Latin American Journal of Solids and Structures
www.lajss.org

Approximate solutions of large amplitude vibration of a string

Abstract
In the present study, the response of a flexible string with large amplitude transverse vibration is studied utilizing amplitude-frequency formulation, improved amplitude-frequency formulation and max-min approach. In order to verify the accuracy of these approaches, obtained results are compared with other methods such as variational approach method, variational iteration method, coupling Newton's method with the harmonic balance method and Hamiltonian approach. It has been found that for this problem, while amplitude-frequency formulation and max-min approach give the same results, improved amplitude frequency formulation is not an appropriate choice.

Keywords
nonlinear vibration, string, amplitude-frequency formulation, improved amplitude-frequency formulation and max-min approach.

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http://dx.doi.org/10.1590/1679-78254122

Received: June 15, 2017
In Revised Form: November 20, 2017
Accepted: November 26, 2017
Available online: February 05, 2018

1 INTRODUCTION

Study of nonlinear vibration of strings with large amplitude has attracted the attention of many researches in the fields of physics and engineering. For instance, long cables used in cranes, ships and bridges might expose to forces that causes large amplitude vibrations (Omran et al., 2013). The nonlinear partial differential equation for transverse vibrations of a flexible string under constant tension is presented in equation (1)(Coulson and Jeffrey, 1997).

$$c^2 \frac{\partial^2 u}{\partial x^2} = [1 + (\frac{\partial u}{\partial x})^2]^2 \frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $c^2 = \tau_0 / \rho_0$, with τ_0 the tension and ρ_0 the mass per unit length. The boundary conditions for the string of length L, vibrating between two fixed end-points are presented in equation (2).

$$u(0,t) = u(L,t) = 0 \tag{2}$$

By considering the transverse vibration as $u(x,t) = V(t)\sin(\frac{\pi}{L}x)$, substituting it into equation (1) and averaging over the string length L, an ordinary second-order differential equation (3) is derived as presented in equations (3) - (6)(Gottlieb, 1990).

$$\frac{d^2 S}{dt^2} + \frac{\beta S}{1 + \frac{S^2}{2} + \frac{S^4}{8}} = 0 \tag{3}$$

where

$$S(t) = \frac{\pi}{L} V(t) \tag{4}$$

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$$\beta = (\frac{\pi c}{L})^2 \tag{5}$$

with initial conditions

$$S(0) = a\ ,\ \frac{dS(0)}{dt} = 0 \tag{6}$$

Many researches have been conducted to solve equation (3) with different methods such as variational approach method, variational iteration method, coupling Newton's method with the harmonic balance method and Hamiltonian approach. In the present study, this equation is solved using amplitude-frequency formulation, improved amplitude-frequency formulation and max-min approach. Furthermore, obtained results are compared with that of the mentioned methods.

2 METHODOLOGY

2.1 Amplitude-frequency formulation (AFF)

To illustrate this approach, a generalized nonlinear oscillator is considered as presented in equation (7)

$$u'' + f(u) = 0\ ,\ u(0) = A\ ,\ u'(0) = 0 \tag{7}$$

Used Trial functions are presented in equation (8) and (9).

$$u_1(t) = A\cos(\omega_1 t) \tag{8}$$

$$u_2(t) = A\cos(\omega_2 t) \tag{9}$$

The residuals are as shown in equations (10) and (11).

$$R_1(\omega t) = -A\omega_1^2\cos(\omega_1 t) + f(A\cos(\omega_1 t)) \tag{10}$$

$$R_2(\omega t) = -A\omega_2^2\cos(\omega_2 t) + f(A\cos(\omega_2 t)) \tag{11}$$

The original frequency-amplitude formulation is presented in equation (12).

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} \tag{12}$$

To find the frequency, He (2008a) used equation (13), Geng and Cai (2007) used another location point which is indicated in equation (14). By substituting ω from these equations into $u(t) = A\cos(\omega t)$ and considering $\cos(\omega_1 t) = \cos(\omega_2 t) = k$, the frequency is obtained.

