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A Modified FEM for Transverse and Lateral Vibration Analysis of Thin Beams Under a Mass Moving with a Variable Acceleration

Abstract

In this paper, a new modified finite element method that can be used in the analysis of transverse and lateral vibrations of the thin beams under a point mass moving with a variable acceleration and constant jerk is presented. Jerk is the change in acceleration over time. In this method, the classical finite element of the beam is modified by the inclusion of the inertial effects of the moving mass. This modification is made using the relations between nodal forces and nodal deflections and shape functions of six DOF beam element. The mass, stiffness, and damping matrices of the modified finite element are determined by forces caused by the corresponding transverse and lateral accelerations and jerks, and transverse Coriolis and centripetal accelerations and jerks, respectively. This method was first applied on a simply supported beam plate to provide a comparison with the previous studies in literature, and it was proved that the results were within acceptable limits. Secondly, it was applied on a CNC type box-framed beam to analyse the dynamic response of the beam in terms of variable acceleration and jerk as well as constant velocity and mass ratios.

Keywords

Finite element, beam vibrations, accelerating mass, jerk.

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1 INTRODUCTION

Dynamic behaviour of mechanical and structural systems with moving masses is a very important engineering problem. In the design of the running systems under very high-speed masses and with a very little working tolerance such as high speed precision metal processing, high-speed rail transportation, robotics applications in microchip production, high-speed projectile firing from a barrel, high–speed rocket launching systems, and etc., some new engineering problems arise together with speed increase. In such systems, the pre-engineering analysis must be accurate in order to prevent possible accidents and failures in the application. For example, the design and development studies of a missile and high-speed rail systems that cost millions of dollars should not only depend on experimental

studies. When an accident is happened, no one retrieve the situation in an experimental trial. In this case, realistic modelling and analysis is vital for such systems. The computation methods given in literature are commonly for low speeds and simple cases where the motion at variable speeds, acceleration and inertia effect of mass are neglected. In this study, a more realistic method of modelling and analysis of such systems is provided without neglecting the damping, inertia effects of the mass in motion, effect of a variable speed, effect of a variable acceleration, and effect of a constant jerk. Due to the variety of its applications, the moving load problem has been widely studied in literature by many researchers. For example, (Gerdemeli et al. 2011, Kahya 2012, Sharbati and Szyszkowski 2011, and Esen 2013) have studied the subject for a constant velocity of a moving mass. An analytical solution of the effects of moving load motion for Timoshenko beams has been provided by Lee 1996. Considering the effect of a variable velocity load on structures (Michaltsos 2002, Wang 2009, Esen 2011, and Dyniewicz and Bajer 2012 have studied the dynamic behaviour of beams under accelerated masses. Inertia effects of the moving mass continue to be a point of interest for bridge dynamics, railroad design, and other high-velocity delicate motion processes and some others (Michaltsos et al. 1996, Michaltsos and Kounadis 2001, and Dehestani et al. 2009) have studied and proposed some simplified analytical solutions of the inertial effects of moving masses. Wu 2005 has studied the vertical and horizontal displacements of an inclined simply supported beam under moving loads for the effects of the velocity and the mass of a moving-load including the Coriolis and centrifugal forces and inclination angle of the beam. Omolofe 2013 has presented a procedure involving spectral Galerkin and integral transformation methods for the problem of the dynamic deflections of beam structure resting on bi-parametric elastic subgrade and subjected to travelling loads. Oni and Awodola 2010, and Awodola 2014 have studied the dynamic response of an elastically supported non-prismatic Bernoulli-Euler beam and the flexural motions of elastically supported rectangular plates carrying moving masses and resting on variable Winkler elastic foundations. For the application of the moving load problem in defence field, a finite element model of a high-speed projectile and barrel interaction has provided by Esen and Koç 2015a, 2015b. Another Finite element application of the moving load on plate dynamics can be found in Esen 2015. The dynamics of a railcar bogie when moving over a flexible bridge has been studied by Mizrak and Esen 2015 as an application in high-speed rail transportation.

Generally, studies in the field of moving loads have been carried out by researchers working on vehicle-bridge interaction. In such problems, the mass of the bridge system is often too large when compared to the mass of the vehicle. For this reason, the mass of the vehicle does not significantly affect the natural frequency of the overall system. However, in mechanical engineering applications, the mass of the moving load can be much larger than the mass of the transport system. This is the main difference of the moving loads in the mechanical engineering applications where the mass of the moving load may significantly affect the natural frequencies of vibration modes of the transport system. In addition, for the vehicle-bridge-interaction, the moving loads on bridges can be modelled as a moving oscillator due to the vehicle's suspension system; thus, an amount of vibration energy dissipates in suspension system. However, the moving load is in tight contact with the beam especially in robotic and CNC systems. Due to the accuracy of the machined work-piece, any relative motion is not allowed between the mass and beam in the vertical direction. Moreover, the vibration-induced vertical position error should be less in the micron level; it is reasonably expected due to the accuracy

of the work performed. Therefore, the velocity, acceleration, and jerk of the mass are vital in terms of the accuracy of such mechanical engineering applications. The main purpose of this study is examining the issue in this perspective.

2 MATHEMATICAL MODELLING

In Figure 1, a simply supported Euler-Bernoulli beam with a lumped mass m_p which is travelling on the beam from the left end to the right end with a variable velocity $v_m(t)$, a variable acceleration $a_m(t)$ and a constant jerk J, is given.

In the formulation the following assumptions will be adopted (Fig. 1)

- The mass inertia is considered.
- The mass is always in contact with the beam.
- The beam is thin and small displacements in the beam are occurred according to thin beam theory.

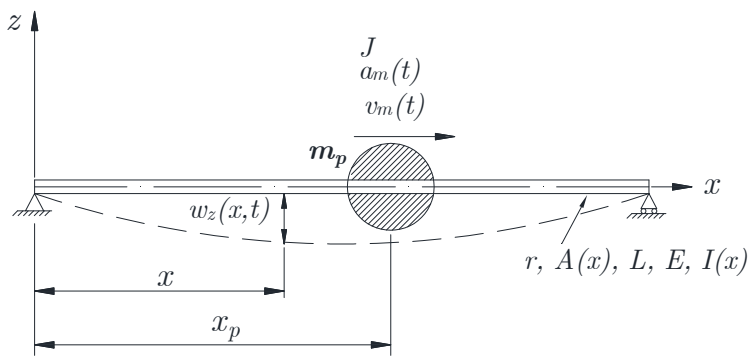


Figure 1: A simply supported beam and a moving mass system.

- The beam can be of variable thickness but the material properties are constant trough length of the beam.
- In vibration, the longitudinal and flexural motions are independent of each other but the axial force due to the acceleration of the mass affects the flexural displacements.
- The trajectory of the mass is defined by time-dependent $x_p(t)$

Based on the above assumptions, the motion equation of the beam due to the effect of the mass located at the time-dependent point x_p on the beam is provided by Eq. (1) Fryba 1999:

$$\begin{aligned} EI(x)\frac{\partial^4 w_z(x,t)}{\partial x^4} - N(x,t)\frac{\partial^2 w_z(x,t)}{\partial x^2} + \rho A(x)\frac{\partial^2 w_z(x,t)}{\partial t^2} + \sigma\frac{\partial w_z(x,t)}{\partial t} = F(x,t), \\ F(x,t) = m_p g\delta(x - x_p(t)) - m_p\left(\frac{\mathrm{d}^2 w_z(x_p,t)}{\mathrm{d}t^2}\right)\delta(x - x_p(t)), \end{aligned} \tag{1}$$

The left hand side of Eq. (1) represents the resisting internal stiffness, inertia and damping forces due to the external forces on the right side with the parameters that: $N(x,t)$ represents the time dependent axial load due to the acceleration of the moving mass; ρ represents the density, $A(x)$ represents the none uniform cross-sectional area, σ represents equivalent viscous damping coefficient, E represents the Young's modulus of elasticity, $I(x)$ represents the area moment of inertia, x represents the central coordinate of the beam, t represents time, $w_z(x, t)$ represents the vertical displacement of the beam, m_p represents the mass of the load, $m_p g\delta(x - x_p(\mathrm{t}))$ represents the force applied to the unit length of the beam by the moving mass, g and δ represent gravitational acceleration and the Dirac delta function, respectively, and $\mathrm{d}^2w(x_p,t)/\mathrm{d}t^2$ represents the acceleration of the beam at the contact point of lumped mass.

The initial and boundary conditions of the beam are:

$$w(x=0,t) = w(x=L,t) = 0,$$
$$\frac{\partial^2 w(x=0,t)}{\partial x^2} = \frac{\partial^2 w(x=L,t)}{\partial x^2} = 0 \tag{2}$$

In order to determine time dependent displacements of the beam, a rough analytical solution of the motion Eq. (1) shown above, can be obtained through simplifications that ignore the effects of inertia and damping; and accepting that the cross-section area is uniform and the mass moves with a constant velocity. For simplified cases, omitting geometric and dynamic nonlinearities such moving load problems have been extensively studied in the literature by numerous researchers. When the mass moves with a constant jerk on the beam, the exact solution of the motion equation using analytical methods will be very difficult; and no study of constant jerky motion of a mass has been reported in literature, yet. This kind of motion problems are faced frequently in today's engineering applications such as high-speed precision machining, precision grinding of hard cutting tools, some robotic applications of precision surgical robots and precision robots used in microchip production; where vibration of the linear run-ways are vital for the accuracy of the work done. For an exact or an admissible solution in the engineering sense, there are needs of some new methods that represent mass motion with all effects including jerk. This paper presents a solution method representing the mass as a time dependent finite element and combining this finite element with the classical finite elements of the beam. This method includes both inertial and damping effects and gives a solution for both transverse and longitudinal vibrations of the beam. From this perspective, the proposed FEM combined method can be very useful for design and control engineers if they pay attention to the requirements of modelling as described below.

2.1 Modified Mass, Damping and Stiffness Matrices of the Beam Element

Figure 2 displays mesh discretion of a thin beam under a lumped moving mass and the beam element s on which the mass m_p applies, at time t. The beam element s has three equivalent nodal forces and displacements at each nodal point. The time-dependent global position of the moving mass in the span is $x_p\ (t)$, while local position on the length of the element s is $x_m\,(t)$. The beam has n elements and $(n+1)$ nodes.

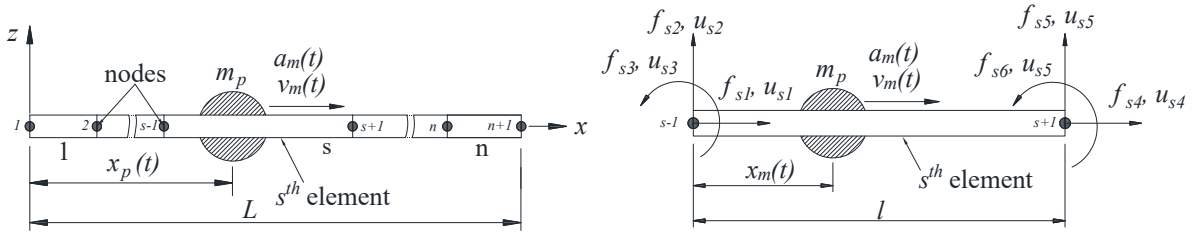


Figure 2: Finite element model of a thin beam with a lumped moving mass and equivalent nodal forces and displacements of the beam element s.

