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# Mechanics of Cable-Suspended Beams

**Abstract**

The cable-suspended bridges differ from the elastic structures because of the inherent nonlinearity of the suspension cables. The primary focus of the available theories is to investigate the effect of nonlinearities associated with the distributed self-weight of the cable and its finite elastic displacements. The main point of departure of this paper is to study the effect of the configurational nonlinearity of the weightless linear elastic suspension cables undergoing small elastic displacements. Based upon authors' theory of weightless elasto-flexible planar sagging cables, a new Beam with Elasto-Flexible Support (BEFS) model of the cable-deck interaction is proposed here. Rate-type constitutive equations and third order differential equations of motion are derived for a simple four-node cable-suspended beam structure undergoing small elastic vertical displacements. Static and dynamic analysis of the cable-suspended structures is carried out to reveal their characteristic configurational response. The proposed theory is critically evaluated in the context of existing theories of suspension bridges.

**Keywords**

Suspension bridges, cable-deck interaction, configurational response, rate-type constitutive equations, third order differential equation of motion, flexural-torsional vibrations.

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## 1 INTRODUCTION

Because of their efficacy in resisting the applied forces, cable-supported bridges are the preferred choice for the construction of long span bridges. Their lightness, slenderness and flexibility renders these structures vulnerable to vibrations and dynamic instabilities when subjected to wind, earthquake and traffic loads. Assurance of adequate safety and satisfactory performance during their service life requires complex analysis. Cable-suspended bridges differ from the conventional elastic structures in that the cable configuration has also to be determined as part of the cable analysis. Of course, certain details like pylon heights, sag/span ratio of the suspension cables, spacing of hangers, etc. are decided a priori based upon the current design and construction practices (Gimsing and

Georgakis, 2012). While the deformed configuration under dead loads is thus specified, the undeformed configuration to achieve this target deformed configuration is still required to be determined (Kim and Lee, 2001). Theoretical prediction of structural behavior of cable-suspended bridges is based upon the assumed cable-deck interaction model which, in turn, is informed by the used theory of the suspension cables.

Extensive survey of the field has revealed that there is a considerable diversity in cable theories dedicated to different applications (Rega, 2004). Still certain aspects of dominant cable theories can be identified. Knowing that the cables lack unique natural passive state, their deformed equilibrium state under self-weight is generally employed to play the role of reference state. The equations of motion are stated in terms of additional dynamic forces and displacements from this reference state (Simpson, 1966; Irvine and Caughey, 1974; Nayfeh and Pai, 2004; Lacabonara, 2013; Greco et al., 2014). Analysis of flexible purely extensible sagging cables is complicated by the physical and geometric nonlinearities associated with their nonlinear tension-extension relations and finite elastic deformations (Lacarbonara, 2013). Reduction in cable diameter due to Poisson's effect (Nayfeh and Pai, 2004) and even exact finite deformation kinematics a la continuum mechanics (Luongo et al., 1984) are incorporated. Many a time, it is sufficient to use the linearized theory, e.g., while conducting, the modal analysis of the structure undergoing small displacements from the reference state (Irvine and Caughey, 1974).

Kinematical description is based upon the position vector as a function of the Lagrangian coordinate along the cable length and the assumed constitutive equation relates the local axial tension with the local stretch (Luongo et al., 1984, Antman, 2005; Lacarbonara, 2013). In the incremental or updated Lagrangian formulations, there exists a diversity of approaches for establishing the tangent stiffness of the cable. Generally, it is determined as the sum of tangent elastic and geometric stiffness matrices, the latter being defined as related to the change in components of the tensile force in the initial reference state (Lacarbonara, 2013). In the finite element formulations, the distributed self-weight of the cable is another source of nonlinearity in the nodal force-displacement relations. In this case, the relevant geometric stiffness matrix is estimated by using the stability functions from the theory of elastic beam-columns (Thai and Kim, 2008). In contrast, some researchers have preferred to obtain the tangent stiffness matrix by inverting the tangent flexibility matrix (Santos and Paulo, 2011; Coarita and Flores, 2015).

Classical deflection theory is popular amongst designers for linear analysis of suspension bridges. Here, the suspension cables are considered to be under the action of distributed hanger forces (Gimsing and Georgakis, 2012). It has been modified by incorporating the effects of nonlinearity associated with axial stretching and nonlinear curvature of the stiffening structure, though such effects are found to be relevant only at very high vibration amplitudes (Abdel- Ghaffar and Rubin, 1983a; 1983b). Other extensions include the spatial and flexural-torsional dynamic response, the axial and shear deformations and rotatory inertia of the deck, and stretch of the hangers (Lacarbonara, 2013). The structural response is governed by Melan fourth order nonlinear differential equations of equilibrium and motion stated in terms of additional static or dynamic loads, horizontal component of cable tension and deck displacements. The approximate linear version of the deflection theory is not valid for long span suspension bridges (West and Robinson, 1967; Gazzola et al., 2014). Parametric section models composed of deck section with a pair of hangers have been

used to simulate the cable deck interaction. Flexural-torsional deck vibrations are found to be coupled with vibrations of the suspension cables (Lepidi and Gattulli, 2014). In the cable-deck interaction models, the deformed configuration of the cable-suspended bridges under dead loads is generally employed as their reference configuration (Niazy et al, 1991; Lacarbonara, 2013; Lepidi and Gattulli, 2014). Tangent stiffness matrices are generally employed for carrying out the incremental analysis of suspension bridges (Altin et al., 2012; Lacarbonara, 2013). An elastic catenary element method for form finding and static analysis of cable structures with taut as well as slack cables is also proposed (Greco et al., 2014).