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = 0) - \omega_2^2 R_1(\omega_1 t = 0)}{R_2 - R_1} \tag{13}$$

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = \pi/3) - \omega_2^2 R_1(\omega_1 t = \pi/3)}{R_2 - R_1} \tag{14}$$

To solve equation (3), $u_1(t) = A\cos t$ and $u_2(t) = A\cos 2t$ are selected as trial functions. Substituting these functions into equation (3), the residuals are computed as follows,

$$R_1 = -a\cos(t) + a\cos(t)\frac{\beta}{1 + \frac{a^2\cos^2(t)}{2} + \frac{a^4\cos^4(t)}{8}} \tag{15}$$

$$R_2 = -4a\cos(2t) + a\cos(2t)\frac{\beta}{1+\frac{a^2\cos^2(2t)}{2}+\frac{a^4\cos^4(2t)}{8}} \tag{16}$$

Substituting equations (15) and (16) in equation (12) and considering $\cos(t)=\cos(2t)=k$ yields equation (17).

$$\omega^2 = \frac{-4ak+\frac{ak\beta}{1+\frac{a^2k^2}{2}+\frac{a^4k^4}{8}} - 4(-ak+\frac{ak\beta}{1+\frac{a^2k^2}{2}+\frac{a^4k^4}{8}})}{-4ak+ak} = \frac{\beta}{1+\frac{a^2k^2}{2}+\frac{a^4k^4}{8}} \tag{17}$$

Parameter k can be drawn by substituting the natural frequency from equation (17) into the integral equation (18) and assuming $u(t)=a\cos(\omega t)$.

$$\int_0^{\frac{T}{4}}(\omega^2 a\cos\omega t - a\cos\omega t\frac{\beta}{1+\frac{a^2\cos^2\omega t}{2}+\frac{a^4\cos^4\omega t}{8}})\cos\omega t dt = 0 \tag{18}$$

2.2 Improved amplitude-frequency formulation (IAFF)

The accuracy of AFF depends upon the location choice. He (2008b) suggested the method of weighted residual to overcome this shortcoming. A generalized nonlinear oscillator is considered as presened in equation (19).

$$u''+f(u)=0,\ u(0)=A,\ u'(0)=0 \tag{19}$$

Respectively, solutions of the following linear oscillator equations are considered as two trial functions $u_1(t)=A\cos t$ and $u_2(t)=A\cos\omega t$ which are respectively represented in equations (20) and (21).

$$u''+\omega_1^2 u=0\quad \omega_1^2=1 \tag{20}$$

$$u''+\omega_2^2 u=0\quad \omega_2^2=\omega^2 \tag{21}$$

where ω is assumed to be the frequency of the nonlinear oscillator in equation (19). The residuals are presented in equations (22) and (23).

$$R_1(t) = -\cos t + f(A\cos t) \tag{22}$$

$$R_2(\omega t) = -\omega^2\cos\omega t + f(A\cos\omega t) \tag{23}$$

In this approach, to overcome the shortcomings, two new residuals are defined as $\tilde{R}_1$ and $\tilde{R}_2$ as presented in equations (24) and (25).

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos(\frac{2\pi}{T_1}t)dt \tag{24}$$

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos(\frac{2\pi}{T_2}t)dt \tag{25}$$

Square of the frequency is approximately determined in the form of equation (26) (He, 2008b)

$$\omega^2 = \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1} \tag{26}$$

To solve equation (3) by this method, trial functions are considered as $u_1(t) = A\cos t$ and $u_1(t) = A\cos \omega t$. Substituting these functions into equation (3), the residuals are computed as presented in equation equation (27) to (30).

$$R_1 = -a\cos t + a\cos t \frac{\beta}{1 + \frac{a^2\cos^2 t}{2} + \frac{a^4\cos^4 t}{8}} \tag{27}$$

$$\tilde{R}_1 = \frac{4}{2\pi}\int_0^{\frac{\pi}{2}} R_1 \cos t dt \tag{28}$$

$$R_2 = -\omega^2 a\cos \omega t + a\cos \omega t \frac{\beta}{1 + \frac{a^2\cos^2 \omega t}{2} + \frac{a^4\cos^4 \omega t}{8}} \tag{29}$$

$$\tilde{R}_2 = \frac{4\omega}{2\pi}\int_0^{\frac{\pi}{2\omega}} R_2 \cos \omega t dt \tag{30}$$