For any time t, the effective transverse (z) force at coordinate x, due to the interaction between the moving mass and the beam which is vibrating and deflected, can be determined as follows using he principle of D'Alembert :

$$\begin{aligned} f_z(x,t) &= m_p\left[g-\frac{\mathrm{d}^2 w_z(x_p,t)}{\mathrm{d}t^2}\right]\delta\left(x-x_p\right),\\ x_p &= x_0+v_0t+\frac{1}{2}a_0t^2+\frac{1}{6}Jt^3,\\ v_m &= \frac{\mathrm{d}x_p}{\mathrm{d}t}=v_0+a_0t+\frac{1}{2}Jt^2,\\ a_m &= \frac{\mathrm{d}^2x_p}{\mathrm{d}t^2}=a_0+Jt, J=\frac{\mathrm{d}a_m}{\mathrm{d}t}, \end{aligned} \tag{3}$$

For the linear motion of a particle object, the instantaneous acceleration is derived using the constant jerk of the motion and integrating it with respect to time for an increment of time Δt from t-Δt to t, as given below:

$$a^t = a^{t-\Delta t}+J\Delta t \tag{4}$$

Considering continuous contact between the mass and beam, when the mass moves on the beam with a variable acceleration and a constant jerk, the instantaneous flexural acceleration of the contact point of the mass can also be determined using the definite integration of the differential acceleration and jerk equation over a small time increment Δt, in the same way as Eq. (4). In such case, the effective transverse force equation will be rearranged as follows:

$$f_z^t(x,t)=m_p\left[g-\left(\frac{\mathrm{d}^2w_z(x_p,t)}{\mathrm{d}t^2}^{t-\Delta t}+\frac{\mathrm{d}^3w_z(x_p,t)}{\mathrm{d}t^3}\Delta t\right)^t\right]\delta\left(x-x_p\right), \tag{5}$$

When the necessary total differentiations of deflection function $w_z(x, t)$ with respect to the time dependent contact point x_p of the moving mass at time t, are determined and after making some mathematical arrangements, Eq.(5) becomes:

$$f_z^t(x,t) = -m_p g + m_p\left(\ddot{w}_z + 2v\dot{w}_z' + v^2 w_z'' + a w_z'\right)^{t-\Delta t} + m_p\left(v^3 w_z''' + 3v^2\dot{w}_z'' + 3vaw_z'' + 3v\ddot{w}_z' + a\dot{w}_z' + Jw_z' + \dddot{w}_z\right)^{t-\Delta} \Delta t\delta\left(x - x_p\right), \tag{6}$$

where the acceleration and jerk are

$$\frac{\mathrm{d}^2 w_z(x_p,t)}{\mathrm{d}t^2} = \ddot{w}_z + 2v\dot{w}_z' + v^2 w_z'' + a w_z',$$
$$\frac{\mathrm{d}^3 w_z(x,t)}{\mathrm{d}t^3} = w_z''' v^3 + 3\dot{w}_z'' v^2 + 3w_z'' va + 3\ddot{w}_z' v + \dot{w}_z' a + Jw_z' + \dddot{w}_z, \tag{7}$$
$$v = \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=x_p}; a = \left.\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right|_{x=x_p}; J = \left.\frac{\mathrm{d}^3 x}{\mathrm{d}t^3}\right|_{x=x_p}$$

where " ′ " and " · " are, respectively, spatial and time derivatives of deflection. Derivation of the acceleration and jerk terms are given in Appendix A. The terms $\left(\ddot{w}_z + 2v\dot{w}_z' + v^2 w_z'' + a w_z'\right)^{t-\Delta t}$ in Eq. (6), represent the inertia, the Coriolis and the centripetal acceleration components at time t-Δt, while $\left(v^3 w_z''' + 3v^2\dot{w}_z'' + 3vaw_z'' + 3v\ddot{w}_z' + a\dot{w}_z' + Jw_z' + \dddot{w}_z\right)^t$ represent jerks for above mentioned acceleration components of the interaction of the mass and beam at time t. Where $\ddot{w}_z$ is inertia; $2v\dot{w}_z'$ is Coriolis, $v^2 w_z'' + a w_z'$ is centripetal acceleration components, while $\ddot{w}_z + 3v\ddot{w}_z'$ is inertia, $3v^2\dot{w}_z'' + a\dot{w}_z'$ is Coriolis and $v^3 w_z''' + 3vaw_z'' + Jw_z'$ is Centripetal jerks of the accelerations respectively. In vibration, the inertia jerk of the beam can be approximated using the average acceleration method proposed by Newmark in the Beta method of integration with in the assumptions of the method. In this case the jerk of the beam is:

$$\dddot{w}_z^{\,t} = \frac{1}{\Delta t}\left[\ddot{w}_z^{\,t} - \ddot{w}_z^{\,t-\Delta t}\right], \tag{8}$$

When the beam is in vibration, the longitudinal (x) force component, between the moving mass and the beam, induced by the vibration and curvature of the deflected beam is

$$f_x^t(x,t) = m_p\left[a^t - \frac{\mathrm{d}^2 w_x(x_p,t)}{\mathrm{d}t^2}\right]\delta(x - x_p) = -m_p\left(a^{t-\Delta t} + J\Delta t\right) + m_p\ddot{w}_x\delta(x - x_p), \tag{9}$$

The equivalent nodal forces of the beam element s under a lumped moving mass that has a variable acceleration and a constant jerk are:

$$f_{si}^t = -\phi_i m_p\left(a^{t-\Delta t} + J\Delta t\right) + \phi_i m_p\left(\ddot{w}_x\right)^t \quad (i = 1,\ 4),$$
$$f_{si}^t = \phi_i m_p\left(\ddot{w}_z + 2v\dot{w}_z' + v^2 w_z'' + a w_z'\right)^{t-\Delta t} + \phi_i m_p\left(w_z''' v^3 + 3\dot{w}_z'' v^2 + 3w_z'' va + 3\ddot{w}_z' v + \dot{w}_z' a + Jw_z' + \frac{1}{\Delta t}\left[\ddot{w}_z^{\,t} - \ddot{w}_z^{\,t-\Delta t}\right]\right)^{t-\Delta t}\Delta t$$
$$-\phi_i m_p g \quad (i = 2,\ 3,\ 5,\ 6), \tag{10}$$

Where, ϕ_i (i=1- 6) are shape functions of the beam element given by Clough and Penzien, 2003; these are cubic Hermitian polynomials that are expressed as

$$\begin{aligned}\phi_1 &= 1-\frac{x}{l}, \phi_2 = 1-3\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)^3, \phi_3 = x\left(1-\frac{x}{l}\right)^2,\\ \phi_4 &= \frac{x}{l}, \phi_5 = 3\left(\frac{x}{l}\right)^2-2\left(\frac{x}{l}\right)^3, \phi_6 = \frac{x^2}{l}\left(\frac{x}{l}-1\right),\end{aligned} \tag{11}$$

where l the length of s-th beam element.

The relation between shape functions and transverse and longitudinal displacements of the s-th beam element at position x and time t, is (Clough and Penzien, 2003):

$$\begin{aligned}w_x(x,t) &= \phi_1 u_{s1} + \phi_4 u_{s4},\\ w_z(x,t) &= \phi_2 u_{s2} + \phi_3 u_{s3} + \phi_5 u_{s5} + \phi_6 u_{s6},\end{aligned} \tag{12}$$

where u_{si} (i = 1–6) are the displacements for the nodes of the s-th beam element on which the moving mass m_p locates at time t.

The property matrices of a beam element without any moving mass, in Fig. 2, having both transverse and longitudinal nodal forces and deflections, are derived from the usage of the principle of virtual works. Introducing the principle of virtual displacements, and applying unit displacements at the nodal points and then equating the work done by the external forces to the work done on the internal forces: $W_E = W_I$ (Clough and Penzien, 2003) for a uniform beam segment using the interpolation functions of Eq. (11), a stiffness equation can be obtained for the stiffness coefficients. For the mass coefficients, an elemental balance equation can be obtained using the relation between nodal accelerations and resisting inertial forces with the help of the same interpolation functions. In the special case of a beam element with uniformly distributed mass and under a mass moving with a variable acceleration over the beam element, the coefficients of modified mass, stiffness and equivalent damping matrices of the beam element s can be obtained by taking into account the contribution of inertial, Coriolis and centripetal forces due to the related accelerations and jerks (7) induced by the moving mass. In such a case, the total force balance equation for the beam element s is obtained in matrix form as below:

$$\mathbf{f}^t = \bar{\mathbf{m}}^t \ddot{\mathbf{u}}_s^{\,t} + \bar{\mathbf{c}}^t \dot{\mathbf{u}}_s^{\,t} + \bar{\mathbf{k}}^t \mathbf{u}_s^{\,t}, \tag{13a}$$

$$\bar{\mathbf{m}}^{\mathbf{t}} = \mathbf{m} + \mathbf{m}_{\ddot{\mathbf{w}}}^{\,t-\Delta t} + \mathbf{m}_{\dddot{\mathbf{w}}}^{\,t-\Delta t}\Delta t, \tag{13b}$$

$$\bar{\mathbf{c}}^{\mathbf{t}} = \mathbf{c} + \mathbf{c}_{\ddot{\mathbf{w}}}^{\,t-\Delta t} + \mathbf{c}_{\dddot{\mathbf{w}}}^{\,t-\Delta t}\Delta t,, \tag{13c}$$

$$\bar{\mathbf{k}}^t = \mathbf{k} + \mathbf{k}_G + \mathbf{k}_{\ddot{\mathbf{w}}}^{\,t-\Delta t} + \mathbf{k}_{\dddot{\mathbf{w}}}^{\,t-\Delta t}\Delta t, \tag{13d}$$

where $\bar{\mathbf{m}}$, $\bar{\mathbf{c}}$ and $\bar{\mathbf{k}}$ are, respectively, the modified mass, damping and stiffness matrices of the beam element s that includes all the effects of the moving mass, while $\mathbf{m}$, $\mathbf{c}$ and $\mathbf{k}$ are consistent mass, equivalent viscous damping and consistent stiffness matrices of the beam element itself respectively,

and $\mathbf{k}_G$ is the geometric stiffness matrix due to the axial force *N(x,t)*. The matrices $\mathbf{m}_{\ddot{w}}$ and $\mathbf{m}_{\dddot{w}}$, $\mathbf{c}_{\ddot{w}}$ and $\mathbf{c}_{\dddot{w}}$, $\mathbf{k}_{\ddot{w}}$ and $\mathbf{k}_{\dddot{w}}$ are derived from the interaction of the moving mass and the beam element and they represent the moving mass with a constant jerk. The components of these matrices vary depending on the motion velocity, acceleration and jerk, and the position of the mass on the beam element. The coefficients of these matrices are obtained from the equivalent nodal force equation (10) of the beam element *s*. One can obtain these matrices using the relation between the nodal displacements and the deflection functions (12) with the interpolation functions (11), and then putting them in the nodal force equation (10). After obtaining the spatial and time derivatives of the deflection function, Eq. (10) is rearranged separately for acceleration and jerk in terms of nodal accelerations, nodal velocities and nodal displacements, thus these matrices are obtained, as given in Appendix B.