The elastic deformations and the resulting inclination of the hangers do not have much effect on the structural behavior of the cable-suspended structures (Abdel- Ghaffar, 1976). Free vertical and torsional vibrations have been found to be coupled. It has been found that the seismic vibration amplitudes of the cable-supported bridges in the lowest vibration modes of the cables and the torsional modes of the deck are negligible. Coupled flexural, torsional and lateral vibration response of suspension bridges is recently simulated by using exact spatial models of the suspension cables and the complete structure. Under the action of harmonic loading, suspension bridges are predicted to exhibit subharmonic resonance, multiple stable and unstable responses, etc. Aero-elastic phenomena like flutter and torsional divergences are also predicted under wind loading (Lacarbonara, 2013).

Recently, the authors have proposed a theory for the static and dynamic analysis of single weightless planar cables with point masses. Their approach is informed by the appreciation that weightless cables lack unique passive state configuration. Such cables are shown to exhibit characteristic configurational nonlinearity distinct from the conventional material and geometrical nonlinearities. The proposed theory involves rate-type constitutive equations and third order equations of motion. Pure and mixed configurational and elastic modes of vibration are predicted (Kumar et al., 2016a). These configurational and elastic vibration modes are shown to be predominantly associated with horizontal and vertical seismic excitations respectively (Kumar et al., 2015).

In this paper, the mechanical foundations of the static and dynamic response of the cable-suspended bridges are explored. A canonical problem of a single span bridge deck suspended from two weightless elasto-flexible planar cables by four vertical rigid hangers is formulated. The initial configuration of the cable-suspended structure undeformed in its passive state is specified. Using the authors' theory of weightless cables, a Beam with Elasto-Flexible Support (BEFS) model is proposed for simulating the cable-deck interaction. Assuming small elastic displacements of the deck and the cable, rate-type constitutive equations and third order equations of motion are derived. The typical elasto-configurational nature of static and dynamic response of the cable-suspended structures is predicted. The approach followed and structural response predicted by the proposed theory of cable-suspended structures are discussed.

## 2 BEAM WITH ELASTO-FLEXIBLE SUPPORT (BEFS) MODEL

A rather simple suspension bridge consist of simply supported weightless deck ADEH suspended from two identical elasto-flexible cables PQRS and P′Q′R′S′ at four nodal points B, C, G, and F (Figure 1) is considered. For the convenience of stating the relevant equations in the matrix form, the nodal points B, C, G, and F on the deck are also identified as nodes 1, 2, 3 and 4 respectively. The corresponding nodal points on the suspension cables are Q, R, Q′, and R′ respectively. The bridge

deck can be considered as a beam with elasto-flexible support. In its passive state, the deck, weightless vertical rigid hangers (QB, RC, Q′G and R′F) and weightless planar cables are stress-free and undeformed.

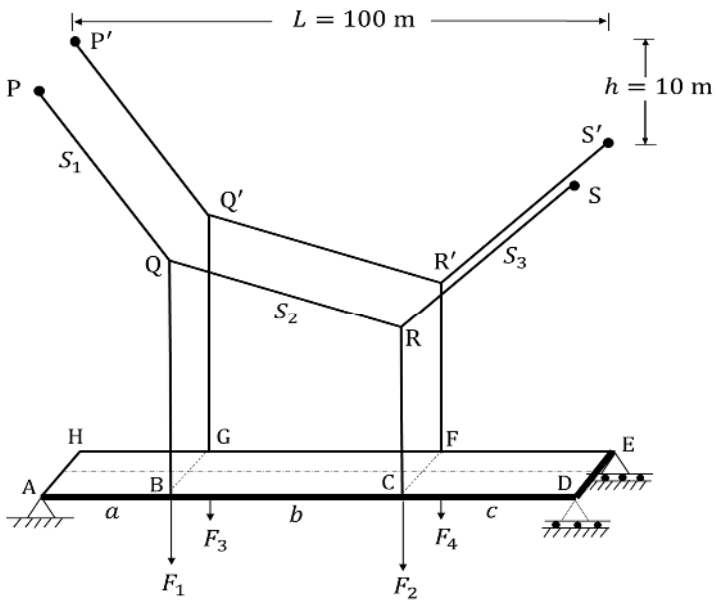



**Figure 1**: Geometrical details of cable-suspended structure.

Under the action of vertical downward nodal forces $(F_1, F_2, F_3, F_4)$, the linear elastic deck undergoes small vertical displacements $(v_1, v_2, v_3, v_4)$. The hangers carry a part of the applied loads as tensile forces $(Q_1, Q_2, Q_3, Q_4)$. The remaining part $(F_{d1}, F_{d2}, F_{d3}, F_{d4})$ of the applied loads is resisted by the deck and transferred to the supports. The point masses are located at the nodes 1, 2, 3 and 4 only. Obviously, each cable is under the action of two vertically downward forces applied through the vertical hangers.

The response of one of these cables PQRS in the BEFS model as shown in Figure 2, is described here. This description is based upon Authors' theory of weightless sagging planar cables (Kumar et al., 2016a). In its passive state, the cable configuration is defined by the coordinates of cable nodes $\mathrm{Q}(x_1, x_2)$ and $\mathrm{R}(x_3, x_4)$ assuming cable as inextensible. Upon loading, the cable is assumed to undergo small nodal elastic displacements $(u_1, u_2)$ and $(u_3, u_4)$ respectively in reference to its equilibrium configuration of inextensible cable. The nodal coordinates $(x)$ are unique functions of the nodal force $(Q)$ while the converse is not true. The following rate-type constitutive equations apply:

$$\begin{aligned} \dot{x} &= D\,\dot{Q} \\ \dot{u} &= f\,\dot{Q} \end{aligned} \tag{1}$$

Here, $D$ and $f$ are tangent configurational and elastic compliance matrices of the cable relating only the vertical nodal configuratioal and elastic velocities ($\dot{x}$ and $\dot{u}$) of the cable with vertical loading rates respectively. The matrix $D$ is singular having only one non-vanishing eigenvalue. The nod-

al coordinates $(x)$, of suspension cables , its small elastic displacements $(u)$, coefficients $D$ and $f$ of compliance matrices are functions homogeneous of order zero, 1, -1 and zero of force $Q_1$ and $Q_2$ respectively.

