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Analytical Approximation of Nonlinear Vibration of Euler-Bernoulli Beams

Abstract

In this paper, the Homotopy Analysis Method (HAM) with two auxiliary parameters and Differential Transform Method (DTM) are employed to solve the geometric nonlinear vibration of Euler-Bernoulli beams subjected to axial loads. A second auxiliary parameter is applied to the HAM to improve convergence in non-linear systems with large deformations. The results from HAM and DTM are compared with another popular numerical method, the shooting method, to validate these two analytical methods. HAM and DTM show excellent agreement with numerical results (the maximum errors in our calculations are about 0.002%), and they additionally provide a simple way to conduct a parametric analysis with different physical parameters in Euler-Bernoulli beams. To show the benefits of this method, the effect of different physical parameters on the amplitude is discussed for a cantilever beam with a cyclically varying axial load.

Keywords

Nonlinear vibration, Euler-Bernoulli beam, Homotopy Analysis Method (HAM), Two auxiliary parameters, Differential Transform Method (DTM).

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1 INTRODUCTION

The governing equations of beam vibration are generally non-linear making it difficult to solve the nonlinear problem via analytical methods. Because of this, many researchers use numerical methods to solve these equations. In recent years, some approximate analytical solutions have been offered to investigate these non-linear equations. Wu and Liu (1999) used differential quadrature to solve the

single-span Bernoulli–Euler beam's buckling equation. He (2006) suggested that the Parameterized-Perturbation Method (PPM) be used to solve strongly nonlinear equations. Qaisi (1993) determined the vibration modes of geometrically nonlinear beams under the various edge conditions by the harmonic balance principle. Moeenfard et al. (2011) developed the homotopy perturbation method (HPM) to analyze the nonlinear free vibration of Timoshenko beams. They also converted the non-linear partial differential governing equation to a non-linear ordinary differential equation using Galerkin's projection method. Sfahani et al. (2011) used the energy balance method (EBM) to study the dynamic response of inextensible beams (neutral axis length is preserved during vibration). Barari et al. (2010) used the Homotopy-Perturbation method to analyze the beam deformation. Pillai and Rao (1992) applied several methods to study the large amplitude free vibrations of simply supported prismatic beams with fixed ends. The methods that they used in their study consist of the elliptic function method, harmonic balance method and methods considering simple harmonic oscillations. Kopmaza and Gündogdub (2003) developed the relation between curvature of an Euler-Bernoulli beam and bending moment. Soldatos and Selvadurai (1985) used a perturbation method to survey the static flexure of a Bernoulli-Euler beam resting on a nonlinear Winkler-type foundation. Ahmadi et al. (2014) devoted to the new classes of analytical techniques called the Iteration Perturbation Method (IPM)and Hamiltonian Approach (HA) for solving the equation of motion governing the nonlinear vibration of Euler-Bernoulli beams. Liu and Gurram (2009) employed He's Variational Iteration Method (VIM) to solve free vibration problems for an Euler–Bernoulli beam with various supporting conditions. They compared VIM results with Adomian decomposition method results. They concluded that VIM is accurate and it provides a simple and efficient approach for solving vibration of uniform Euler–Bernoulli beams. Rashidi et al. (2012) derived the equations of the motion for a rectangular isotropic plate. They considered the effect of shear deformation and rotary inertia and used the Homotopy Perturbation Method (HPM) to solve the nonlinear equation.

In 1992, Liao (1992) proposed the homotopy analysis method (HAM) to solve nonlinear equations. Many researchers used HAM to solve various nonlinear problems. Hoseini et al. (2008) studied the nonlinear free vibration of a conservative oscillator via HAM. They showed that the HAM leads to an accurate analytical solution, which is valid for a wide range of system parameters. Mohammadpour et al. (2012) studied deflection of beams with nonlinear Winkler type foundations using HAM. They showed that the HAM improves convergence of nonlinear problems.

The Differential Transform Method (DTM) was developed by Taylor's series expansion. DTM applied to solve linear and nonlinear differential equations. This method can be used for solution of ordinary and partial differential equations. Ayaz (2003) solved the initial value problem for partial differential equations (PDE) using a two-dimensional differential transform method. Using this method reduces computational time. Ayaz (2004) studied PDE by a three-dimensional differential transform method. Hassan (2002) solved eigenvalue problems by the DTM method. Kuang Chen and Shin Chen (2004) applied DTM to solve the free vibrations of a conservative oscillator and also compared the results with the Runge-Kutta method. In addition, Shin Chen and Kuang Chen (2009) investigated the free vibrations of strongly non-linear oscillators by DTM and proposed this method to solve the non-linear problem because of DTMs improved accuracy.