During the integration of motion equation of the overall system, the instantaneous values of these matrices are calculated at every time step Δt, an then added to the values of the matrices of the beam element s on which the moving mass locates at time t. Because, they are derived from the relation between nodal displacements and shape functions of the two-node beam element, the dimensions of these matrices are equal to the dimensions of the mass, damping, stiffness matrices of the beam element. Hence, the beam element has three displacements DOF at each end nodal point; the dimensions of these modification matrices will also be 6x6.

The consistent mass, equivalent damping and consistent stiffness matrices **m**, **c**, and **k**, of the beam element s are calculated using their coefficients m_{ij}, $c_{ij,}$ and k_{ij}, as given below

$$
\begin{aligned}
m_{ij} &= \int_0^l m(x)\phi_i(x)\phi_j(x)dx, (i,j=1,\ 2,\ 3,\ 4,\ 5,\ 6),\\
c_{ij} &= \psi\int_0^l m(x)\phi_i(x)\phi_j(x)dx - \tau\int_0^l EA\phi_i'(x)\phi_j'(x)dx, (i,j=1,\ 4),\\
c_{ij} &= \psi\int_0^l m(x)\phi_i(x)\phi_j(x)dx + \tau\int_0^l EI(x)\phi_i''(x)\phi_j''(x)dx, (i,j=2,\ 3,\ 5,\ 6),\\
k_{ij} &= -\int_0^l EA\phi_i'(x)\phi_j'(x)dx, (i,j=1,\ 4),\\
k_{ij} &= \int_0^l EI(x)\phi_i''(x)\phi_j''(x)dx, (i,j=2,\ 3,\ 5,\ 6),
\end{aligned}
\tag{14}
$$

Where ψ and τ are the coefficients for the equivalent viscous damping. They can be determined by using Rayleigh damping theory in which the coefficients of damping matrix is proportional to the combination of coefficients of the mass and stiffness matrices, as given by

$$
\begin{aligned}
c_{ij} &= \psi m_{ij} + \tau k_{ij},\\
\psi &= \frac{2\omega_i\omega_j\left(\zeta_i\omega_j - \zeta_j\omega_i\right)}{\omega_j^{\,2} - \omega_i^{\,2}}, \tau = \frac{2\left(\zeta_j\omega_j - \zeta_i\omega_i\right)}{\omega_j^{\,2} - \omega_i^{\,2}},
\end{aligned}
\tag{15}
$$

where ζ_i and ζ_j are the damping ratios of the structural system for any corresponding natural frequencies ω_i and ω_j.

In order to include the flexural effect of an axial load $N(x,t)$ due to the acceleration of the mass, that is negative for positive acceleration and vice versa, the geometric stiffness matrix $\mathbf{k}_G$ are obtained with the following coefficients:

$$k_{Gi,j} = \mp \int_0^l N(x,t)\phi_i'(x)\phi_j'(x)dx, \quad (i,j = 2,\ 3,\ 5,\ 6), \tag{16}$$

where $k_{Gi,j}$ are the coefficients of consistent *geometric stiffness matrix* and the Hermitian interpolation functions. Eqs. (11) are used in deriving the geometric stiffness coefficients.
In the time dependent elemental equation of motion of the beam element s, in Eq. (13a), the force $\mathbf{f}^t$ is as given below:

$$\begin{aligned}
\mathbf{f}^t &= \begin{bmatrix} f_{s1} & f_{s2} & f_{s3} & f_{s4} & f_{s5} & f_{s6} \end{bmatrix}^T, \\
f_{s1} &= m_p \left(a^{t-\Delta t} + J\Delta t \right) \phi_1, \\
f_{s2} &= m_p g \phi_2, \\
f_{s3} &= m_p g \phi_3, \\
f_{s4} &= m_p \left(a^{t-\Delta t} + J\Delta t \right) \phi_4, \\
f_{s5} &= m_p g \phi_5, \\
f_{s6} &= m_p g \phi_6,
\end{aligned} \tag{17}$$

The nodal acceleration, velocity, and displacement vectors, $\ddot{\mathbf{u}}_s^{\,t}, \dot{\mathbf{u}}_s^{\,t}$, and $\mathbf{u}_s^{\,t}$ of the beam element s are given by

$$\begin{aligned}
\ddot{\mathbf{u}}_s^{\,t} &= \begin{bmatrix} \ddot{u}_{s1} & \ddot{u}_{s2} & \ddot{u}_{s3} & \ddot{u}_{s4} & \ddot{u}_{s5} & \ddot{u}_{s6} \end{bmatrix}^T, \\
\dot{\mathbf{u}}_s^{\,t} &= \begin{bmatrix} \dot{u}_{s1} & \dot{u}_{s2} & \dot{u}_{s3} & \dot{u}_{s4} & \dot{u}_{s5} & \dot{u}_{s6} \end{bmatrix}^T, \\
\mathbf{u}_s^{\,t} &= \begin{bmatrix} u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5} & u_{s6} \end{bmatrix}^T,
\end{aligned} \tag{18}$$

2.2 Equation of Motion of the entire System

The equation of motion for the multiple degree of freedom damped structural system, shown in Fig. 2, is given by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}, \tag{19}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are, respectively, the overall mass, damping and stiffness matrices, while $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$ and $\mathbf{u}$ are respectively, the acceleration, velocity and displacement vectors. Besides, $\mathbf{F}$ is the overall external force vector of the system at time t.

In general, such a structural system shown in Fig. 2, one can obtain overall mass $\mathbf{M}$, stiffness $\mathbf{K}$ and $\mathbf{C}$ matrices by assembling its element matrices and imposing given boundary conditions. If there is a mass moving with a variable acceleration over the structure, the mass, stiffness and damping matrices of entire system can be obtained by taking into account the contribution of inertial, Coriolis

and centripetal forces, and jerks induced by moving mass. This is done by modifying only the elemental matrices of the beam element s on which the moving mass locates at time t, and keeping the elemental matrices of the other elements the same as original. The instantaneous overall force vector is also time-depended. The coefficients of overall force vector are equal to zero except the nodal forces of the beam element *s*. Thus, the instantaneous overall force vector of entire system becomes as below:

$$\begin{gathered}\mathbf{F}=\begin{bmatrix}0 & \ldots & f_{s1} & f_{s2} & f_{s3} & f_{s4} & f_{s5} & f_{s6} & \ldots & 0\end{bmatrix}^{T},\\ f_{s\,i}=m_p g\phi_i \quad (i=2,\ 3,\ 5,\ 6),\\ f_{s\,i}=m_p(a(t-\Delta t)+J\Delta t)\phi_i \quad (i=1,4)\end{gathered} \tag{20}$$

In evaluation of the matrices of the beam element s, for the instantaneous position of the mass, the term x in the Eq. (11) is replaced with $x_m(t)$. The instantaneous values of $x_m(t)$ and s can be determined as follows:

$$\begin{gathered}x_m(t)=x_p(t)-(s-1)l,\\ s=(\text{integer part of}\ \frac{x_p(t)}{l})+1, \quad s=(1-n),\end{gathered} \tag{21}$$

2.3 Solution of Equation of Motion

If the added matrices of $\mathbf{m}_{\ddot{w}}, \mathbf{m}_{\ddot{\theta}}, \mathbf{k}_{\ddot{w}}, \mathbf{k}_{\ddot{\theta}}, \mathbf{k}_G$ and the equivalent force vector $\mathbf{f}$ are zero, the undamped natural frequencies and vibration mode-shapes of the beam are obtained from homogenous solution of (19). In such a case, the Equation (19) reduces to

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=0, \tag{22}$$

The solution of (22) can be assumed as

$$\mathbf{u}=\mathbf{A}e^{i\omega t}, \tag{23a}$$

$$\ddot{\mathbf{u}}=-\omega^2\mathbf{A}e^{i\omega t}, \tag{23b}$$

Introducing the latter $\mathbf{u}$ and $\ddot{\mathbf{u}}$ into (22) gives

$$(\mathbf{K}-\omega^2\mathbf{M})\boldsymbol{\Psi}e^{i\omega t}=0, \tag{24}$$

This is a set of homogeneous equations, for which a nontrivial solution exists if

$$\det(\mathbf{K}-\omega^2\mathbf{M})=0, \tag{25}$$

The above determinant equation is satisfied for a set of n values of frequency $\omega_1, \omega_2, .., \omega_n$. The frequency ω_i is called the i-th natural frequency. Substituting all the ω_i into (24) yields the corresponding set of vectors $\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, ..., \boldsymbol{\Psi}_n$ that satisfy this equation. The ith vector $\boldsymbol{\Psi}_i$ corresponding to the ith natural frequency is called the ith natural mode, or ith mode shape [6].

For such a system, given in (19), one can obtain a solution by using a numerical integration method like Newmark's β method [5]. Using the Newmark integration method, equation (19) is transformed to equivalent stiffness and force equations. Thus, for the n-th time step at time t_n ($=t_{n-1}+\Delta t$):

$$\widehat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}, \tag{26}$$

$$\begin{aligned}\widehat{\mathbf{F}}(t_n) = \mathbf{F}(t_n) + \mathbf{M}\left[a_0\mathbf{u}(t_{n-1}) + a_2\dot{\mathbf{u}}(t_{n-1}) + a_3\ddot{\mathbf{u}}(t_{n-1})\right] \\ +\mathbf{C}[a_1\mathbf{u}(t_{n-1}) + a_4\dot{\mathbf{u}}(t_{n-1}) + a_5\ddot{\mathbf{u}}(t_{n-1})].\end{aligned} \tag{27}$$

where $\ddot{\mathbf{u}}(t_{n-1})$, $\dot{\mathbf{u}}(t_{n-1})$, $\mathbf{u}(t_{n-1})$ are, respectively, the initial conditions for the accelerations, velocities and displacements of the structural system at time $t = t_n$. Displacements at time t_n

$$\mathbf{u}(t_n) = \widehat{\mathbf{K}}^{-1}\widehat{\mathbf{F}}(t_n), \tag{28}$$

Accelerations and velocities at time t_n

$$\ddot{\mathbf{u}}(t_n) = \left(a_0\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\right) - a_2\dot{\mathbf{u}}(t_{n-1}) - a_3\ddot{\mathbf{u}}(t_{n-1}), \tag{29}$$

$$\dot{\mathbf{u}}(t_n) = \dot{\mathbf{u}}(t_{n-1}) + a_6\ddot{\mathbf{u}}(t_{n-1}) + a_7\ddot{\mathbf{u}}(t_n). \tag{30}$$

where a_0, a_1, a_2, a_3, a_4, a_5, a_6 and a_7 are the integration constants in the Newmark-β method; that are

$$\begin{aligned} a_0 &= \frac{1}{\beta\Delta t^2}, \quad a_1 = \frac{\gamma}{\beta\Delta t}, \\ a_2 &= \frac{1}{\beta\Delta t}, \quad a_3 = \frac{1}{2\beta} - 1, \\ a_4 &= \frac{\gamma}{\beta} - 1, \quad a_5 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right), \\ a_6 &= \Delta t(1-\gamma), \quad a_7 = \gamma\Delta t. \end{aligned} \tag{31}$$

where β and γ are the integration parameters. In this study, β=0.25 and γ=0.5 are used for integration accuracy and stability.