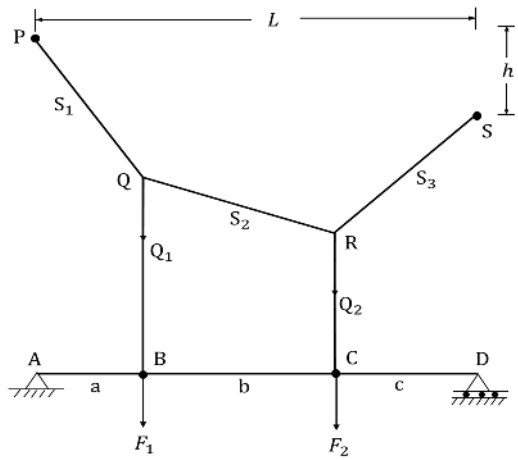



**Figure 2**: BEFS model.

To recapitulate, in its passive state, the cable-suspended structure is stress-free and undeformed. The cable configuration in this state is defined by its nodal coordinates $(x_0)$. Upon application of nodal loads $(F_1, F_2)$ on the deck, both the cable and the deck undergo vertical elastic displacements. Additionally, the cable configuration changes as evidenced by changes in its nodal coordinates $(x)$. Thus, the cable undergoes horizontal as well as vertical configurational and elastic displacements. However, here only its vertical nodal displacements in the direction of vertical nodal loads are being considered. The compatibility condition relating the nodal displacements of the cable and the deck is stated as

$$\begin{aligned} (x - x_0) + u &= v \\ \dot{x} + \dot{u} &= \dot{v} \end{aligned} \tag{2}$$

Here, $v$ represents the nodal elastic displacements of the deck. The total displacements of the cable node, Q and R, of the hangers are composed of two components, viz., the configurational displacements $(x - x_0)$ from the natural state $(x_0)$ of the cable and the elastic displacements $(u)$ from its instantaneous natural state $(x)$. Using Eq. 1 and 2, the constitutive equation at the cable nodes takes the form as

$$\begin{aligned} \dot{x} + \dot{u} = (D + f)\,\dot{Q} = N\dot{Q}\dot{x} + \dot{u} = \dot{v} \\ \dot{v} = N\dot{Q} \\ \dot{Q} = k_c\,\dot{v} \end{aligned} \tag{3}$$

Here, $k_c = N^{-1}$ represents the tangent elasto-flexible stiffness matrix of the cables. The above analysis applies to the other cable as well. The tangent compliance matrices $(N)$ for the two parallel

cables in the total structure are identical but uncoupled. To be specific, the coefficients of the matrix $N$ representing the coupling between the nodes Q and R of one cable and nodes Q′ and R′ of the second cable vanish.

The deck assumed as linear elastic, its stiffness matrix ($k_d = {f_d}^{-1}$) plays the role of both the secant and tangent stiffness matrices. As a matter of fact, the flexibility matrix ($f_d$) of the beam is determined by applying the classical theory of linear elastic structures. From skeleton structure shown in Figure 3, the deck slab is modelled as a beam with flexural and torsional rigidity values $EI$ and $GJ$ respectively. Under the action of nodal forces ($F_{d1}, F_{d2}, F_{d3}, F_{d4}$) each applied at eccentricity $e$ equal to half the deck width, the deck undergoes flexural and torsional nodal displacements.

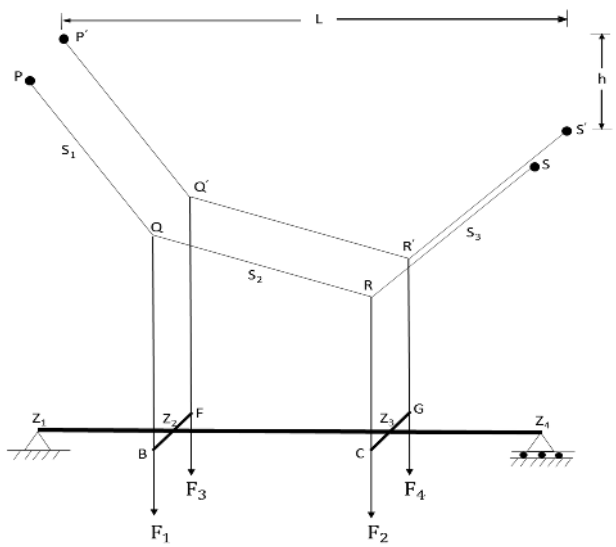



**Figure 3**: Modelling of Cable-suspended structures.

At both the ends of the beam, the supports are capable of mobilizing torque, but not any flexural moment. The constitutive equations relating the nodal loading and their time rates with nodal displacements and their time rates are stated as:

$$\begin{aligned} F_d &= k_d\, v \\ \dot{F}_d &= k_d\, \dot{v} \\ \dot{Q} + \dot{F}_d &= k_c\, \dot{v} + k_d\, \dot{v} = (k_c + k_d)\, \dot{v} = K\, \dot{v} \end{aligned} \tag{4}$$

