\rho_2 r$$

Appendix 4

$$D_1=\frac{re_{111}}{\rho_2},D_2=\rho_2 re_{222},D_3=\frac{e_{333}}{r},D_4=-\frac{r\eta_{11}}{\rho_2},D_5=\rho_2 r\eta_{22},D_6=-\frac{\eta_{33}\rho_2}{r},D_7=-\frac{rg_{11}}{\rho_2},D_8=\rho_2 rg_{22},$$

$$D_9=-\frac{g_{33}\rho_2}{r},D_{10}=\frac{re_{112}}{\rho_2},D_{11}=\frac{re_{113}}{\rho_2},D_{12}=e_{212}\rho_2 r,D_{13}=e_{223}\rho_2 r,D_{14}=\frac{e_{313}\rho_2}{r},D_{15}=\frac{e_{323}\rho_2}{r},$$

$$D_{16}=r(e_{221}+e_{212}),D_{17}=(e_{331}+e_{331}),D_{18}=r(e_{112}+e_{112}),D_{19}=(e_{113}+e_{313}),D_{20}=(e_{123}+e_{312}),$$

$$D_{21}=r(e_{123}+e_{213}),D_{22}=\rho_2(e_{332}+e_{323}),D_{23}=\rho_2(e_{213}+e_{312}),D_{24}=\rho_2 e_{223},D_{25}=-r(\eta_{12}+\eta_{12}),$$

$$D_{26}=-(\eta_{13}+\eta_{13}),D_{27}=-\rho_2(\eta_{23}+\eta_{23}),D_{28}=-r(g_{12}+g_{12}),D_{29}=-(g_{13}+g_{13}),D_{30}=-\rho_2(g_{23}+g_{23}),$$

$$D_{31}=((\frac{re_{111}}{\rho_2})_{,\psi}+e_{331}\cos\psi-\frac{re_{112}}{\rho_2}+(re_{112})_{,z}),D_{32}=((re_{221})_{,\psi}+(\rho_2 re_{222})_{,z}+re_{112}+\rho_2 e_{332}\sin\psi),$$

$$D_{33}=((e_{331})_{,\psi}+(\rho_2 e_{332})_{,z}+(\frac{\rho_2 e_{333}}{r})_{,\theta}-\frac{e_{313}\rho_2}{r}\cos\psi-\frac{e_{323}\rho_2}{r}\sin\psi),D_{34}=(\frac{re_{111}}{\rho_2}+e_{331}\sin\psi+(\frac{re_{112}}{\rho_2})_{,\psi}$$

$$+(e_{212}r)_{,z}+e_{312,\theta}),D_{35}=((re_{112})_{,\psi}+\rho_2 e_{332}\cos\psi+(e_{212}\rho_2 r)_{z}-e_{212}r+(e_{312}\rho_2)_{,\theta}),D_{36}=((e_{113})_{,\psi}+(e_{213}\rho_2)_{,z}$$

$$+\frac{\rho_2 e_{333}}{r}\cos\psi-e_{312}+(\frac{e_{313}\rho_2}{r})_{,\theta}),D_{37}=(-e_{113}\cos\psi+(\frac{re_{113}}{\rho_2})_{,\psi}-e_{123}\sin\psi+(e_{213}r)_{,z}+(e_{313})_{,\theta}),$$

$$D_{38}=((e_{123})_{,\psi}+(e_{223}\rho_2)_{,z}+\frac{\rho_2 e_{333}}{r}\sin\psi+(\frac{e_{323}\rho_2}{r})_{,\theta}),D_{39}=((re_{123})_{,\psi}-e_{213}\rho_2\cos\psi-e_{223}\rho_2\sin\psi+(e_{223}\rho_2 r)_{,z}$$

$$+(e_{323}\rho_2)_{,\theta}),D_{40}=-((\frac{r\eta_{11}}{\rho_2})_{,\psi}+(r\eta_{12})_{,z}+(\eta_{13})_{,\theta}),D_{41}=-((r\eta_{12})_{,\psi}+(\rho_2 r\eta_{22})_{,z}+(\eta_{23}\rho_2)_{,\theta}),D_{42}=-((\eta_{13})_{,\psi}$$