By changing variable to $s = \omega t$, the residual $\tilde{R}_2$ can be rewritten as presented in equation (31).

$$\tilde{R}_2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} R_2 \cos(s) ds \tag{31}$$

Frequency can be obtained by substituting $\tilde{R}_1$ and $\tilde{R}_2$ into equation (26).

2.3 Max-min Approach (MMA)

In this approach, after finding the maximal and minimal solution thresholds, an approximate solution of the nonlinear equation is deduced using He-Chengtian's interpolation (He, 2008c). For instance, a generalized nonlinear oscillator is considered in the form of equation (32),

$$\ddot{u} + u f(u) = 0 \quad u(0) = A \quad \dot{u}(0) = 0 \tag{32}$$

where $f(u)$ is a non-negative function of u. The trial function is considered to be $u(t) = A\cos(\omega t)$, where ω is the frequency. Considering the linear oscillator, equation (33),

$$\ddot{u} + u f_{\min} = 0,\ u(0) = A,\ \dot{u}(0) = 0 \tag{33}$$

where $f_{\min}$ is the minimum value of the function $f(u)$, the square of frequency in equation (32) is never less than that of equation (33) with the solution in the form of $u_1(t) = A\cos(\sqrt{f_{\min}}t)$. In addition, the square of frequency never outdoes that of the oscillator in equation (34) which has the solution in the form of $u_1(t) = A\cos(\sqrt{f_{\max}}t)$.

$$\ddot{u} + u f_{\max} = 0,\ u(0) = A,\ \dot{u}(0) = 0 \tag{34}$$

where $f_{\max}$ is the maximum value of the function $f(u)$. Hence, equation (35) can be concluded. (Bayat et al., 2012)

$$f_{\min} < \omega^2 < f_{\max} \tag{35}$$

He-Chentian (He, 2008c) interpolation for the square of the frequency is mentioned in equation (36).

$$\omega^2 = \frac{mf_{\min} + nf_{\max}}{m+n} \tag{36}$$

where m and n are weighting factors. By defining $k = n/m$, the previous equation can be rewritten as equation (37).

$$\omega^2 = \frac{f_{\min} + kf_{\max}}{1+k} \tag{37}$$

By substituting the frequency from equation (37) into the assumed solution of equation (34), the residual is obtained as equation (38).

$$R = -\omega^2 A\cos(\omega t) + (A\cos(\omega t)).f(A\cos(\omega t)) \tag{38}$$

Solving equation (39), k is obtained as indicated in equation (40).

$$\int_0^{\frac{2\pi}{\omega}} R\cos\sqrt{\frac{f_{\min} + kf_{\max}}{1+k}}t\,dt = 0 \tag{39}$$

$$k = \frac{f_{\max} - f_{\min}}{1 - \sqrt{\frac{A}{\pi}\int_0^{\pi}\cos^2 x.f(A\cos x)dx}} \tag{40}$$

Substituting k from equation (40) into equation (37), frequency can be obtained (Bayat et al., 2012).

To apply this method for the nonlinear partial differential equation for transverse vibrations of a flexible string, equation (1) is rewritten in the form of equation (41).

$$\ddot{u} + u\,f\,(u) = 0 \tag{41}$$

where $f(u) = \frac{\beta u}{1 + \frac{u^2}{2} + \frac{u^4}{8}}$. The trial function to solve equation (41) is considered as $u(t) = A\cos\omega t$. Frequency is computed utilizing equation (37), in which $f_{\min}$ and $f_{\max}$ are substituted from equations (42) and (43).

$$f_{\min} = \frac{\beta}{1 + \frac{a^2}{2} + \frac{a^4}{8}} \tag{42}$$

$$f_{\max} = \beta \tag{43}$$

Substituting equations (42) and (43) into equation (37), square of the frequency is obtained as indicated in equation (44).