From dynamic equilibrium equation for the damped structure derived from Newton's law of motion stated below, yields the value of the hanger forces:

$$\begin{aligned} Q &= F(t) - C\, \dot{v} - M\, \ddot{v} - F_d \\ F(t) &= F_0 + F_L\, sin\, \omega t \end{aligned} \tag{5}$$

Here, $F(t)$ represent the applied nodal forces with constant force component $F_0$ and harmonic component $F_L$. This equation of motion cannot be stated in the familiar form as $Q \neq k_c v$, but only as $\dot{Q} = k_c\, \dot{v}$. Since the cables are known to possess little damping, the constant classical damping

matrix is considered to be determined by only the initial stiffness matrix $K_0$ of the structure and the nodal mass matrix as

$$C = a_0\, M + a_1 K_0 \tag{6}$$

Thus, using the constitutive equations the following third order differential equation of motion can be stated for the damped cable-suspended deck:

$$M\, \ddot{v} + C\, \ddot{v} + K\, \dot{v} = \dot{F} \tag{7}$$

For specified initial nodal displacements $v(0) = \bar{v}$ and velocities $\dot{v}(0)$, the nodal initial accelerations are expressed below. Here, initial forces $\bar{F}$ correspond to the initial nodal displacements $\bar{v}$.

$$\ddot{v}(0) = M^{-1}[F(0) - C\dot{v}(0) - \bar{F}] \tag{8}$$

At any instant the condition of dynamic equilibrium (Eq. 5.7) is satisfied. The hanger forces $(Q)$ yield $x$ and $y$ satisfy the compatibility condition as

$$v(t) = x(t) - x(0) + u(t) \tag{9}$$

## 3 STATIC ANALYSIS

Static analysis involves the determination of the structural response to applied nodal loads. The analysis cannot be started from the natural state because, in the absence of the hanger forces, the tangent configurational and elastic compliance matrices of the suspension cables cannot be defined. However, the internal forces as well as the nodal elastic displacements introduced by dead loads are already known. Thus, the changes in the internal forces and nodal displacements introduced by additional nodal loads $(\Delta F = F - F_0)$ are to be determined. As per the proposed BEFS Model, only rate-type constitutive equations are valid. For conducting quasi-static analysis of this cable-suspended structure, these rate-type constitutive equations are restated in the following incremental form as:

$$\begin{gathered} dx + du = dv \\ dQ = k_c\ dv \\ dF = K\, dv \end{gathered} \tag{10}$$

One set of such nodal forces $(F_0 = [3500; 4000; 3500; 4000]\ \text{kN})$ is identified here as being caused by dead load. The other structural details are as follows:

**Deck:** $L = 100$ m, $a = 25$ m, $b = 40$ m, $c = 35$ m,
$E_d = 31.6 \times 10^9 \text{N/m}^2$, $I = 0.6\ \text{m}^4$, $G = 13.7 \times 10^9 \text{N/m}^2$, $J = 1.13\ \text{m}^4$

**Cables:** $S_1 = 31.0754$ m, $S_2 = 53.3715$ m, $S_3 = 25.5531$ m, $h = 10$ m,
$A_0 = 0.01\ \text{m}^2$, $E_c = 2 \times 10^{11} \text{N/m}^2$

It can be observed that upon proportional variation of dead loads, internal forces and nodal elastic forces vary linearly. Thus, the elastic displacements ($v$ and $u$) of the deck and so the cable as

well as the internal forces ($Q$ and $F_d$) are determined corresponding to a particular set of dead load nodal forces $F_0$. Here, the nodal loads $F_0$ caused by the dead load of the structure corresponding to its specified cable configuration are determined. The converse problem, faced by the bridge designers, the requirements of establishing the configuration of the suspension cables for known dead loads acting at specified locations on the deck is not addressed in this paper.

It should be appreciated that these incremental constitutive equations are valid only for infinitesimally small nodal load increments, but not for the finite load increments $\Delta F$. Such happens to be the case because the instantaneous tangent stiffness matrix $k_c$ and so $K$, is to be updated after application of each load increment. In view of this fact, the following method of static analysis is proposed: Consider that the additional nodal loads $\Delta F$ are applied in $n$ small load increments $dF$. Determine $dv = K^{-1}\, dF$, $dQ = k_c\, dv$ and $dF_d = k_d\, dv$. Then, determine $dx = D\, dQ$ and $du = f_c\, dQ$. Update the magnitudes of $x$, $u$, $v$, $Q$ and $F_d$. Using $Q$ and $x$, determine $D$, $f_c$, $N$, $k_c$ and $K$. These tangent stiffness matrices $k_c$ and $K$ are valid for the next small load increment $dF$. Proceeding in this manner, the structural response for any applied load set can be determined.

Here, only the nodal vertical coordinates ($x_0$) defining the cable configuration and the vertical elastic displacements of the cables ($u_0$) and the deck ($v_0$) are presented. Under the above specified dead loads and given structural parameters, the following equilibrium configuration and displacements are obtained:

$$x_0 = [18.4575; 25.5531; 18.4575; 25.5531]\ \text{m}$$
$$u_0 = v_0 = [0.3438; 0.4594;\ 0.3438; 0.4594]\ \text{m}$$

The structural response is determined in the form of some equilibrium state of the structure. The cable configuration ($x_0$) and the elastic displacements ($u_0$ and $v_0$) correspond to the equilibrium sustained nodal forces ($F_0$). The predicted dynamic response is presented in terms of configurational displacements of the cable ($x_e = x - x_0$), elastic displacements ($u_e = u - u_0$) of the cable and the elastic displacements ($v_e = v - v_0$) of the deck from this equilibrium state. Figure 4(a) shows the structural response when a specific set of nodal loads is applied along the proportional load path on the structure in its passive natural state. The configurational displacements experienced by the cable imply non-proportional variation of the hanger forces.