$$+(\rho_2\eta_{23})_{,z}+(\frac{\eta_{33}\rho_2}{r})_{,\theta}),D_{43}=-((\frac{rg_{11}}{\rho_2})_{,\psi}+(rg_{12})_{,z}+(g_{13})_{,\theta}),D_{44}=-((rg_{12})_{,\psi}+(\rho_2 rg_{22})_{,z}+(g_{23}\rho_2)_{,\theta}),$$

$$D_{45}=-((g_{13})_{,\psi}+(\rho_2 g_{23})_{,z}+(\frac{g_{33}\rho_2}{r})_{,\theta}),D_{46}=((e_{331})_{,\psi}\cos\psi-e_{331}\sin\psi-(\frac{re_{112}}{\rho_2})_{,\psi}+(\rho_2 e_{332})_{,z}\cos\psi-(e_{212}r)_{,z}$$

$$+(\frac{\rho_2 e_{333}}{r})_{,\theta}\cos\psi-e_{312,\theta}),D_{47}=((\frac{re_{111}}{\rho_2})_{,\psi}+(e_{331})_{,\psi}\sin\psi+e_{331}\cos\psi+(re_{112})_{,z}+(\rho_2 e_{332})_{,z}\sin\psi$$

$$+(\frac{\rho_2 e_{333}}{r})_{,\theta}\sin\psi),D_{48}=(-(e_{113})_{,\psi}\cos\psi+e_{113}\sin\psi-(e_{123})_{,\psi}\sin\psi-e_{123}\cos\psi-(e_{213}\rho_2)_{,z}\cos\psi$$

$$-(e_{223}\rho_2)_{,z}\sin\psi-(\frac{e_{313}\rho_2}{r})_{,\theta}\cos\psi-(\frac{e_{323}\rho_2}{r})_{,\theta}\sin\psi)$$

Appendix 5

$$E_1=\frac{rd_{111}}{\rho_2},E_2=\rho_2 rd_{222},E_3=\frac{d_{333}}{r},E_4=-\frac{rg_{11}}{\rho_2},E_5=\rho_2 rg_{22},E_6=-\frac{g_{33}\rho_2}{r},E_7=-\frac{r\mu_{11}}{\rho_2},E_8=\rho_2 r\mu_{22},E_9=-\frac{\mu_{33}\rho_2}{r}$$

$$E_{10}=\frac{rd_{112}}{\rho_2},E_{11}=\frac{rd_{113}}{\rho_2},E_{12}=d_{212}\rho_2 r,E_{13}=d_{223}\rho_2 r,E_{14}=\frac{d_{313}\rho_2}{r},E_{15}=\frac{d_{323}\rho_2}{r},E_{16}=r(d_{221}+d_{212}),E_{17}=(d_{331}+d_{331}),$$

$$E_{18}=r(d_{112}+d_{112}),E_{19}=(d_{113}+d_{313}),E_{20}=(d_{123}+d_{312}),E_{21}=r(d_{123}+d_{213}),E_{22}=\rho_2(d_{332}+d_{323}),E_{23}=\rho_2(d_{213}+d_{312}),$$

$$E_{24}=\rho_2 d_{223},E_{25}=-r(g_{12}+g_{12}),E_{26}=-(g_{13}+g_{13}),E_{27}=-\rho_2(g_{23}+g_{23}),E_{28}=-r(\mu_{12}+\mu_{12}),E_{29}=-(\mu_{13}+\mu_{13}),$$

$$E_{30}=-\rho_2(\mu_{23}+\mu_{23}),E_{31}=((\frac{rd_{111}}{\rho_2})_{,\psi}+d_{331}\cos\psi-\frac{rd_{112}}{\rho_2}+(rd_{112})_{,z}),E_{32}=((rd_{221})_{,\psi}+(\rho_2 rd_{222})_{,z}$$