In this paper, two new analytical methods are proposed to solve the nonlinear vibration of an Euler-Bernoulli beam. First, the governing equation is obtained, and then the Galerkin Method is used to obtain the corresponding ordinary differential equation. Then, the HAM with two auxiliary parameters and DTM are employed to solve this equation. Finally, the results of HAM and DTM are compared with numerical method results and good agreement is achieved.

2 GOVERNING EQUATION

In order to obtain the governing equation, we consider three assumptions: the plane deformation is negligible so it is neglected, transverse shear strains are small so they are neglected, and rotation of the cross section is only due to bending. The schematic of Euler-Bernoulli beam subjected to the axial loading is shown in Fig. 1.

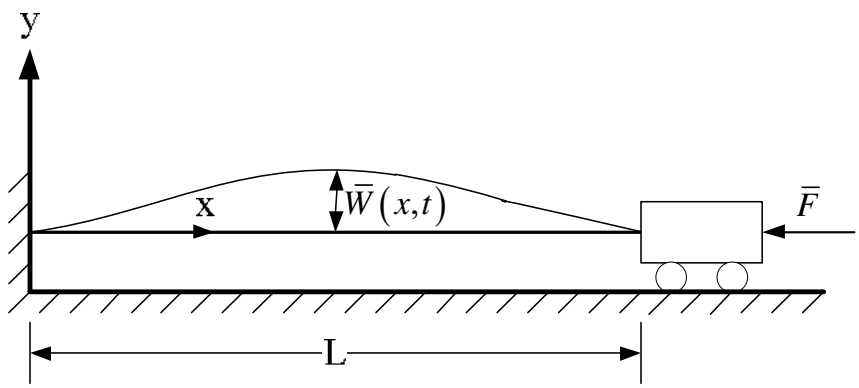

Figure 1: A schematic of an Euler-Bernoulli beam subjected to an axial load.

The equation of motion including the effects of mid-plane stretching is introduced as (Rao, 2007):

$$EI\frac{\partial^4\overline{W}}{\partial\overline{X}^4}+\overline{F}\frac{\partial^2\overline{W}}{\partial\overline{X}^2}+m_b\frac{\partial^2\overline{W}}{\partial\overline{t}^2}+\frac{EA_b}{2L}\frac{\partial^2\overline{W}}{\partial\overline{X}^2}\int_0^L\left(\frac{\partial\overline{W}}{\partial\overline{X}}\right)^2 d\overline{X}=0, \tag{1}$$

where L is the length of beam, A_b is the cross-sectional area, I is the moment of inertia, E is the young modulus, m_b is the mass per unit length of beam and $.\overline{F}.$ is magnitude of axial force. For simplification, the following non-dimensional variables are used:

$$X=\frac{\overline{X}}{L},W=\frac{\overline{W}}{R},t=\bar{t}\sqrt{\frac{EI}{m_bL^4}},F=\frac{\overline{F}L^2}{EI}, \tag{2}$$

where $R=\sqrt{\frac{I}{A_b}}$ is the radius of gyration of the cross-section. Therefore, Eq. (1) can be written as follows:

$$\frac{\partial^4 W}{\partial X^4} + F\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial t^2} - \frac{1}{2}\frac{\partial^2 W}{\partial X^2}\int_0^1 \left(\frac{\partial W}{\partial X}\right)^2 dX = 0. \tag{3}$$

Assuming that $W(X,t) = T(t)\, w(X)$, where $w(X)$ is the first Eigen-mode of the beam. By applying the Galerkin method, the equation of motion is obtained as follows:

$$\ddot{T}(t) + \beta T(t) + \lambda T^3(t) = 0, \tag{4}$$

where $\beta = \beta_1 + F\beta_2$. The above equation is the governing equation of nonlinear vibration of Euler-Bernoulli beams. The β_1, β_2 and λ parameters are as follows:

$$\beta_1 = \frac{\int_0^1 T^{iv} T dx}{\int_0^1 T^2 dx}, \tag{5a}$$

$$\beta_2 = \frac{\int_0^1 T^{"} T dx}{\int_0^1 T^2 dx}, \tag{5b}$$

$$\lambda = \frac{-0.5\int_0^1 \left(T^{"}\int_0^1 T^{'2} dx\right) T dx}{\int_0^1 T^2 dx}, \tag{5c}$$

The center of the beam are subjected to the following initial conditions:

$$T(0) = A, \qquad \dot{T}(0) = 0, \tag{6}$$

where A denotes the non-dimensional maximum amplitude of oscillation.