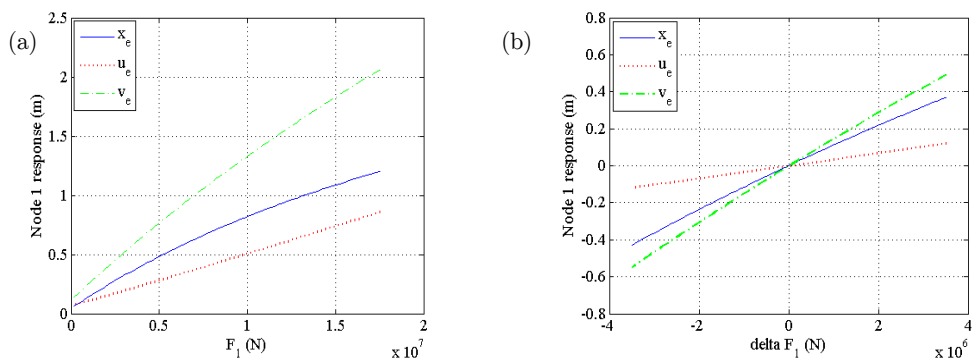



**Figure 4**: Node 1 response (a) general proportional loading and (b) proportional loading.

The consequent nonlinearity in the cable response is reflected in the nonlinear increase of its elastic response as well as of the total displacement at the same node with general proportional increase in applied loading. It has been verified through not presented here that, if the dead load is proportionately increased, no change in cable configuration occurs and the elastic displacements of the cable and the deck increases in a linear manner. Figure 4(b) represents the structural response to a particular case of non-proportional loading achieved by increasing equally nodal loads at nodes 1 and 3 keeping the other nodal loads constant. As observed by others as well (Lacarbonara, 2013), the nonlinear variation of deck displacement at node 1 implies hardening of the structure with increase in loads, while decrease of loads causes its softening. In all these cases, the configurational displacements $(x_e)$ of the suspension cables are of the same order as the elastic displacements.

## 4 DYNAMIC ANALYSIS

The dynamic response of the structure is determined using third order differential equation of motion expressed in Eq. 7. Runge-Kutta method is applied for solving third order differential equation of motion. Newmark-Beta method with constant rate of acceleration is also used in validation of Runge-Kutta method. Nodal response at the end of small discrete time interval is obtained. The tangential stiffnessmatrix is assumed to remain constant during this small time-step. The stiffness and damping matrices corresponding to the nodal forces $\bar{F}$ required to introduce the specified initial displacement $v(0)$ are relevant for the first time-step. Here, the cables are assumed to always be in tension. For the theoretical prediction presented in this paper, such has been verified to be the case. The damping ratio $(\xi)$ for the lowest two modes is considered as 0.05 for obtaining the damping matrix.

The dynamic response predicted at the end of each time-step includes nodal velocity $(\dot{v})$, acceleration $(\ddot{v})$ and triacceleration $(\dddot{v})$. The nodal velocities and the acceleration so predicated are related to magnitudes of the hanger forces $(Q)$ acting on the cables as per Eq. 5 of dynamic equilibrium. The configurational and the elastic displacements $(x, u)$ of the cables pertain to the upper ends of the hangers. Then, there is a compatibility from Eq. 2 has also to be satisfied. The two unknowns, $Q$ and $v$ can be obtained by solving these two nonlinear simultaneous equations. However, this approach is not followed here. Instead, using the Runge-Kutta method, nodal displacements $v(i)$ at the end of $i^{th}$ time step are directly calculated from solving third order equation of motion. Then, the hanger forces is determined using Eq. 5 which yields the configuration $(x)$ and elastic displacements $(u)$ of the cables. The tangent configurational and elastic flexibility matrices then determined and so the stiffness matrix is calculated at the end of each time step required for the next time step. Here, the compatibility equations relating $x$, $u$ and $v$ are not explicitly used, but it is verified for many cases that the compatibility conditions are indeed satisfied.

Under the dead load, the linear natural frequencies and the mode shapes corresponding to the antisymmetric flexural (ASF), symmetric flexural (SF), antisymmetric torsional (AST) and symmetric torsional (ST) modes respectively are presented in Table 1.

In the present case, linear modal analysis pertains to small amplitude vibrations of the corresponding undamped structure about its equilibrium state under dead loads. The equation of motion, Eq. 7 can be recast as an eigenvalue problem as $\ddot{v} - \omega^2 \dot{v} = 0$. Here, the eigenvalues of the matrix $M^{-1}K$ represent the squares of the linear natural modal frequencies $\omega$ of the structure. The corre-

sponding eigenvectors in the form of modal velocity vectors represent the respective mode shapes. In view of Eq. 7, these modal velocity vectors correspond to the respective modal rates of loading. When the structure under dead loads is perturbed by any of these modal velocity vectors, it executes 'free' vibrations about the equilibrium state at the corresponding linear natural modal frequency.

| Modes | Angular frequency (rad/s) | Mode shape (Eigenvectors) |
|---|---|---|
| ASF | 3.5290 | (-0.4964; 0.5036; -0.4964; 0.5036) |
| SF | 6.0402 | (0.5355; 0.4618; 0.5355; 0.4618) |
| AST | 6.8642 | (0.1766; -0.6847; -0.1766; 0.6847) |
| ST | 7.3666 | (0.6898; 0.1556; -0.6898; -0.1556) |

**Table 1**: Modes of vibration.

The structure can be set into such modal vibrations by perturbing it by initial displacement vector proportional to the modal velocity vectors. The structure can be forces into modal vibrations by subjecting it to harmonic loading with peak sinusoidal force vector $(F_L)$ proportional to the chosen modal velocity vector. It needs to be understood that, in the symmetric flexural and torsional modes, the cables experience purely elastic displacements, while in the anti- symmetric flexural and torsional modes are characterized by purely configurational displacements. The natural frequencies of the deck alone for the anti-symmetric flexural and torsional modes (3.36 rad/s and 6.86 rad/s) can be observed to be approximately equal to those $\sqrt{M^{-1}k_d}$ of the cable supported structure. In contrast, the absence of the cable results in considerable decrease in the symmetric flexural and torsional frequencies (0.96 rad/s and 4.35 rad/s) of the structure.