$$+rd_{112}+\rho_2 d_{332}\sin\psi),E_{33}=((d_{331})_{,\psi}+(\rho_2 d_{332})_{,z}+(\frac{\rho_2 d_{333}}{r})_{,\theta}-\frac{d_{313}\rho_2}{r}\cos\psi-\frac{d_{323}\rho_2}{r}\sin\psi),E_{34}=(\frac{rd_{111}}{\rho_2}+d_{331}\sin\psi$$

$$+(\frac{rd_{112}}{\rho_2})_{,\psi}+(d_{212}r)_{,z}+d_{312,\theta}),E_{35}=((rd_{112})_{,\psi}+\rho_2 d_{332}\cos\psi+(d_{212}\rho_2 r)_z-d_{212}r+(d_{312}\rho_2)_{,\theta}),E_{36}=((d_{113})_{,\psi}+(d_{213}\rho_2)_{,z}$$

$$+\frac{\rho_2 d_{333}}{r}\cos\psi-d_{312}+(d\frac{e_{313}\rho_2}{r})_{,\theta}),E_{37}=(-d_{113}\cos\psi+(\frac{rd_{113}}{\rho_2})_{,\psi}-d_{123}\sin\psi+(d_{213}r)_{,z}+(d_{313})_{,\theta}),$$

$$E_{38}=((d_{123})_{,\psi}+(d_{223}\rho_2)_{,z}+\frac{\rho_2 d_{333}}{r}\sin\psi+(\frac{d_{323}\rho_2}{r})_{,\theta}),E_{39}=((rd_{123})_{,\psi}-d_{213}\rho_2\cos\psi-d_{223}\rho_2\sin\psi+(d_{223}\rho_2 r)_{,z}+(d_{323}\rho_2)_{,\theta})$$

$$E_{40}=-((\frac{rg_{11}}{\rho_2})_{,\psi}+(rg_{12})_{,z}+(g_{13})_{,\theta}),E_{41}=-((rg_{12})_{,\psi}+(\rho_2 rg_{22})_{,z}+(g_{23}\rho_2)_{,\theta}),E_{42}=-((g_{13})_{,\psi}+(\rho_2 g_{23})_{,z}+(\frac{g_{33}\rho_2}{r})_{,\theta})$$

$$E_{43}=-((\frac{r\mu_{11}}{\rho_2})_{,\psi}+(r\mu_{12})_{,z}+(\mu_{13})_{,\theta}),E_{44}=-((r\mu_{12})_{,\psi}+(\rho_2 r\mu_{22})_{,z}+(\mu_{23}\rho_2)_{,\theta}),E_{45}=-((\mu_{13})_{,\psi}+(\rho_2\mu_{23})_{,z}+(\frac{\mu_{33}\rho_2}{r})_{,\theta})$$

$$E_{46}=((d_{331})_{,\psi}\cos\psi-d_{331}\sin\psi-(\frac{rd_{112}}{\rho_2})_{,\psi}+(\rho_2 d_{332})_{,z}\cos\psi-(d_{212}r)_{,z}+(\frac{\rho_2 d_{333}}{r})_{,\theta}\cos\psi-d_{312,\theta})$$

$$E_{47}=((\frac{rd_{111}}{\rho_2})_{,\psi}+(d_{331})_{,\psi}\sin\psi+d_{331}\cos\psi+(rd_{112})_{,z}+(\rho_2 d_{332})_{,z}\sin\psi+(\frac{\rho_2 d_{333}}{r})_{,\theta}\sin\psi)$$

$$E_{48}=(-(d_{113})_{,\psi}\cos\psi+d_{113}\sin\psi-(d_{123})_{,\psi}\sin\psi-d_{123}\cos\psi-(d_{213}\rho_2)_{,z}\cos\psi-(d_{223}\rho_2)_{,z}\sin\psi-(d\frac{e_{313}\rho_2}{r})_{,\theta}\cos\psi$$

$$-(\frac{d_{323}\rho_2}{r})_{,\theta}\sin\psi)$$