If the mass travels on the beam, the instantaneous mass and stiffness matrices should be used for the solution of instantaneous natural frequencies of the entire system. The added matrices of $\mathbf{m}_{\ddot{w}}$, $\mathbf{m}_{\dot{w}}$, $\mathbf{k}_{\ddot{w}}$, $\mathbf{k}_{\dot{w}}$, $\mathbf{k}_G$ are included in Eq. (19). In this case:

$$\bar{\mathbf{M}}\ddot{\mathbf{u}} + \bar{\mathbf{K}}\mathbf{u} = 0, \tag{32}$$

For the frequency solution of (32), one can use (33). That is:

$$\det(\overline{\mathbf{K}} - \overline{\omega}_i^{\,2}\overline{\mathbf{M}}) = 0, (i = 1 - n) \tag{33}$$

where $\overline{\omega}_i$ is the ith forced vibration frequency of the entire system. If the mass and stiffness matrices, used in (33) are time-dependent, the frequency solution will also be time dependent.

For the calculation of the instantaneous overall mass and stiffness matrices of the entire system at every time step of Δt, one may use following steps:

1. Determine the mass and stiffness matrices of each beam element.
2. For time t, determine the element s on which the moving mass locates with (21).
3. Determine $x_m(t)$ which is the time dependent position of the moving mass on the s-th element with (21).
4. Calculate the time dependent shape functions with (11) by substituting the value $x=x_m(t)$ which is defined in the previous step.
5. Calculate the modified mass $\overline{\mathbf{m}}$, damping $\overline{\mathbf{c}}$ and stiffness $\overline{\mathbf{k}}$ matrices of the s-th element with the help of (14), and Appendix B.
6. Calculate the instantaneous overall mass and stiffness matrices of the entire system by combining the mass and stiffness matrices of each beam element. Then impose boundary conditions. Eigen solution of these matrices gives instantaneous natural frequency of the entire system at time t.

For $t+\Delta t$ go to step 2

3 RESULTS AND DISCUSSION WITH NUMERICAL SOLUTIONS

In this article, the Newmark β method of integration [27] with the time step size =0.00001 s is used to obtain the solution of Eq. (19). The integration constants that manage the sensitiveness and stability of the Newmark procedure, β=0.25 and γ=0.5 are imposed. It has been reported that when β takes 0.25 value and γ 0.5, this numerical procedure is unconditionally stable Szilard 2004.

Example 1. In order to compare present method with the others at the same assumptions, excluding the inertia effects of the mass and accepting the mass as moving load that has been studied in literature, let us take a simple supported isotropic beam-plate under a $F = 4.4$ N moving load. The dimensional and material specifications of the plate are identical with those preferred in Meirovitch 1967, i.e. $l_x = 10.36$ cm; $l_y = 0.635$ cm, $h = 0.635$ cm; $E = 206.8$ GPa, $\rho = 10686.9$ kg/m^3; T_f=8.149 s, where T_f is the fundamental period. In Table 1, dynamic amplification factors (DAF), which are defined as the ratio of the maximum dynamic deflection to the maximum static deflection, are compared with several previous numerical, analytical, and experimental results available in literature. It is noted that T is the required time for moving load to travel the beam. It is seen that the results obtained by the current finite element (column 3) are very close to the analytical solution Meirovitch 1967, and the results of first order shear deformation theory (FSDT) method Michaltsos 2002.

V (m/s)	T_f / T	1	2	3	4	5	6
15.6	0.125	1.040	1.025	1.055	1.063	1.045	1.040
31.2	0.25	1.352	1.121	1.112	1.151	1.350	1.090
62.4	0.5	1.265	1.258	1.252	1.281	1.273	1.183
93.6	0.75	1.574	1.572	-	1.586	1.572	1.194
124.8	1	1.704	1.701	1.7	1.704	1.704	1.549
156	1.25	1.717	1.719	-	1.727	1.716	1.527
187.2	1.5	1.547	-	-	-	-	1.471
250	2	1.543	1.548	1.54	1.542	1.542	1.374

Table 1: Dynamic amplification factors (DAF) versus velocity.

(1) Present method.
(2) Analytical solution from Meirovitch 1967.
(3) From Yoshida and Weaver 1971.
(4) From Kadivar and Mohebpour 1998.
(5) From Michaltsos 2002.
(6) From Mohebpour et al 2011

Example 2: For another example, a frame box-section beam that can be preferable for a CNC system due to the lightweight and rigidity, with length $L = 1$ m, height h=0.1 m and width $b = h = 0.1$, and a wall thickness of 7.5 mm, was chosen. Since $h / L = 10$, it can be assumed that it is a thin Euler Bernoulli beam. In order to satisfy the desired computational accuracy in the Finite Element Method, firstly the number of elements should be determined. For this reason in Tables 2-5, the first six natural frequencies depending on the number of elements are given. In this paper, the number of elements is preferred as 100 since its results in the frequency calculation are very close to the exact ones. However, one can prefer 10 elements for an acceptable result in the analysis, in that case, for the sixth frequency in Table 2 the relative error percentage is about 0.8 percent. Where, the relative error=100(69797.8459-69247.9106)/ 69247.9106=0.79, and the error will be very little and insignificant for the lower frequencies.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
n=10	1923.5660	7695.0358	17321.2284	30827.8381	48278.6284	69797.8459
n=20	1923.5539	7694.2642	17312.5668	30780.1430	48101.3136	69284.9137
n=60	1923.5531	7694.2129	17311.9850	30776.8902	48088.9834	69248.3777
n=100	1923.5531	7694.2124	17311.9786	30776.8545	48088.8471	69247.9712
Exact	1923.5531	7694.2123	17311.9776	30776.8492	48088.8268	69247.9106

Table 2: Free vibration frequencies for pinned-pinned thin beam.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
n=10	3005.0065	9739.6533	20332.4606	34817.3588	53271.1778	75835.1555
n=20	3004.9601	9738.0906	20318.4813	34748.9140	53034.6549	75186.0291
n=60	3004.9571	9737.9867	20317.5411	34744.2364	53018.1413	75139.3886
n=100	3004.9570	9737.9856	20317.5308	34744.1850	53017.9588	75138.8693
Exact	3004.9169	9737.9856	20317.5308	34744.1850	53017.9588	75138.8693

Table 3: Free vibration frequencies for fixed-pinned beam.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
n=10	4360.6300	12022.9618	23586.9257	39054.7738	58520.3968	82140.5762
n=20	4360.4884	12020.0258	23565.1534	38958.6131	58209.5755	81329.8426
n=60	4360.4790	12019.8305	23563.6874	38952.0250	58187.7582	81270.8722
n=100	4360.4789	12019.8284	23563.6713	38951.9525	58187.5168	81270.2153
Exact	4360.4789	12019.8284	23563.6713	38951.9525	58187.5168	81270.2153

Table 4: Free vibration frequencies for fixed-fixed beam.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
n=10	685.2603	4294.5913	12027.6434	23585.8362	39050.1464	58501.5603
n=20	685.2597	4294.4581	12024.7787	23564.8533	38958.5563	58209.2783
n=60	685.2597	4294.4492	12024.5843	23563.3997	38952.0408	58187.7569
n=100	685.2598	4294.4491	12024.5822	23563.3837	38951.9685	58187.5160
Exact	685.2598	4294.4491	12024.5822	23563.3837	38951.9685	58187.5160

Table 5: Free vibration frequencies for fixed-free beam.

In moving mass problem, the variation of the natural frequencies of the beam is an interesting phenomenon that every design engineer should take into account of this effect of the mass. When the mass moves on the beam depending on the position of the mass, the natural frequencies undergo a continuous change, thus the vibration of the beam has to change as can be seen in Figures 3-7 in which normalized frequencies ***fr*** versus normalized position $x_p\,(t)/L$ of the mass over the beam are depicted for different end support conditions. Where f_r is ratio of the frequencies in cases of with and without mass on the beam.

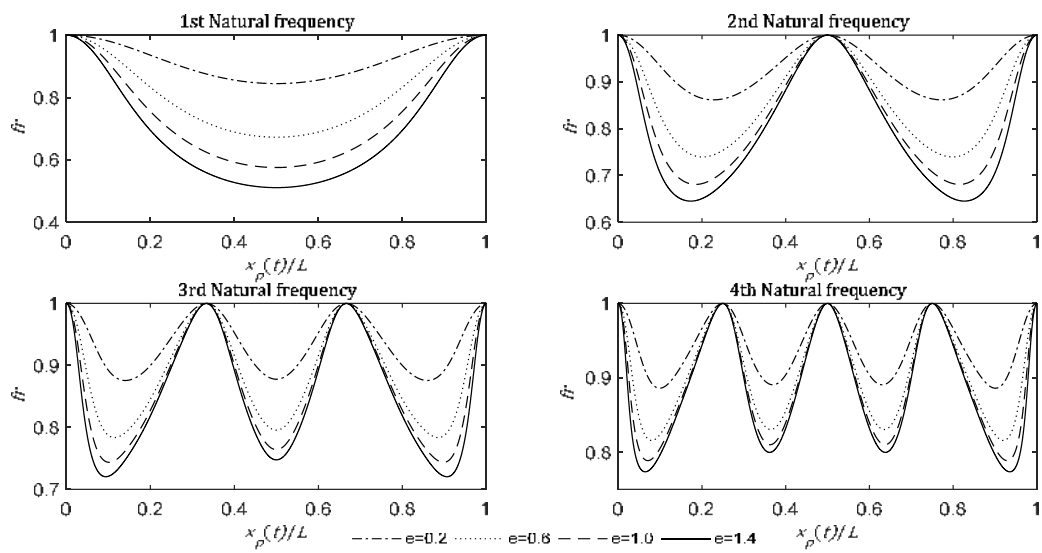


Figure 3: Variations in natural frequencies for pinned-pinned beam.