Obviously, the cables contribute to the elastic resistance of the structure only in the symmetric modes of deformation and vibration. When the structure deforms or vibrates in the anti- symmetric modes, the contribution of the cables to the elastic resistance of the cable-suspended structure is minimal. Further, it has been estimated that about 96% of the dead load is resisted by the cables. It must be realized that the structural response of the cable-suspended structure under dead load corresponds to its symmetric flexural mode. Thus, in this purely elastic mode, the contribution of the cables to the elastic stiffness matrix of the cable-suspended structure is many times than that of the deck. Still, cables do not have much effect on the anti- symmetric configurational modes of the structure.

Typical steady state forced wave-forms for the anti-symmetric and symmetric flexural modes are presented respectively in Figure 5(a) and Figure 5(b). The anti- symmetric flexural response at node 1 can be observed to be mainly configurational with little elastic response of the cable. In contrast, the symmetric flexural response at the same node is mainly associated with elastic displacements of the cable and with little change in cable configuration. Even though the peak amplitude in the anti-symmetric flexural mode is higher, the cable experiences smaller tensile forces in this case as shown in Figure 5(c) for cable segment PQ.

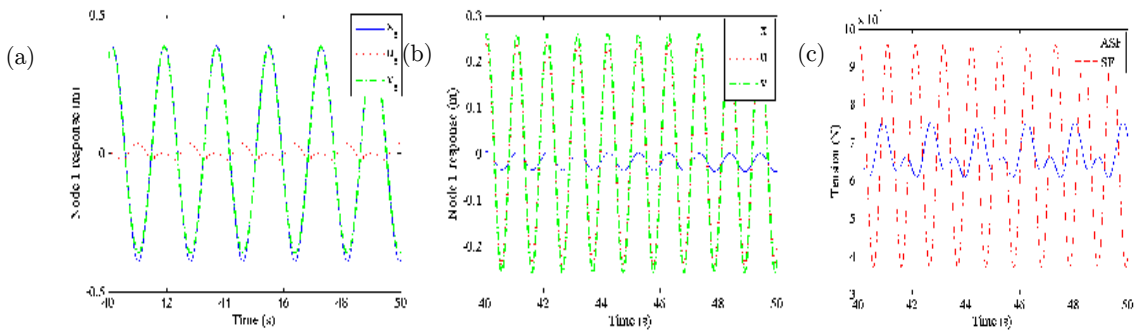
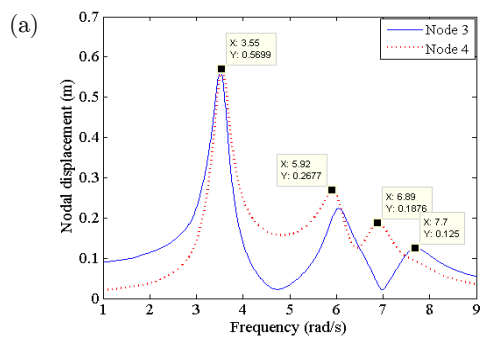
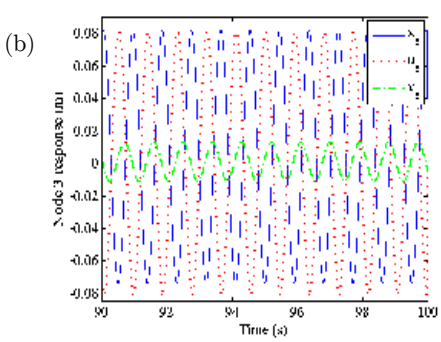



**Figure 5**: (a) Anti- symmetric modal response, (b) symmetric flexural modal response, and (c) modal tensile forces.

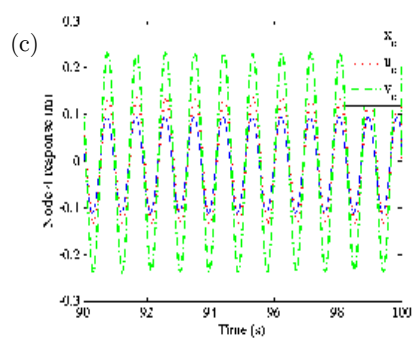
Frequency domain response (FDR) under certain general sinusoidal loading as depicted in Figure 6(a) exhibits a peculiar phenomenon. The node 3 does not register peak response corresponding to anti- symmetric response at forcing frequency of 6.89 rad/s. This phenomenon can be explained by comparing the response waveforms for nodes 3 and 4 as depicted in Figure 6(b) and Figure 6(c) respectively at the same frequency. The configurational and elastic vibration amplitudes of the cable can be observed to be out of phase at node 3 while they are in phase at node 4. Because of the lower predicted displacements of the deck at the former node, no resonance peak is observed. As the predicted response at the latter node is higher, a resonance peak is observed. Similar explanation holds for the absence of the resonance peak for node 4 at frequency of 7.7 rad/s.