3 APPLICATION OF HAM

Consider the suitable initial approximation, in order to solve Eq. (4). (for more information about HAM steps, see Refs. (Abolbashari et al. (2014), Abolbashari et al. (2015), Jafari and Freidoonimehr(2015)):

$$T_0(t) = A\mathrm{Cos}(\alpha t), \tag{7}$$

where α is the secondary auxiliary parameter, which is used to accelerate the convergence of the series. The auxiliary linear operator $\mathcal{L}_T(T)$ becomes:

$$\mathcal{L}_T\left(T\right)=\frac{\partial^2 T}{\partial \xi^2}+\alpha^2 T, \tag{8}$$

which the following relation is satisfied

$$\mathcal{L}_T\left(c_1\,\mathrm{Sin}\left(\alpha t\right)+c_2\,\mathrm{Cos}\left(\alpha t\right)\right)=0, \tag{9}$$

where c_1 and c_2 are the arbitrary constants. According to the Eq. (4), the nonlinear operator is introduced as:

$$\mathcal{N}_T\left[\hat{T}\left(t;p\right)\right]=\frac{\partial^2 \hat{T}\left(t;p\right)}{\partial t^2}+\beta\hat{T}\left(t;p\right)+\lambda\left(\hat{T}\left(t;p\right)\right)^3, \tag{10}$$

The zero-order deformation equation is introduced as:

$$\left(1-p\right)\mathcal{L}_f\left[\hat{T}\left(t;p\right)-T_0\left(t\right)\right]=p\hbar_T\mathcal{H}_T\left(t\right)\mathcal{N}_T\left[\hat{T}\left(t;p\right)\right], \tag{11}$$

where p is an embedded parameter and $\hbar_T$ is an auxiliary nonzero parameter. The auxiliary functions are introduced as:

$$\mathcal{H}_T\left(t\right)=1. \tag{12}$$

Due to the boundary conditions

$$\begin{aligned}&\hat{T}\left(0;p\right)=A,\\&\frac{d\hat{T}\left(0;p\right)}{dt}=0.\end{aligned} \tag{13}$$

By Taylor's theorem, we have

$$\hat{T}\left(t;p\right)=T_0\left(t\right)+\sum_{m=1}^{\infty}T_m\left(t\right)p^m, \tag{14}$$

Where

$$T_m\left(t\right)=\frac{1}{m!}\frac{\partial^m\hat{T}\left(t;p\right)}{\partial p^m}\bigg|_{p=0}. \tag{15}$$

Consider that $\hbar_T$ is chosen such that the Taylor series expansion of Eq. (14) has converged at $p=1$:

$$T\left(t\right)=T_0\left(t\right)+\sum_{m=1}^{\infty}T_m\left(t\right), \tag{16}$$

$$\mathcal{L}_T\left[T_m(t)-\chi_m\ T_{m-1}(t)\right]=\hbar_T\,\mathcal{H}_T(t)R_{T,m}(t), \tag{17}$$

where

$$R_{T,m}(t)=\frac{\partial^2 T_{m-1}(t)}{\partial t^2}+\beta\,T_{m-1}(t)+\lambda\sum_{j=0}^{m-1}\left(T_{m-1-j}(t)\sum_{i=0}^{j}\left(T_i(t)T_{j-i}(t)\right)\right), \tag{18}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{19}$$

The mth order deformation equations (Eq. (17)) were solved using MATHEMATICA software. The result of first order solution of HAM is determined as:

$$T_1(t)=\frac{A\hbar_T\mathrm{Sin}(t\alpha)[2t\alpha(4\beta+3\lambda A^2-4\alpha^2)+\lambda A^2\mathrm{Sin}(2t\alpha)]}{16\alpha^2}. \tag{20}$$

It is important to select a proper value of auxiliary parameter to control the speed of convergence of the approximation series by the help of the so-called $\hbar_T$ – curve. The optimal values of $\hbar_T$ are selected from the valid region in straight line. The $\hbar_T$ – curve of $T''(0)$ obtained by the different order of approximation is shown in Fig. 2. The averaged residual error is introduced as Eq. (21):

$$Res_T=\frac{d^2T(t)}{dt^2}+\beta\,T(t)+\lambda\left(T(t)\right)^3. \tag{21}$$

In order to select the optimal value of the auxiliary parameter, the averaged residual error (for more details, see Refs. (Liao (2010), Rashidi and Abbasbandy(2011)) is defined as:

$$\Delta_m=\frac{1}{K}\sum_{i=0}^{K}\left[Res\left(\sum_{j=0}^{m}T_j(i\Delta x)\right)\right]^2, \tag{22}$$

where $\Delta x=5/K$ and $K=10$. For a given order of approximation m, the optimal value of $\hbar_T$ is given by the minimum of Δ_m, corresponding to nonlinear algebraic equations

$$\frac{d\Delta_m}{dh_T}=0. \tag{23}$$

In order to check the accuracy of the HAM method with two auxiliary parameters, the residual errors for 5th order HAM solutions of Eq. (21) is shown in Fig. 3.