$$\omega^2 = \frac{\frac{\beta}{1 + \frac{a^2k^2}{2} + \frac{a^4k^4}{8}} + k\beta}{1+k} \tag{44}$$

Substituting equation (44) into the equation (41), the residual is presented in equation (45).

$$R = a\omega^2\cos\omega t - a\cos\omega t\frac{\beta}{1 + \frac{a^2\cos^2\omega t}{2} + \frac{a^4\cos^4\omega t}{8}} \tag{45}$$

By using equation (46) and equation (44), the frequency can be attained. The obtained results are compared in the next section.

$$\int_0^{\frac{T}{4}} \left(-\omega^2 a\cos\omega t + a\cos\omega t \frac{\beta}{1+\frac{a^2\cos^2\omega t}{2}+\frac{a^4\cos^4\omega t}{8}}\right)\cos\omega t dt = 0 \tag{46}$$

3 Results and discussion

3.1 Free vibration response of the cable

The response of the cable under free vibration is compared by different methods, including VAM, Fourth order Runge-Kutta, IAFF, AFF and MaxMin, for different amplitudes in Figure 1 to Figure 5.

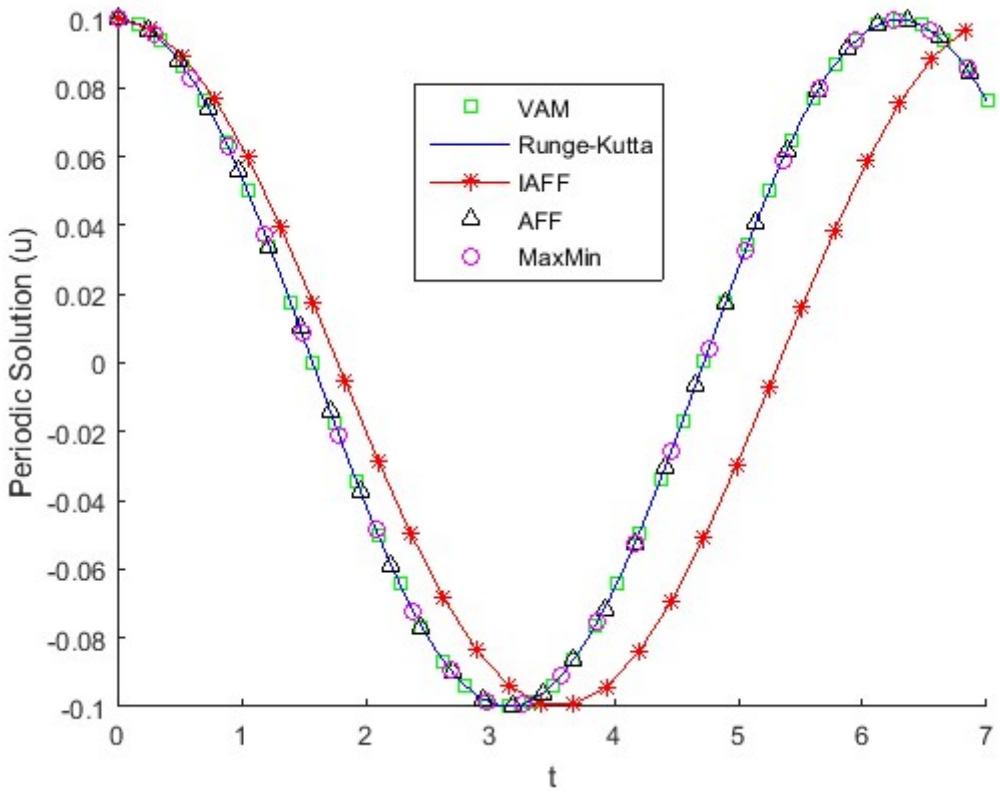

Figure 1: *Response comparison for* $a = 0.1$ *and* $\beta = 1$.