(a) Typical FDR




(b) Response components of cable node 3




(c) Response components of cable node 4

**Figure 6**: (a) Typical FDR of and nodal waveforms at forcing frequency 6.89 rad/s.

In another case of loading ($F_0 = [7000; 8000; 0; 8000]$ kN), the highest two linear modal frequencies (ASF: 3.6680, SF: 6.0231, ST: 7.0386, AST: 7.5322) rad/s turned out to be almost equal. Under harmonic loading ($F_L$: [227.8; -659.2; 195.1; 689.8]) kN corresponding to the anti- symmetric torsion (AST) mode but at the symmetric torsion linear modal frequency, this structure with much lesser damping is predicted to exhibit transient beating phenomenon. Like the elastic displacements of the deck, the elastic and configurational displacements are also observed from Figure 7(a) to exhibit similar beating phenomenon. Fast Fourier Transform (FFT) plot depicted in Figure 7(b), the beat frequency (0.5 rad/s) equals the difference between the adjacent linear modal frequencies. The harmonic response frequency gradually varies from the lower to the higher of the adjacent frequencies as confirmed by the corresponding peaks appearing in the FFT plot.

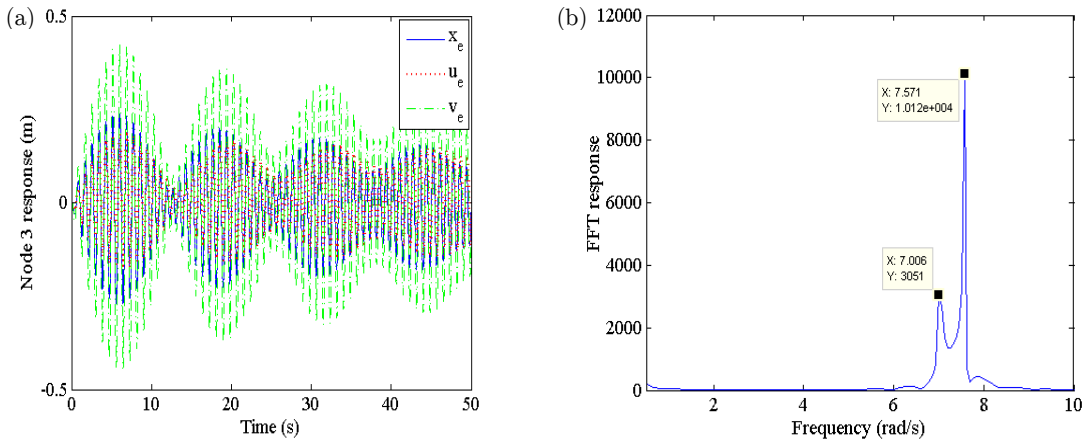



**Figure 7**: (a) Typical beating phenomenon and (b) FFT plot of cable node 3.

Inextensible cables can be modelled as elastic cables with very high axial elastic stiffness. Such an inextensible cable-suspended structure will have different configuration under the same dead loads. Its modal frequencies of a particular structure under dead loads are obtained as (3.55, 7.11, 187.71, 187.75) rad/s. The largest two frequencies in fact correspond approximately to the 'expected infinitely high frequencies' for the symmetric flexural and torsional modes of structure with inextensible cables. It has been verified that the lowest two frequencies indeed pertain to the anti- symmetric flexural and torsional modes. The frequency domain response of such a structure subjected to a particular harmonic loading shown in Figure 8(a) exhibits resonance peaks corresponding to these anti- symmetric flexural and torsional modes. The waveform depicted in Figure 8(b) confirms the expectation that the structural response is purely configurational with the elastic response being vanishingly small.

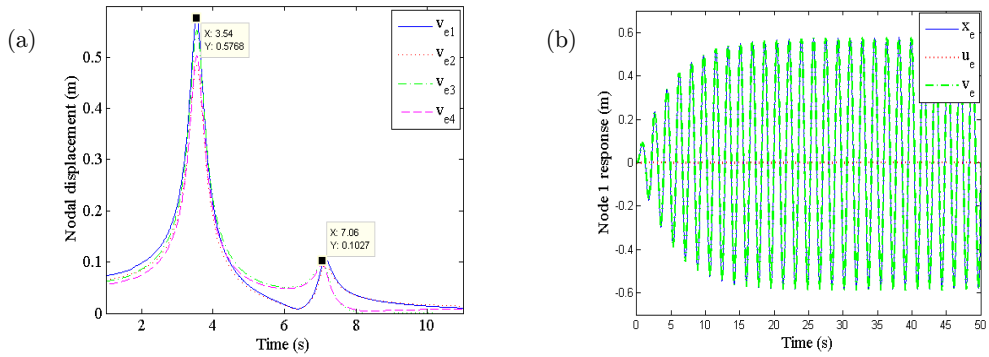



**Figure 8**: (a) Typical FDR and (b) response components for cable node 1 (at 3.54 rad/s) for general loading.

## 5 DISCUSSION

In this paper, a new theory of cable-suspended structures with weightless cables and linear elastic deck both undergoing small elastic displacements is proposed. Also, the cables are assumed to be linear elastic in tension. Thus, the usually investigated sources of nonlinearity, viz., the effect of self-weight, physical nonlinearity and finite elastic displacements of the cable on its stiffness, are absent in the present case. The nonlinearity of such cable-suspended structures is caused only by the dependence of natural state of the suspension cables upon the applied nodal loads. Changes in the configurational-defining coordinates of the cable nodes with non-proportional variations in applied loads are called configurational displacements. Based upon authors' theory of weightless sagging planar cables, a new Beam with Elasto-Flexible Support (BEFS) model is proposed for simulating the cable-deck interaction. Its main point of departure is the distinction between the elastic and configurational components of the suspension cable response (Kumar et al., 2016a).