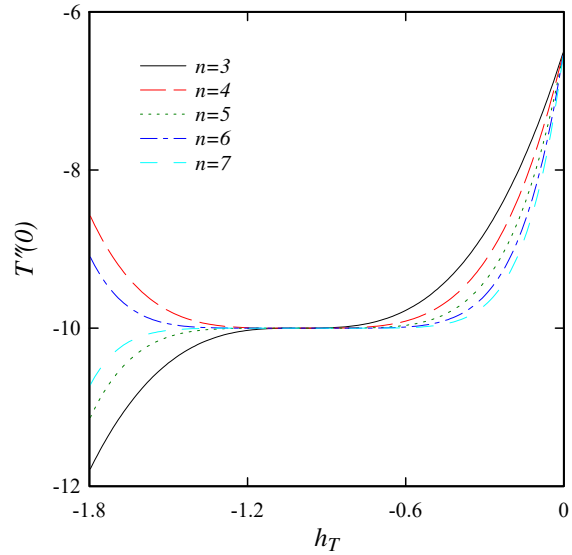

Figure 2: The $\hbar_T-$curve of $T''(0)$ obtained by different orders of approximation of HAM when $A=2,\ \beta=1,\ \lambda=1$ and $\alpha=1.8$.

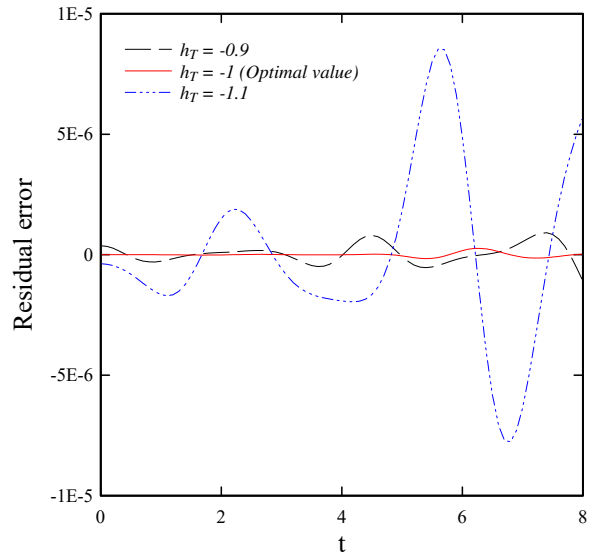

Figure 3: The residual error for Eq. (21) using 5th-order of approximations when $A=1,\ \beta=0.5$ and $\lambda=0.1$.

4 APPLICATION OF DTM

Taking the differential transform of Eq. (4), one can obtain (for more details, see Refs. (Freidoonimehr (2015), Rashidi et al. (2015), Rashidi and Freidoonimehr (2014)):

$$(k+1)(k+2)T(k+2)+\beta T(k)+\lambda\sum_{r=0}^{k}\left(\sum_{s=0}^{r}\left(T(s)T(r-s)T(k-r)\right)\right)=0, \tag{24}$$

where $T(k)$ is the differential transforms of $T(t)$ and displayed as:

$$T(t)=\sum_{k=0}^{\infty}T(k)\eta^{k}, \tag{25}$$

$$\begin{aligned} T(0)&=A\\ T(1)&=0, \end{aligned} \tag{26}$$

Eq. (26) provides the transformed boundary conditions. By substituting Eq. (26) into Eq. (24) and combining with Eq. (25), the recursive method yields the values of $T(t)$:

$$T(t)=A+\frac{1}{2}\left(-\beta A-\lambda A^{3}\right)t^{2}+\frac{1}{12}\left(-\frac{1}{2}\beta\left(-\beta A-\lambda A^{3}\right)-\frac{3}{2}\lambda A^{2}\left(-\beta A-\lambda A^{3}\right)\right)t^{4}+\cdots. \tag{27}$$

It is important to note that the number of required terms is determined by the convergence of the numerical values up to one's desired accuracy.