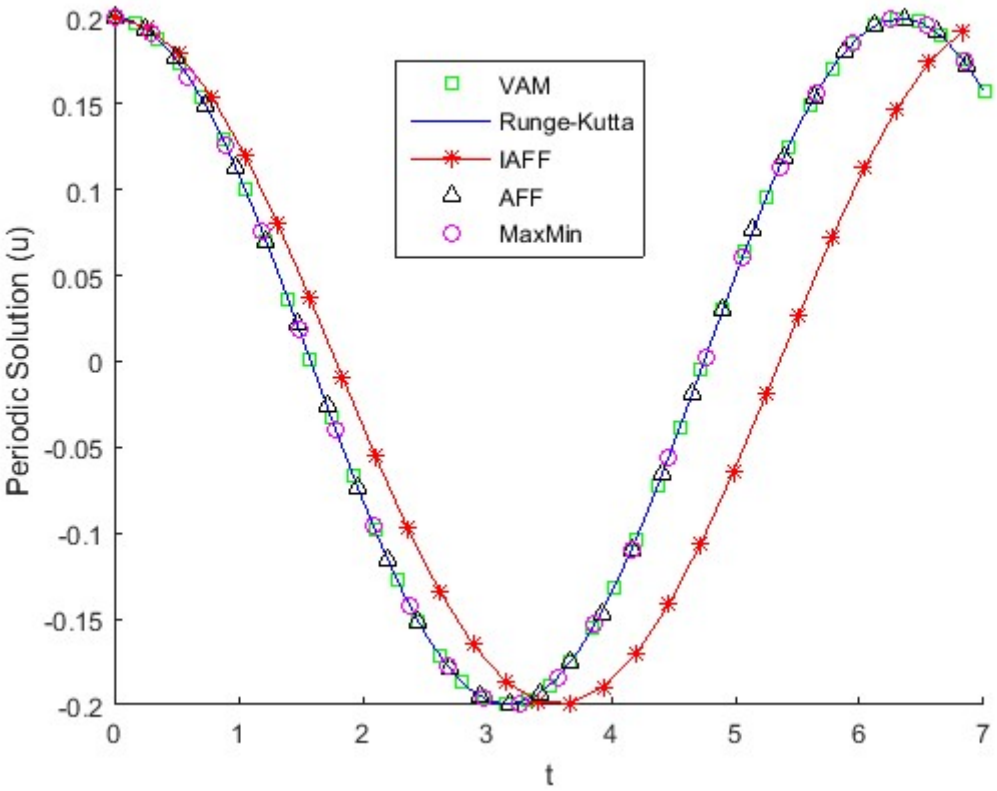

Figure 2: *Response comparison for* $a = 0.2$ *and* $\beta = 1$.