As discussed in Authors' earlier paper, the weightless elastic cables lack unique passive state but their equilibrium state under specified loads is unique. Also, for each equilibrium state, there corresponds a unique natural state attained by proportional unloading. Also, this natural state is identical with the equilibrium state $(x)$ of the corresponding inextensible cable. The nodal elastic displacements $(u)$ are measured relative to this natural state (Kumar et al., 2016a). In contrast to single sagging cables, the cable-suspended structures possess a unique natural passive state. It is this natural state which plays the role of reference state for defining the configurational displacements of the cables and the elastic displacements of the deck, while the elastic displacements of the suspension cables are measured form their respective natural states under the hanger forces.

Modal response to smaller sinusoidal forces relative to sustained forces is predicted to be linear. Such linearity is confirmed here by the independence of the resonance frequencies of the magnitudes of the sinusoidal forces and by the proportionality between the resonance peak amplitudes with these peak sinusoidal forces. However, these structures exhibits their nonlinearity in the form of dependence of its linear natural frequencies on the sustained loads. Even the crossover phenomenon investigated earlier for sagging cables (Irvine and Caughey, 1974) is predicted here as well for the cable-suspended structures. Beating phenomenon is also predicted at forcing frequencies closer to adjacent natural frequencies.

Even though the configurational displacements of the cables are quite small, they are of the same order as the elastic displacements of the cables and the deck. It has also been shown that, in the antisymmetric flexural and torsional modes, the configurational response dominates the elastic response of the suspension cables. In some cases, or even at some nodes, these configurational and elastic amplitudes happen to be out of phase resulting in smaller deck response. To cite an extreme example, the configurational displacement (-0.3997 m) and elastic displacement (0.3997 m) of the cable at point Q are of opposite sense and so the elastic displacement of the deck at node 2 becomes zero. It has also been demonstrated that higher nodal amplitudes of the deck do not always imply higher tensile forces in the suspension cables. Structures with inextensible suspension cables are predicted to exhibit only configurational vibrations in the antisymmetric flexural and torsional modes. Irrespective of their axial elastic stiffness, the suspension cables are observed not to affect the natural frequencies in these antisymmetric modal vibrations of the cable-suspended structures.

The methods intended for incorporating the effect of large displacements of cable-suspended bridges on their behavior are oblivious of the distinction between the configurational nonlinearity and the conventional geometric nonlinearity associated with the large elastic displacements. In fact, the cables under only one vertical load chosen in one case for illustrating the concept of tangent stiffness matrix is incapable of exhibiting the configurational displacements under (only proportional) load increments (Altin, et al., 2012). Of course, the kinematic description of the deformed state of the cable in terms of position vector as a function of the Lagrangian coordinate (Luongo, et al., 1984; Antman, 2005; Locarbonara, 2013; Greco et al., 2014) includes, perhaps unknowingly so, the configurational response in addition to the effect of finite elastic displacements.

To recapitulate, following authors' approach for the single sagging cables, rate-type constitutive equations and third order differential equations of motion are derived for the cable-suspended structure. Thus, an initial value problem distinct from the conventional initial value problem involving second order differential equation of motion has been formulated. In place of a single set of conventional coupled second order differential equations of motion, the governing equations for the cable-suspended structure is involve three sets of coupled equations: coupled third order differential equations of motion, the equations of dynamic equilibrium and the compatibility equations. Also, apart from the well-known initial nodal displacements and velocities, the third order equations of motion require the specification of the initial nodal accelerations. It is argued here that the required initial nodal accelerations are determined by specified initial nodal displacements and velocities as per Eq. 10. A simple canonical cable-suspended structure is selected for illustrating the main concepts and predictions of the proposed theory.

It is claimed that the proposed theory is the only one wherein the configurational response of the suspension cables is incorporated in an explicit manner. However, its scope is restricted to small vertical displacements of the deck suspended from only two physically-linear planar parallel weightless cables undergoing small elastic displacements. Its direct comparison with the existing theories cannot be undertaken because the complex cable-suspended bridges chosen for analysis by the contemporary researchers have not been analyzed here. The proposed theory can be extended by incorporating in it the physical and geometrical nonlinearity associated with finite elastic displacements, self-weight of the suspension cables of the structure capable of spatial displacements and vibrations. It is suggested that theoretical investigations be undertaken to delineate the aspects in which such

an extended theory is equivalent with the existing theories. Such an exercise is expected to lead to a better theory of cable-suspended structures. Only then will it make any sense to undertake its empirical validation and explore possible applications.

## 6 CONCLUSIONS

It has earlier been established by the authors that the static and dynamic response of the elasto-flexible sagging planar cables is composed of configurational and elastic components. Following the same approach, a new BEFS model involving rate-type constitutive equations and third order differential equations of motion for cable-suspended structures undergoing small elastic displacements is proposed in this paper. Static and dynamic analysis reveals considerable role played by the configurational response of the planar cables. Structural response to small additional loads is approximately linear, but linear modal frequencies are determined by sustained loads. In particular, the vibration response in the antisymmetric flexural and torsional modes is mainly contributed by the configurational response of the cables. Cable configuration is predicted to remain substantially invariable in the symmetric modes. When the configurational and elastic components of cable response happen to be out of phase, lesser vibration amplitudes of the deck are predicted but tensile forces in the suspension cables can still be higher. Structure with inextensible cables is found to exhibit only configurational response. The initial value problem posed and solved here for cable-suspended structures is distinct from the conventional formulations. Only further comparative analysis will establish the appropriateness of the proposed theory vis-a-vis the extant theories.

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