5 RESULTS AND DISCUSSIONS

First of all, comparisons have been done between the results of the current study; the results of numerical solution via the shooting method are used to validate the proposed analytical methods (HAM & DTM) in Fig. 4. Excellent agreement can be observed between them.

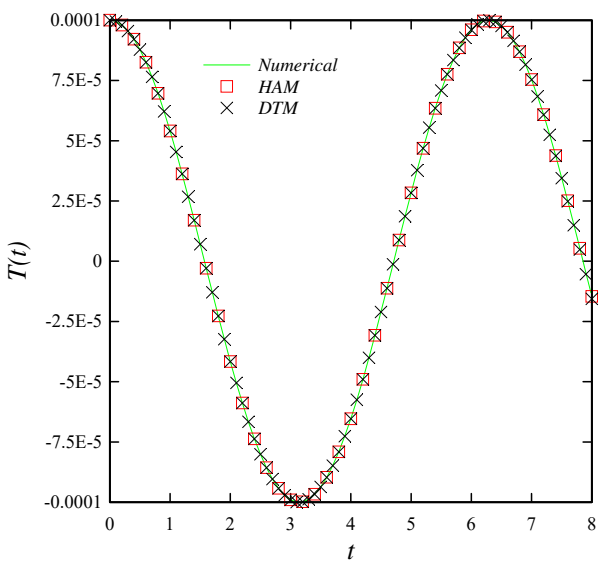

Figure 4: Deflection versus time results comparing HAM, DTM, and a numerical method at $A=0.0001$, $\beta=1$ and $\lambda=0.01$.

Fig. 5 displays the result of HAM solution with and without considering the second auxiliary parameter. As mentioned previously, the second auxiliary parameter is used to increase the convergence performance of the HAM. Thus, the HAM with two auxiliary parameters provides solutions that have good agreement with the numerical solution. In addition, it is obvious that by increasing the value of λ, which represents the value of non-linearity of the Eq. 4, the HAM with one auxiliary parameter diverge faster. Thus for large values of the non-linear parameter, the HAM with two auxiliary parameter must be applied to ensure the series convergence. Tables 1-3 present the comparison between the HAM results with one and two auxiliary parameters with the numerical method results for the various values of the amplitude, β, λ and time. As it is obvious form these tables, good agreement can be seen between the results of the HAM with two auxiliary parameters and the numerical method results.

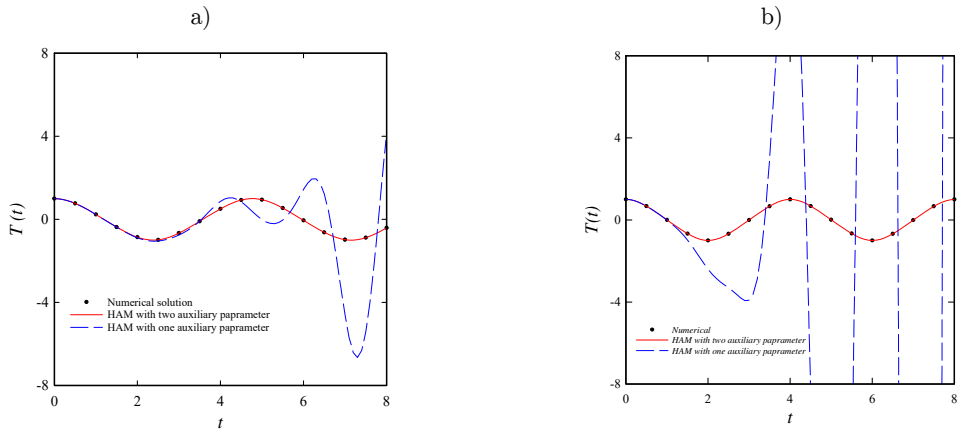

Figure 5: Amplitude oscillation comparison of HAM and numerical solution for the one and two auxiliary parameters at $A=1,\ \beta=1$ a) $\lambda=1$ and b) $\lambda=2$.