Figure 3 displays the variation of the first four natural frequencies depending on the mass position on the beam and the mass ratio of the moving mass and the mass of the beam. Because of this variation in frequencies, the vibration behaviour of the beams under moving loads are very complicated and depends upon various parameters of the system such as mass, velocity, acceleration, and jerk of the moving mass including end support conditions. The variation of a particular natural frequency is governed by its corresponding natural mode shape. For example, as can be seen from Figure 3 the variation of the first natural frequency resembles its first natural mode shape. This is also valid for the other natural frequencies, 2nd, 3rd, 4th and so on. Another interesting behaviour is that at the location of the nodes, there are no changes of the frequencies even for higher mass ratios. Between the nodal points of the corresponding mode shapes, the rate of change is increased by the increasing mass ratio. These variations are also nonlinear, and have a tendency to the left or right as seen from the graphics depending on the first and last nodal point that are near to the left and right ends. The other nodal points except for these there are change but there is no tendency to the right or to the left. This event is valid for other end support conditions that are presented through Figures 3-7. The determination of the change in the natural frequencies is easy by using the proposed method in this study. Because the added matrices of $\mathbf{m}_{\ddot{w}}$, $\mathbf{m}_{\ddot{w}}$, $\mathbf{c}_{\ddot{w}}$, $\mathbf{c}_{\ddot{w}}$, $\mathbf{k}_{\ddot{w}}$, $\mathbf{k}_{\ddot{w}}$ and $\mathbf{k}_G$ are always added to the left side of the motion equation (19) of the whole system. This is the most advantageous property of the proposed method that one can directly model a dynamic system by including the inertia, Coriolis and centripetal forces and jerks of the moving mass as internal forces, and imposing the gravitational force to the right as an external force. The above matrices represent the dynamic added mass, damping and stiffness properties of the beam under a moving mass. The main difference of the assumption of the moving mass is this modification of the unloaded beam matrices. In the moving load assumption, all these matrices are zero except for the stiffness matrix $\mathbf{k}_G$ due to axial force effect in the flexural deflection of the beam if there is an accelerating mass. Including second and third derivations of the deflection function one can analyse the effects of mass moving with constant and variable velocity on a beam using the current method.

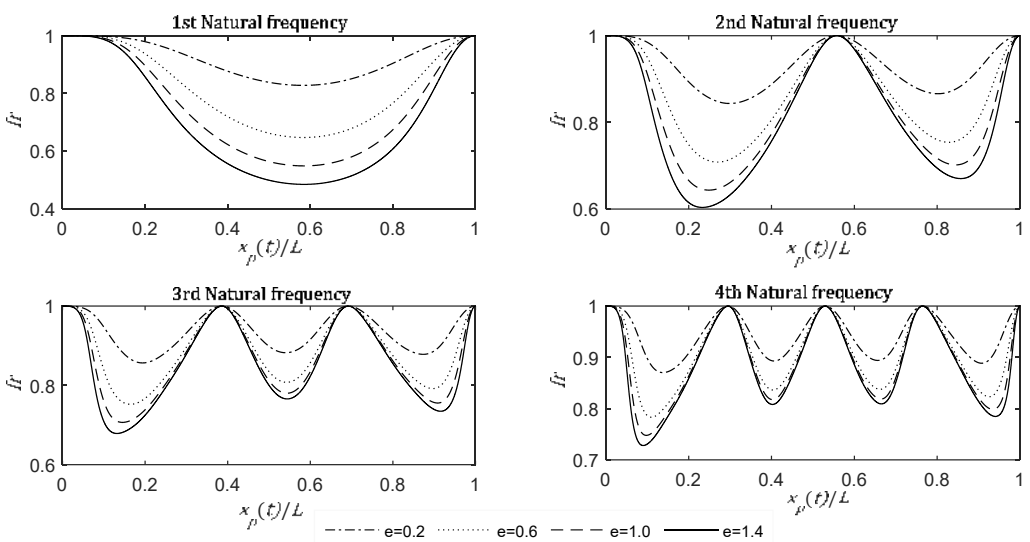


Figure 4: Variations in natural frequencies for fixed-pinned beam.

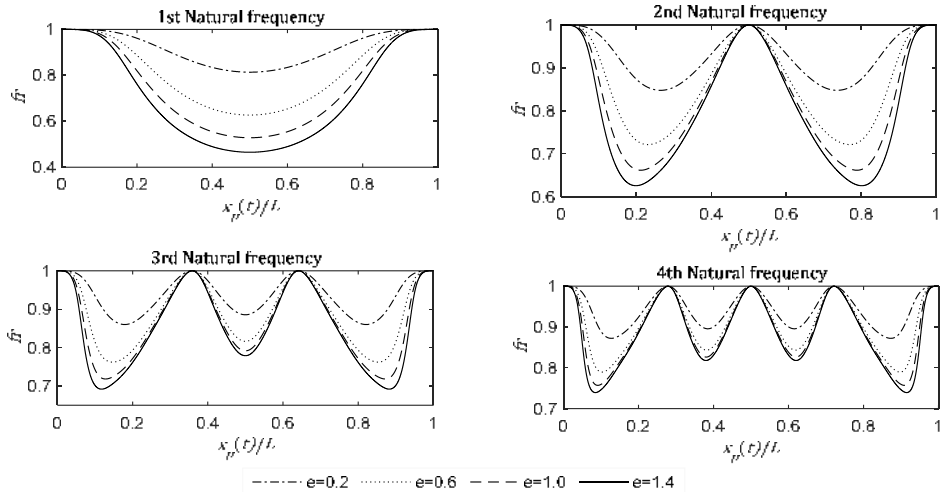


Figure 5: Variations in natural frequencies for fixed-fixed beam.

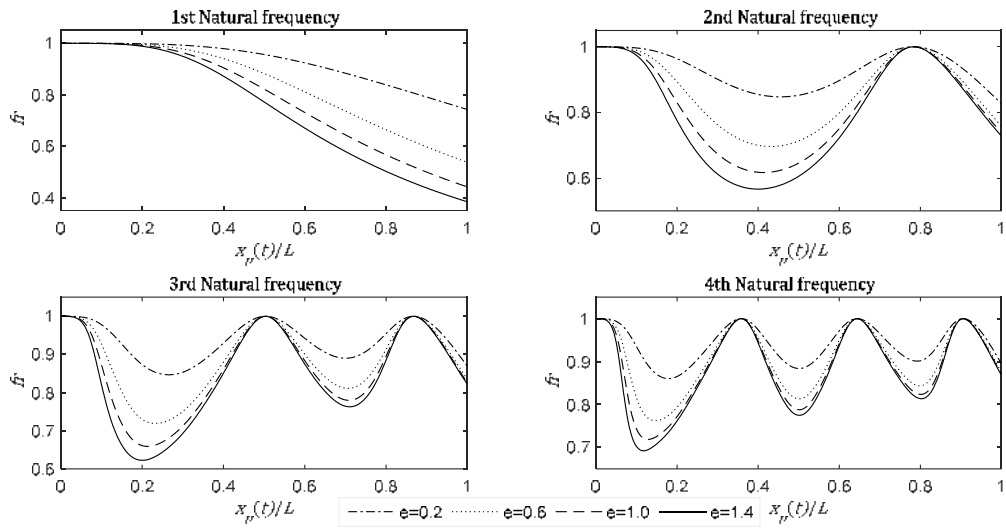


Figure 6: Variations in natural frequencies for fixed-free beam.

For the same mass ratio, i.e., e=0.5 the effect of the velocity of the mass on the change of the natural frequencies is given in Figure 7, for a fixed-free support condition. While the velocity is small, the effect of the velocity in the added matrices will also be very small because the added terms from velocity are too small compared to the high flexural rigidity *EI* of the beam. For the case, that the beam flexural rigidity is higher and velocity of the moving mass is constant the effect of the velocity on the natural frequency can be omitted even for 100 m/s the effect will be at the level of one percent. If the beam is slender or for the case of cable transport the effect of the velocity should be taken into account even though the velocity is small. For barrel and projectile interaction in a weapon system, that has a longer barrel and high exit velocity of the projectile, and rocket launching structures the effect of the velocity is very important due to high rate of acceleration and jerk.

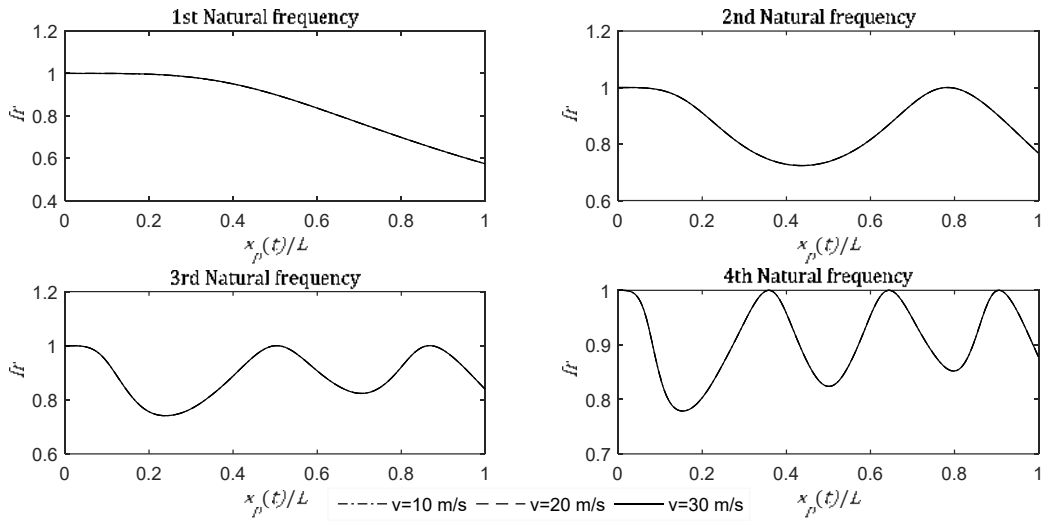
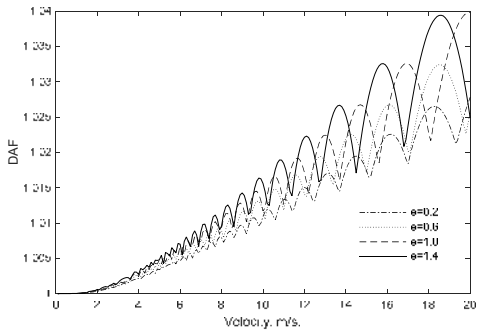


Figure 7: Variations in natural frequencies for fixed-free beam for e=0.5 and v=10, 20 and 30 m/s.

Moving load problems have been studied widely in literature, but most of these studies are for vehicle bridge interaction; and the most of them have omitted the inertial effect of the mass and accepted only the gravitational force of the moving load. Some of them have been modelled as moving oscillator that the main mass of the load is modelled as a sprung mass, and the unsprung mass of the suspension system is omitted. However, for the application of CNC applications of the moving mass where the accuracy of the work piece is important all the mass of the moving load is firmly in contact with the beam, the dynamic amplification factor (DAF) for different masses of the moving load may be significant for design engineers. Where DAF= maximum dynamic displacement of the beam/static displacement of the beam when the mass is at the middle. Figure (8) displays the DAF's, and vibration acceleration and velocities of the midpoint for different constant travelling velocities of the mass and different mass ratios e, where e= mass of the moving load/mass of the beam. The mass of the given beam is 22.2 kg; and for e=0.2, 0.6, 1.0 and 1.4, the masses of the moving load are: 4.44, 13.32, 22.2 and 31.08 kg, such a mass ratio applies to CNC applications. Figure (8a) is for lower velocities while the others are for higher velocities. The figures (8e, 8f and 8g) are for moving load assumptions which the inertia, Coriolis and centripetal effects of the moving mass are omitted and only gravitational effect is considered, while Figures (8a to 8e) are for moving mass assumption that consider all the effects of the moving mass. From the figures, one can realize that they are very different in terms of the mentioned assumptions. The maximum DAF is about 1.7 for moving load assumption, but for the moving mass assumption it is about 6 that is higher about 3.53 times. This is due to the effect of mass with higher travelling velocities. For very low travelling velocities the difference is not considerably high when the rigidity of the beam is high in comparison with the velocity due to its little effect in the system mass, stiffness and damping matrices. The velocity increment in Figure (8a) is 0.001 m/s, while it is 5 m/s for the others. When Figure 8a, is examined closely, it is observed that there are many resonance peaks and cancellation points.