	A=0.1			A=1		
t(s)	Numerical	HAM I	HAM II	Numerical	HAM I	HAM II
0	0.1	0.1	0.1	1	1	1
1	0.0758123053	0.0758123290	0.0758123286	0.5685690107	0.5685689981	0.5685689775
2	0.0150531766	0.0150538194	0.0150531827	-0.2698650107	-0.2698651913	-0.2698653491
3	-0.0529409197	-0.0529280477	-0.0529409341	-0.9325348519	-0.9325366270	-0.9325366021
4	-0.0954464431	-0.0954993211	-0.0954465014	-0.8144739389	-0.8145389037	-0.8144742206
5	-0.0918065561	-0.0921086256	-0.0918066163	-0.0495672950	-0.0494860857	-0.0495654188
6	-0.0438010779	-0.0452883159	-0.0438010357	0.7466652141	0.7462369047	0.7466614051
7	0.0252801484	0.0192031534	0.0252802685	0.9673342245	0.9678434110	0.9673458074
8	0.0822077164	0.0708355335	0.0822078377	0.3654183660	0.3658140840	0.3654169333

Table 1: Comparison between HAM I & HAM II with numerical method for various amplitude (A) and time at $\beta=0.5$ and $\lambda=0.5$.

	$\beta = 0.1$			$\beta = 1$		
t(s)	Numerical	HAM I	HAM II	Numerical	HAM I	HAM II
0	0.1	0.1	0.1	1	1	1
1	0.7186891367	0.7186891073	0.7186891335	0.40916881872	0.4091518362	0.4091672994
2	0.0535511899	0.0535782139	0.0535511963	-0.5645435043	-0.5655204125	-0.5645564307
3	-0.6384940390	-0.6372673322	-0.6384939736	-0.9788739015	-0.9795816619	-0.9788746001
4	-0.9937463714	-0.9965024750	-0.9937463911	-0.2459090069	-0.2431123411	-0.2459589233
5	-0.7906133618	-0.7915504288	-0.7906133964	0.7066586628	0.6990286507	0.7066308351
6	-0.1600941627	-0.1554930423	-0.1600942406	0.9175055751	0.9269407103	0.9175718354
7	0.5510944923	0.4920791463	0.5510944937	0.0786457186	0.0524578451	0.0786884118
8	0.9750893151	1.0122147510	0.97508940017	-0.8286355281	-0.7764330162	-0.8286467371

Table 2: Comparison between HAM I & HAM II with numerical method for various β and time at $A = 1$ and $\lambda = 0.5$.

	$\lambda = 0.1$			$\lambda = 1$		
t(s)	Numerical	HAM I	HAM II	Numerical	HAM I	HAM II
0	0.1	0.1	0.1	1	1	1
1	0.7342918452	0.7342912558	0.7342918201	0.3744464684	0.3744424296	0.3744464636
2	0.1830444159	0.1824732174	0.1830319625	-0.6801752854	-0.6807760172	-0.6801750785
3	-0.4042240428	-0.3834091289	-0.4042415967	-0.9263208907	-0.9281230482	-0.9263222111
4	-0.8902880363	-0.9402908042	-0.8903411584	-0.0279417717	-0.0205402843	-0.0279408781
5	-0.9574603761	-0.9960507470	-0.9575074874	0.9018985316	0.8831000485	0.9018923741
6	-0.5386693482	-0.3560895267	-0.5386867113	0.7219217679	0.74510167918	0.7219283281
7	0.04154669596	-0.6011096589	0.0416248723	-0.3214271794	-0.3801147965	-0.3214297723
8	0.6141523989	1.5550866375	0.6140954363	-0.9981268334	-0.9332048104	-0.9981113550

Table 3: Comparison between HAM I & HAM II with numerical method for various λ and time at $A = 1$ and $\beta = 0.5$.

A parametric study was conducted on a cantilever beam with cyclically varying axial load. The nonlinear vibration analysis of deflection $T(t)$ is shown in Figs. 6-7 for a wide range of amplitudes (A) and time (t) and different values of the constant physical parameters via two auxiliary parameter of HAM solution. The difference between these two figures is in their physical constant β and λ, where in Fig.6 $\beta = \pi$, $\lambda = 0.15$, $\beta = 0.15$ and $\lambda = 0.5$ in Fig. 7.

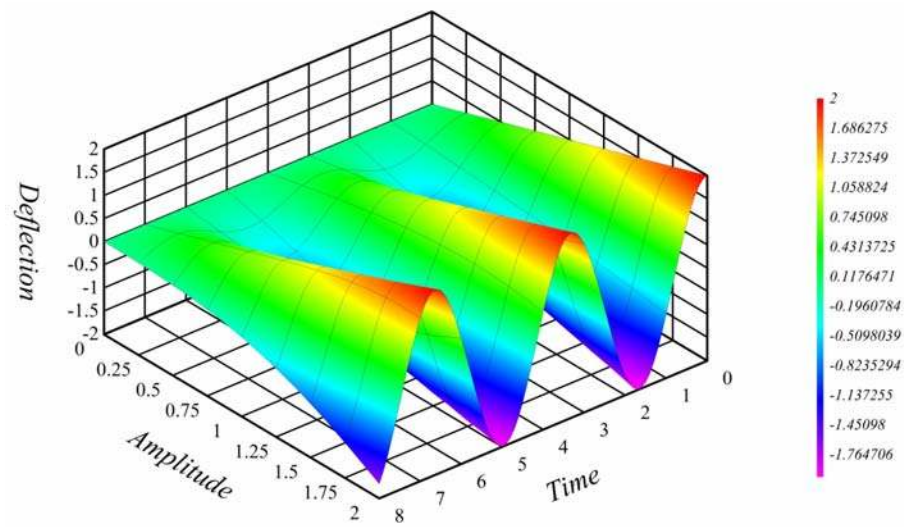

Figure 6: 3D plot of deflection $T(t)$ for a wide range of amplitude $\left(A\right)$ and time (t), where $\beta = \pi$ and $\lambda = 0.15$.

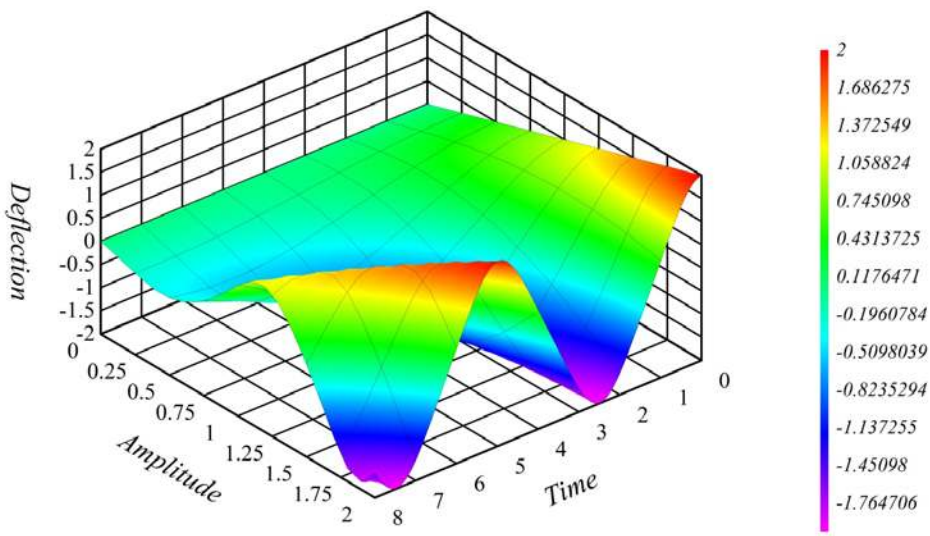

Figure 7: 3D plot of deflection $T(t)$ for a wide range of amplitude $\left(A\right)$ and time (t), where $\beta = 0.15$ and $\lambda = 0.5$.

Influence of λ on $T\left(t\right)$ behavior is demonstrated in Fig. 8, where A and β are equal to 1. Also, the effect of β on the behavior of $T\left(t\right)$ is shown in Fig. 9, where A and λ are equal to 1. It is worth mentioning that the two auxiliary parameters of HAM solution are used in Figs. 8-9.

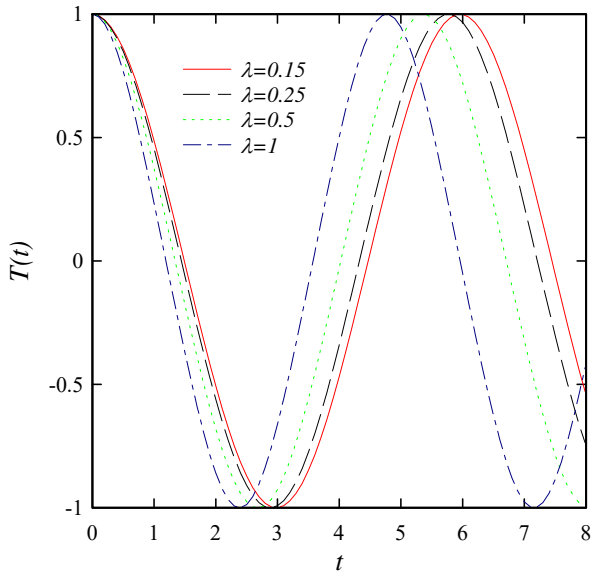

Figure 8: HAM solution to show the effect of λ on $T(t)$ behavior at $A=1$ and $\beta=1$.