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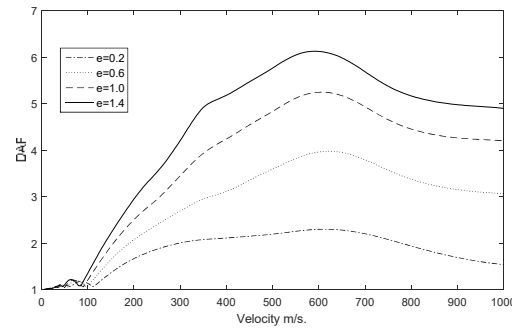

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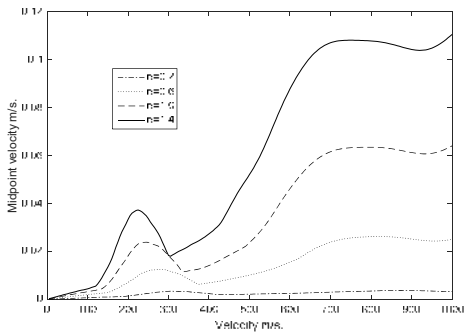

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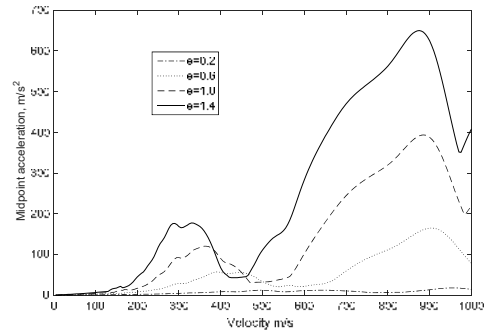

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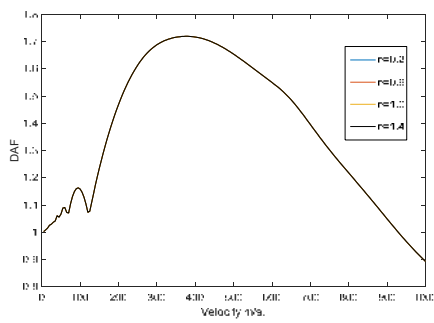

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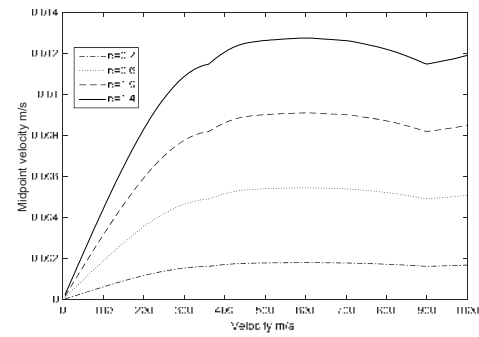

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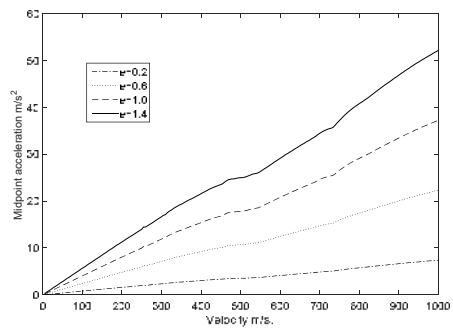


g

Figure 8: Comparisons of DAFs in moving mass (a, b, c and d) and moving load (e, f and g) assumptions.

In both cases the vibration acceleration and velocities are very different from each other. In the case of the moving mass in Fig. 8d, the maximum is ten times higher when compared to the moving load case in Fig. 8g. In addition, although the increase in the mass ratio is linear, the increase in the maximum of the moving mass is not linear, but the increase in the moving load is linear in acceleration and speed, but no increase is observed in the DAF that there is no difference with increasing mass ratio e. The effect of the mass on the frequency variation of the beam given in Figure 3-8 is valid for the moving mass assumption. At constant acceleration of the mass, the displacements and rotations at the midpoint of the beam are given in Fig. 9. The increase in displacement seems to be very high at high acceleration of motion. However, such high acceleration cannot be the case for CNC applications. In addition, from the standpoint of existing bridge engineering problems, high acceleration does not apply in existing applications. However, the acceleration effect can be accounted for in high speed rail transport applications. Moreover, if the projectile is considered that accelerations to be higher than the given examples, the impact of the exposure should be considered in such applications, in the case of projectile and barrel interaction.

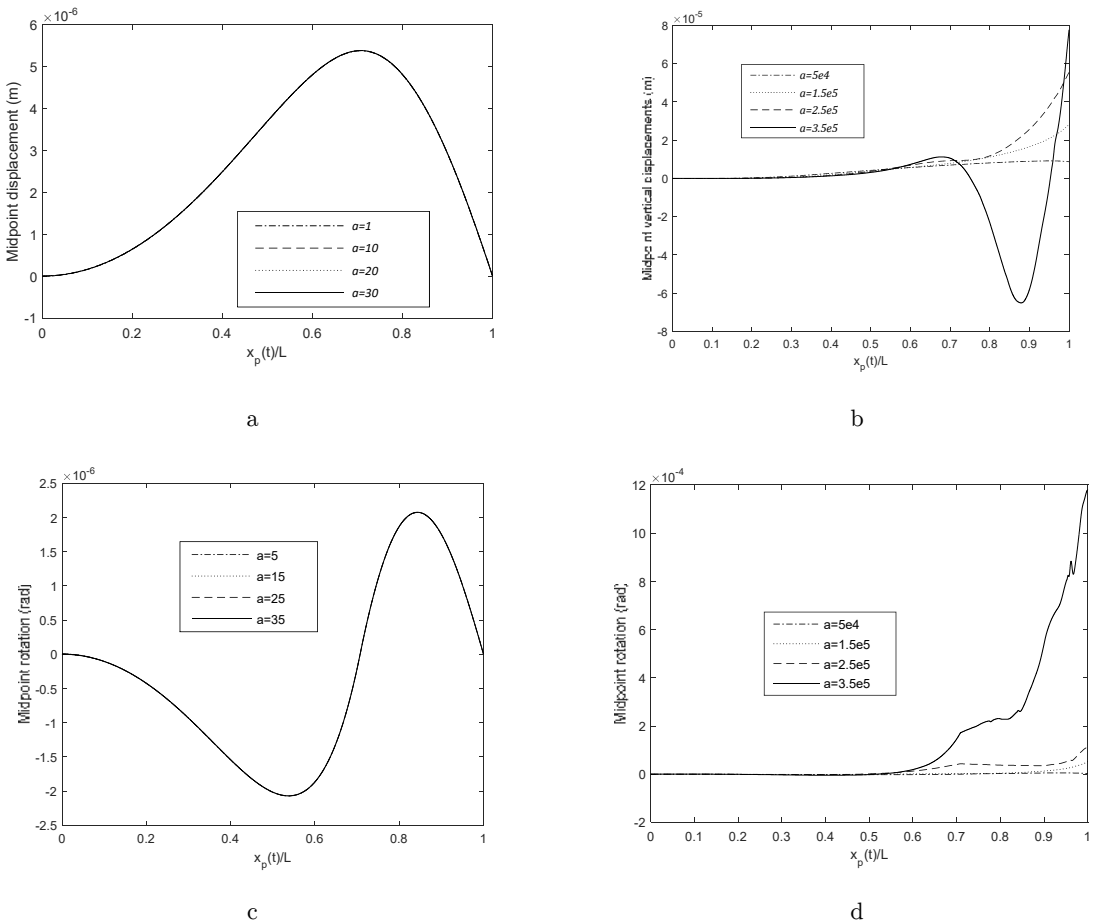
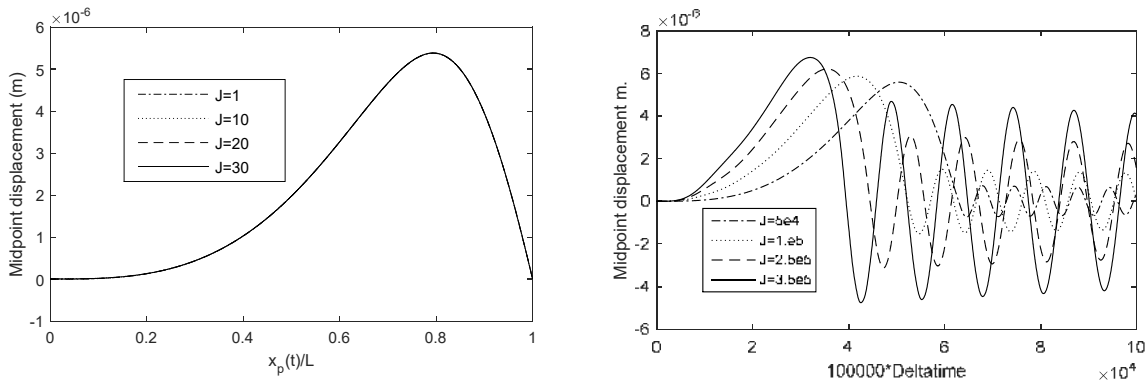


Figure 9: Midpoint displacements and rotations for e=1, and different constant acceleration.

The jerk can be valid for a sudden acceleration increase takes place in braking or accelerating due to the applied force chances over time. Starting with zero acceleration at different jerks, displacements of the midpoint of the beam are illustrated in Figure 10. For the movement with little jerks, Figure 10 a, the displacements are not significantly changed but for higher jerks, Figure 10b, as it can be in projectile motion in a barrel it is observed that displacements are considerably increased.



a b

Figure 10: Midpoint displacements for e=1, and different constant Jerks.

When the mass moves with an acceleration due to an axial force is applied at the contact point of the mass, the beam vibrates in axial direction. Figure 11 shows the axial displacements of the midpoint for e=1, and case of different lower accelerations in (a), and case of different higher accelerations in (b). Higher accelerations are valid for projectile and barrel interaction of a weapon system like a tank that the exit velocity of the projectile is about 1750 m/s. One can calculate the average acceleration of the projectile that starts from zero velocity to a 1750 m/s exit velocity in a 6 m long barrel, the average acceleration is 2.55e5 m/s^2. For lower accelerations that can be applicable for CNC applications the midpoint displacements are increased by the increasing of the motion acceleration of the mass. The rate of displacement increase should be considered when the accuracy of the work is important.