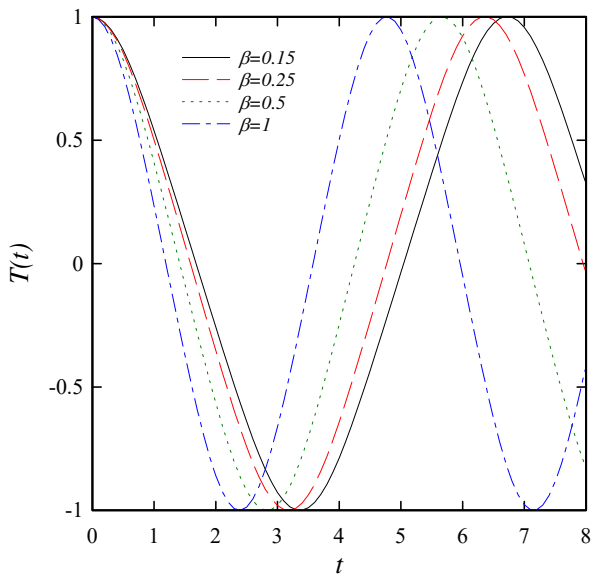

Figure 9: HAM solution to show the effect of β on $T(t)$ behavior at $A=1$ and $\lambda=1$.

The DTM response depends on the order of solutions. The DTM response at $A=0.0001$, $\beta=0.5$ and $\lambda=0.5$ is shown in Fig. 10 for different orders of solutions. According to Fig. 10 there is good agreement with numerical results at the 20th order of solution. The solutions based on DTM and numerical method for $t=1$ are compared in Table. 4. In this table, the values of the λ

and β are equal to 0.5. It can be observed that the maximum errors of HAM and DTM solutions with numerical results are less than 0.002%.

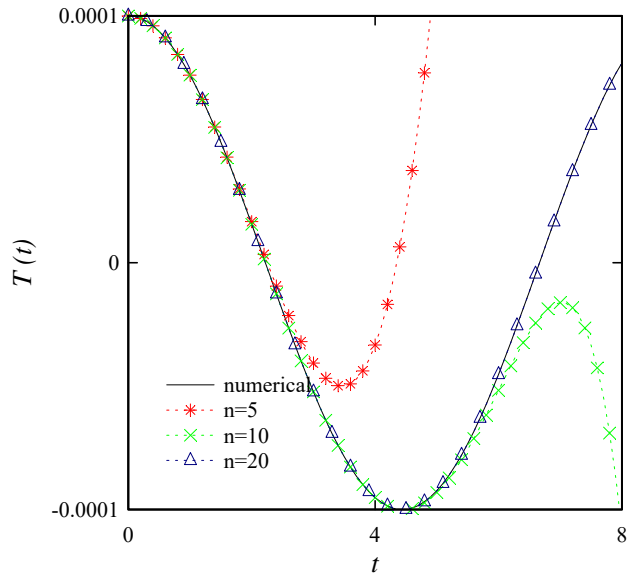

Figure 10: The $T(t)$ behavior at $A = 0.0001$, $\beta = 0.5$ and $\lambda = 0.5$, based on DTM for different order of solutions.

A	Numerical	DTM	HAM	Error (DTM)	Error (HAM)
0.0001	0.0000760376	0.0000760245	0.0000760245	0.000173043	0.000173043
0.01	0.0076022228	0.0076022336	0.0076022336	-1.4201E-6	-1.42009E-6
0.1	0.0758123053	0.0758123286	0.0758123306	-3.0736E-7	-3.34017E-7
0.3	0.2223960659	0.2223960863	0.2223961065	-9.186E-8	-1.82744E-7

Table 4: Comparison between DTM & HAM with numerical method for various amplitude (A) at $t = 1(s)$, $\beta = 0.5$ and $\lambda = 0.5$.

6 CONCLUSIONS

In this paper, nonlinear vibration of an Euler-Bernoulli beam subjected to a cyclically varying axial load is investigated by analytical methods. The governing equation of nonlinear vibration of the beam is in the form of a partial differential equation. The Galerkin method is employed to reduce the governing equation to ordinary differential equation. The homotopy analysis method (HAM) with two auxiliary parameters and differential transform method (DTM) are used to express the response of the axially loaded beam. In this problem, the second auxiliary parameter is used to accelerate the convergence of the Taylor series expansion, and it was shown that this parameter im-

proves the analysis of highly nonlinear systems. Excellent agreement was found in comparing the HAM, DTM and numerical solution results.

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