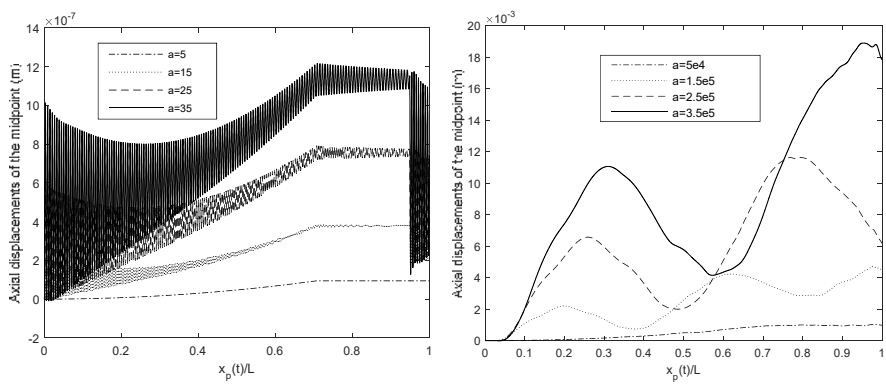


Figure 11: Axial displacements of midpoint for e=1, and different constant accelerations.

In terms of bridge engineering applications, the effect of axial forces may be significantly lower due to the motion acceleration of the vehicle, but this effect may be important for asphalt life. Another application of accelerated motion is rocket launching, where a massive mass can experience very high acceleration because of the ignition of a rocket.

4 CONCLUSIONS

In this study, the inertial, Coriolis and centripetal effects of the moving mass are determined in terms of variable acceleration and constant jerk. The classical beam element equations are modified from the terms that come from the moving mass. The variable acceleration and jerk of the mass are combined the accelerations and jerks of the beam element on which the mass locates at time t, at the variable contact point of the mass. This method, in which only the contact element is replaced and the other elements of the beam remain the same, is a very advantageous tool for calculating many motion patterns, including jerk effect. In the formulation of the time-dependent force equation of the contacted beam element, the effects of velocity, acceleration and jerk are separately determined in order to investigate separately the effects of them. Some interesting results of the different cases are presented in terms of the usage field of the moving load problem, including the possible application of the theory in design of runways of CNCs, vehicle-bridge-interaction, in design of the barrels and rocket launching structures under the effect higher acceleration and jerks, etc. The general approach in Bridge design, the DAF is criticized, in terms of omitting the mass inertia and accepting the moving mass as a moving load that accepts the only gravitational effect of the mass. It is shoved that, for lower velocity of travelling of the mass, this approach makes not very much difference, but for higher velocities and accelerations, including the inertial effect, the DAF can be three times greater depending on size of the velocity, acceleration, and jerk. One drawback of the proposed method is that it is not usable for hand calculations due to the modification of the system matrices in each time step of the integration.

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APPENDIX A

Derivation of Eqs. (6) and (9):

When the beam is in vibration, the longitudinal (x) force component, between the moving mass and the beam, induced by the vibration and curvature of the deflected beam is

$$f_x^t(x,t) = m_p\left(a^t - \frac{\mathrm{d}^2 w_x(x_p,t)}{\mathrm{d}t^2}\right)\delta(x-x_p) = -m_p\left(a^{t-\Delta t} + J\Delta t\right) + m_p \ddot{w}_x \delta(x-x_p), \tag{A.1}$$

The axial acceleration of the beam is derived from the second order total differentiation of its deflection function $w_x = w_x(x,t)$ with respect to time dependent contact point x_p as given below:

$$\begin{aligned}
\frac{\mathrm{d}w_x(x,t)}{\mathrm{d}t} &= \frac{\partial w_x(x,t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w_x(x,t)}{\partial t}\bigg|_{x=x_p} = v w'(x,t)_x + \dot{w}_x(x,t) \approx \dot{w}_x(x,t),\\
\frac{\mathrm{d}^2 w_x(x,t)}{\mathrm{d}t} &= \frac{\partial \dot{w}_x(x,t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \dot{w}_x(x,t)}{\partial t}\bigg|_{x=x_p} = v\dot{w}'_x(x,t) + \ddot{w}_x(x,t) \approx \ddot{w}_x(x,t),
\end{aligned} \tag{A.2}$$

When the beam is in vibration, thetransverse (z) force component, between the moving mass and the beam, induced by the vibration and curvature of the deflected beam is

$$f_z^t(x,t) = m_p\left[g - \left(\frac{\mathrm{d}^2 w_z(x_p,t)}{\mathrm{d}t^2}^{t-\Delta t} + \frac{\mathrm{d}^3 w_z(x_p,t)}{\mathrm{d}t^3}\Delta t\right)^t\right]\delta\left(x-x_p\right), \tag{A.3}$$

The vertical acceleration of the beam is derived from the second order total differentiation of its deflection function $w_z = w_z(x,t)$ with respect to time dependent contact point x_p and the jerk from its time derivative as given below:

$$\begin{aligned}
\frac{\mathrm{d}w_z(x,t)}{\mathrm{d}t} &= \frac{\partial w_z(x,t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w_z(x,t)}{\partial t}\bigg|_{x=x_p},\\
\frac{\mathrm{d}^2 w_z(x,t)}{\mathrm{d}t^2} &= \frac{\partial^2 w_z(x,t)}{\partial t^2} + 2\frac{\partial^2 w_z(x,t)}{\partial x\,\partial t}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial^2 w_z(x,t)}{\partial x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{\partial w_z(x,t)}{\partial x}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\bigg|_{x=x_p},\\
&= \ddot{w}_z + 2v\dot{w}'_z + v^2 w''_z + a w'_z,\\
\frac{\mathrm{d}^3 w_z(x,t)}{\mathrm{d}t^3} &= \frac{\partial^3 w_z}{\partial x^3}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + 3\frac{\partial^3 w_z}{\partial x^2\partial t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 3\frac{\partial^2 w_z}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 3\frac{\partial^3 w_z}{\partial t^2\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\\
&\quad + \frac{\partial^2 w_z}{\partial x\,\partial t}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\partial w_z}{\partial x}\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + \frac{\partial^3 w_z}{\partial x^3}\bigg|_{x=x_p},\\
&= w'''_z v^3 + 3\dot{w}''_z v^2 + 3w''_z va + 3\ddot{w}'_z v + \dot{w}'_z a + J w'_z + \dddot{w}_z,\\
v &= \frac{\mathrm{d}x}{\mathrm{d}t}\bigg|_{x=x_p};\ a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\bigg|_{x=x_p};\ J = \frac{\mathrm{d}^3 x}{\mathrm{d}t^3}\bigg|_{x=x_p}
\end{aligned} \tag{A.4}$$

APPENDIX B

Lateral and transverse Force equation of the beam element s can be obtained as summing up all the forces including the accelerations and jerk terms of Eqs. (A.2 and A.4)

$$\begin{aligned} f_{si}^{t} &= \left[\int_0^l m(x)\phi_i(x)\phi_j(x)dx\right]\left\{\ddot{u}_{si}\right\} \\ &+\left[\psi\int_0^l m(x)\phi_i(x)\phi_j(x)dx-\tau\int_0^l EA\phi_i'(x)\phi_j'(x)dx\right]\left\{\dot{u}_{si}\right\} \\ &+\left[-\int_0^l EA\phi_i'(x)\phi_j'(x)dx\right]\left\{u_{si}\right\} \\ &+\phi_i m_p\left(a^{t-\Delta t}+J\Delta t\right)-\phi_i m_p\frac{\partial^2}{\partial t^2}\left(\sum_j \phi_j u_{sj}\right)^{t}, \\ &(i,j=1,\ 4), \end{aligned} \tag{B.1a}$$

$$\begin{aligned} f_{si}^{t} &= \left[\int_0^l m(x)\phi_i(x)\phi_j(x)dx\right]\left\{\ddot{u}_{si}\right\}+\left[\psi\int_0^l m(x)\phi_i(x)\phi_j(x)dx-\tau\int_0^l EI(x)\phi_i''(x)\phi_j''(x)dx\right]\left\{\dot{u}_{si}\right\} \\ &+\left[\int_0^l EI(x)\phi_i''(x)\phi_j''(x)dx\right]\left\{u_{si}\right\}+\phi_i m_p\frac{\partial^2}{\partial t^2}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}+2v\phi_i m_p\frac{\partial^2}{\partial x\partial t}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t} \\ &+v^2\phi_i m_p\frac{\partial^2}{\partial x^2}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}+a\phi_i m_p\frac{\partial}{\partial x}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}+v^3\phi_i m_p\frac{\partial^3}{\partial x^3}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\Delta t \\ &+3v^2\phi_i m_p\frac{\partial^3}{\partial x^2\partial t}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\Delta t+3va\phi_i m_p\frac{\partial^2}{\partial x^2}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\Delta t \\ &+3vN_i m_p\frac{\partial^3 w_z}{\partial t^2\partial x}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\Delta t+a\phi_i m_p\frac{\partial^2 w_z}{\partial x\ \partial t}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\Delta t \\ &+J\phi_i m_p\frac{\partial w_z}{\partial x}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\Delta t+\phi_i m_p\left[\frac{\partial^2}{\partial t^2}\left(\sum_i \phi_i u_{si}\right)^{t}-\frac{\partial^2}{\partial t^2}\left(\sum_j \phi_j u_{sj}\right)^{t-\Delta t}\right]-\left(\phi_i m_p g\right)^{t} \\ &(i,j=2,\ 3,\ 5,\ 6), \end{aligned} \tag{B.1b}$$

In Eq. (B.1b), the first three terms are for a two-node beam element that has three DOF at each nodal point with the axial, transverse displacements and slope. They represent the mass, damping and stiffness property matrices of the beam element; can be found in [30]. The other terms stand for the interaction with the moving mass with a constant jerk. In such a case, the equivalent force equation for the beam element that includes only these terms can be obtained as below:

$$
\begin{aligned}
f_{si}^{t} = &\begin{pmatrix} m_p\phi_i\begin{bmatrix}\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6\end{bmatrix}^{t-\Delta t} \\ +\phi_i m_p \Delta t\begin{bmatrix}0 & 3v\phi_2'+\Gamma_2 & 3v\phi_3'+\Gamma_3 & 0 & 3v\phi_5'+\Gamma_5 & 3v\phi_6'+\Gamma_6\end{bmatrix}^{t}\end{pmatrix} \\
&\begin{Bmatrix}\ddot{u}_{s1} & \ddot{u}_{s2} & \ddot{u}_{s3} & \ddot{u}_{s4} & \ddot{u}_{s5} & \ddot{u}_{s6}\end{Bmatrix}^{T} \\
+&\begin{pmatrix} 2vm_p\phi_i\begin{bmatrix}0 & \phi_2' & \phi_3' & 0 & \phi_5' & \phi_6'\end{bmatrix}^{t-\Delta t} \\ +\phi_i m_p \Delta t\begin{bmatrix}0 & 3v^2\phi_2''+a\phi_2' & 3v^2\phi_3''+a\phi_3' & 0 & 3v^2\phi_5''+a\phi_5' & 3v^2\phi_6''+a\phi_6'\end{bmatrix}^{t}\end{pmatrix} \\
&\begin{Bmatrix}\dot{u}_{s1} & \dot{u}_{s2} & \dot{u}_{s3} & \dot{u}_{s4} & \dot{u}_{s5} & \dot{u}_{s6}\end{Bmatrix}^{T} \\
+&\begin{pmatrix} m_p\phi_i\begin{bmatrix}0 & v^2\phi_2''+a\phi_2' & v^2\phi_3''+a\phi_2' & 0 & v^2\phi_5''+a\phi_5' & v^2\phi_6''+a\phi_6'\end{bmatrix}^{t-\Delta t} \\ +\phi_i m_p \Delta t\begin{bmatrix}0 & \Gamma_7 & \Gamma_8 & 0 & \Gamma_9 & \Gamma_{10}\end{bmatrix}^{t} \mp N(x,t)\phi_i'\begin{bmatrix}0 & \phi_2' & \phi_3' & 0 & \phi_5' & \phi_6'\end{bmatrix}^{t}\end{pmatrix} \\
&\begin{Bmatrix}u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5} & u_{s6}\end{Bmatrix}^{T} \\
&-\Pi \\
&(i=1,\,2,\,3,\,4,\,5,\,6), \\
&\Pi = \left(\phi_i m_p g\right)^{t}\ (i=2,\,3,\,5,\,6), \\
&\Pi = \phi_i m_p\left(a^{t-\Delta t}+J\Delta t\right)(i=1,\ 4), \\
&\Gamma_2 = \frac{1}{\Delta t}\left[\left(\phi_2\right)^{t}-\left(\phi_2\right)^{t-\Delta t}\right];\Gamma_3 = \frac{1}{\Delta t}\left[\left(\phi_3\right)^{t}-\left(\phi_3\right)^{t-\Delta t}\right]; \\
&\Gamma_5 = \frac{1}{\Delta t}\left[\left(\phi_5\right)^{t}-\left(\phi_5\right)^{t-\Delta t}\right];\Gamma_6 = \frac{1}{\Delta t}\left[\left(\phi_6\right)^{t}-\left(\phi_6\right)^{t-\Delta t}\right]; \\
&\Gamma_7 = v^3\phi_2'''+3va\phi_2''+J\phi_2' \\
&\Gamma_8 = v^3\phi_3'''+3va\phi_3''+J\phi_3' \\
&\Gamma_9 = v^3\phi_5'''+3va\phi_5''+J\phi_5' \\
&\Gamma_{10} = v^3\phi_6'''+3va\phi_6''+J\phi_6' \\
&N(x,t) = m_p a,
\end{aligned}
\tag{B.2}
$$

When the Eq. (B.2) is arranged in the matrix form, the following elemental equation is obtained with the property matrices given in Eqs. (B.4 – B10).

$$
\begin{aligned}
\mathbf{f}^{t} &= \bar{\mathbf{m}}^{t}\ddot{\mathbf{u}}^{t}+\bar{\mathbf{c}}^{t}\dot{\mathbf{u}}^{t}+\bar{\mathbf{k}}^{t}\mathbf{u}^{t}, \\
\bar{\mathbf{m}}^{\mathbf{t}} &= \mathbf{m}+\mathbf{m}_{\ddot{\mathbf{w}}}{}^{t-\Delta t}+\mathbf{m}_{\ddot{\mathbf{w}}}{}^{t-\Delta t}\Delta t, \\
\bar{\mathbf{c}}^{\mathbf{t}} &= \mathbf{c}+\mathbf{c}_{\ddot{\mathbf{w}}}{}^{t-\Delta t}+\mathbf{c}_{\ddot{\mathbf{w}}}{}^{t-\Delta t}\Delta t, \\
\bar{\mathbf{k}}^{t} &= \mathbf{k}+\mathbf{k}_{G}+\mathbf{k}_{\ddot{\mathbf{w}}}{}^{t-\Delta t}+\mathbf{k}_{\ddot{\mathbf{w}}}{}^{t-\Delta t}\Delta t, \\
\mathbf{f}^{t} &= \begin{bmatrix}f_{s1} & f_{s2} & f_{s3} & f_{s4} & f_{s5} & f_{s6}\end{bmatrix}^{T}, \\
\ddot{\mathbf{u}}^{t} &= \begin{bmatrix}\ddot{u}_{s1} & \ddot{u}_{s2} & \ddot{u}_{s3} & \ddot{u}_{s4} & \ddot{u}_{s5} & \ddot{u}_{s6}\end{bmatrix}^{T}, \\
\dot{\mathbf{u}}^{t} &= \begin{bmatrix}\dot{u}_{s1} & \dot{u}_{s2} & \dot{u}_{s3} & \dot{u}_{s4} & \dot{u}_{s5} & \dot{u}_{s6}\end{bmatrix}^{T}, \\
\mathbf{u}^{t} &= \begin{bmatrix}u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5} & u_{s6}\end{bmatrix}^{T},
\end{aligned}
\tag{B.3}
$$

$$\mathbf{m}_{\ddot{\mathbf{w}}} = m_p \begin{bmatrix} \phi_1^{\,2} & 0 & 0 & \phi_1\phi_4 & 0 & 0 \\ 0 & \phi_2^{\,2} & \phi_2\phi_3 & 0 & \phi_2\phi_5 & \phi_2\phi_6 \\ 0 & \phi_3\phi_2 & \phi_3^{\,2} & 0 & \phi_3\phi_5 & \phi_3\phi_6 \\ \phi_4\phi_1 & 0 & 0 & \phi_4^{\,2} & 0 & 0 \\ 0 & \phi_5\phi_2 & \phi_5\phi_3 & 0 & \phi_5^{\,2} & \phi_5\phi_6 \\ 0 & \phi_6\phi_2 & \phi_6\phi_3 & 0 & \phi_6\phi_5 & \phi_6^{\,2} \end{bmatrix}, \tag{B.4}$$

$$\mathbf{c}_{\dot{\mathbf{w}}} = 2m_p v \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_2\phi_2' & \phi_2\phi_3' & 0 & \phi_2\phi_5' & \phi_2\phi_6' \\ 0 & \phi_3\phi_2' & \phi_3\phi_3' & 0 & \phi_3\phi_5' & \phi_3\phi_6' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_5\phi_2' & \phi_5\phi_3' & 0 & \phi_5\phi_5' & \phi_5\phi_6' \\ 0 & \phi_6\phi_2' & \phi_6\phi_3' & 0 & \phi_6\phi_5' & \phi_6\phi_6' \end{bmatrix}, \tag{B.5}$$

$$\mathbf{k}_{\ddot{\mathbf{w}}} = m_p \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v^2\phi_2\phi_2'' + a\phi_2\phi_2' & v^2\phi_2\phi_3'' + a\phi_2\phi_3' & 0 & v^2\phi_2\phi_5'' + a\phi_2\phi_5' & v^2\phi_2\phi_6'' + a\phi_2\phi_6' \\ 0 & v^2\phi_3\phi_2'' + a\phi_3\phi_2' & v^2\phi_3\phi_3'' + a\phi_3\phi_3' & 0 & v^2\phi_3\phi_5'' + a\phi_3\phi_5' & v^2\phi_3\phi_6'' + a\phi_3\phi_6' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v^2\phi_5\phi_2'' + a\phi_5\phi_2' & v^2\phi_5\phi_3'' + a\phi_5\phi_3' & 0 & v^2\phi_5\phi_5'' + a\phi_5\phi_5' & v^2\phi_5\phi_6'' + a\phi_5\phi_6' \\ 0 & v^2\phi_6\phi_2'' + a\phi_6\phi_2' & v^2\phi_6\phi_3'' + a\phi_6\phi_3' & 0 & v^2\phi_6\phi_5'' + a\phi_6\phi_5' & v^2\phi_6\phi_6'' + a\phi_6\phi_6' \end{bmatrix}, \tag{B.6}$$

$$\mathbf{m}_{\ddot{\dot{\mathbf{w}}}} = m_p \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & m_{23} & 0 & m_{25} & m_{26} \\ 0 & m_{32} & m_{33} & 0 & m_{35} & m_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{52} & m_{53} & 0 & m_{55} & m_{56} \\ 0 & m_{62} & m_{63} & 0 & m_{65} & m_{66} \end{bmatrix}, m_{ij} = \left(3v\phi_i\phi_j'\right)^t + \frac{1}{\Delta t}\left[\left(\phi_i\phi_j\right)^t - \left(\phi_i\phi_j\right)^{t-\Delta t}\right], \tag{B.7}$$

$$\mathbf{c}_{\ddot{\dot{\mathbf{w}}}} = m_p \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3v^2\phi_2\phi_2'' + a\phi_2\phi_2' & 3v^2\phi_2\phi_3'' + a\phi_2\phi_3' & 0 & 3v^2\phi_2\phi_5'' + a\phi_2\phi_5' & 3v^2\phi_2\phi_6'' + a\phi_2\phi_6' \\ 0 & 3v^2\phi_3\phi_2'' + a\phi_3\phi_2' & 3v^2\phi_3\phi_3'' + a\phi_3\phi_3' & 0 & 3v^2\phi_3\phi_5'' + a\phi_3\phi_5' & 3v^2\phi_3\phi_6'' + a\phi_3\phi_6' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3v^2\phi_5\phi_2'' + a\phi_5\phi_2' & 3v^2\phi_5\phi_3'' + a\phi_5\phi_3' & 0 & 3v^2\phi_5\phi_5'' + a\phi_5\phi_5' & 3v^2\phi_5\phi_6'' + a\phi_5\phi_6' \\ 0 & 3v^2\phi_6\phi_2'' + a\phi_6\phi_2' & 3v^2\phi_6\phi_3'' + a\phi_6\phi_3' & 0 & 3v^2\phi_6\phi_5'' + a\phi_6\phi_5' & 3v^2\phi_6\phi_6'' + a\phi_6\phi_6' \end{bmatrix}, \tag{B.8}$$

$$\mathbf{k}_{\ddot{\mathbf{w}}} = m_p \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ 0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\ 0 & k_{62} & k_{63} & 0 & k_{65} & k_{66} \end{bmatrix}, k_{ij} = v^3 \phi_i \phi_j''' + 3v^2 a \phi_i \phi_j'' + J \phi_i \phi_j', \tag{B.9}$$

$$\mathbf{k}_G = N(x,t) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_2'\phi_2' & \phi_2'\phi_3' & 0 & \phi_2'\phi_5' & \phi_2'\phi_6' \\ 0 & \phi_3'\phi_2' & \phi_3'\phi_3' & 0 & \phi_3'\phi_5' & \phi_3'\phi_6' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_5'\phi_2' & \phi_5'\phi_3' & 0 & \phi_5'\phi_5' & \phi_5'\phi_6' \\ 0 & \phi_6'\phi_2' & \phi_6'\phi_3' & 0 & \phi_6'\phi_5' & \phi_6'\phi_6' \end{bmatrix} \tag{B.10}$$