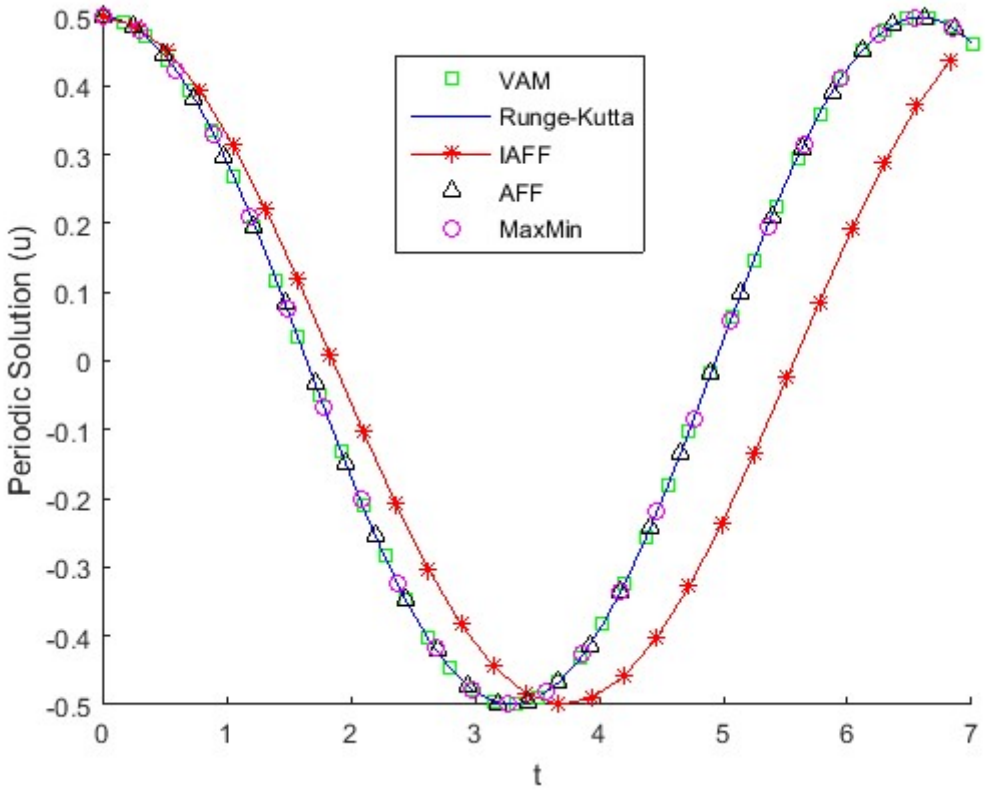

Figure 3: *Response comparison for* $a = 0.5$ *and* $\beta = 1$.

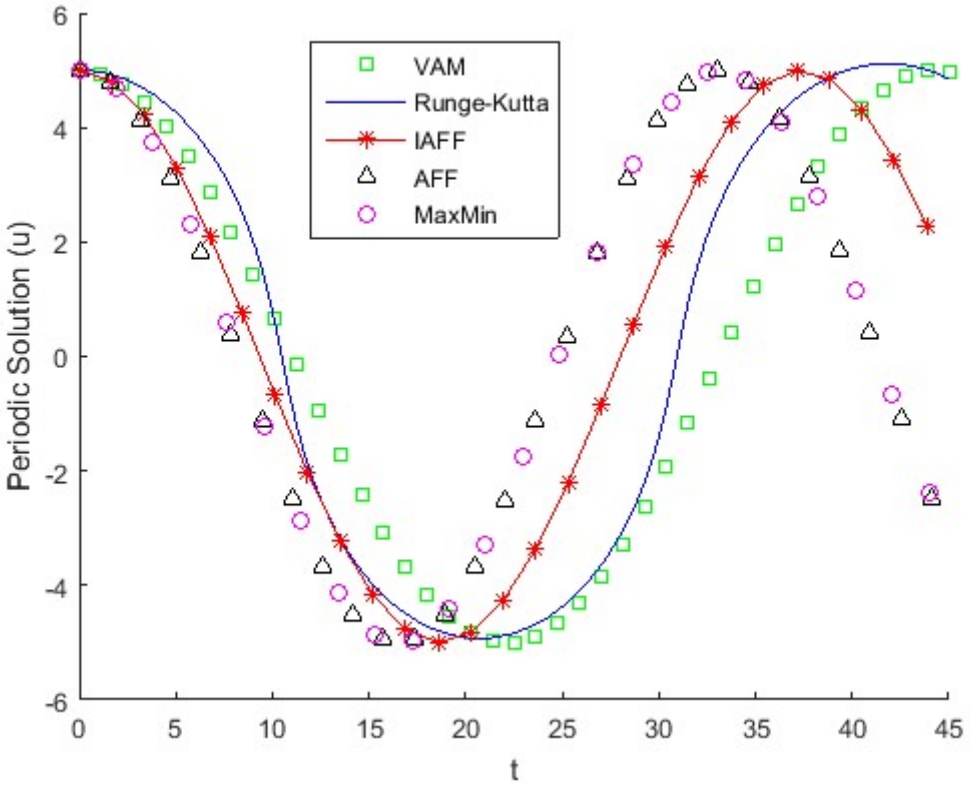

Figure 4: *Response comparison for* $a = 5$ *and* $\beta = 1$.

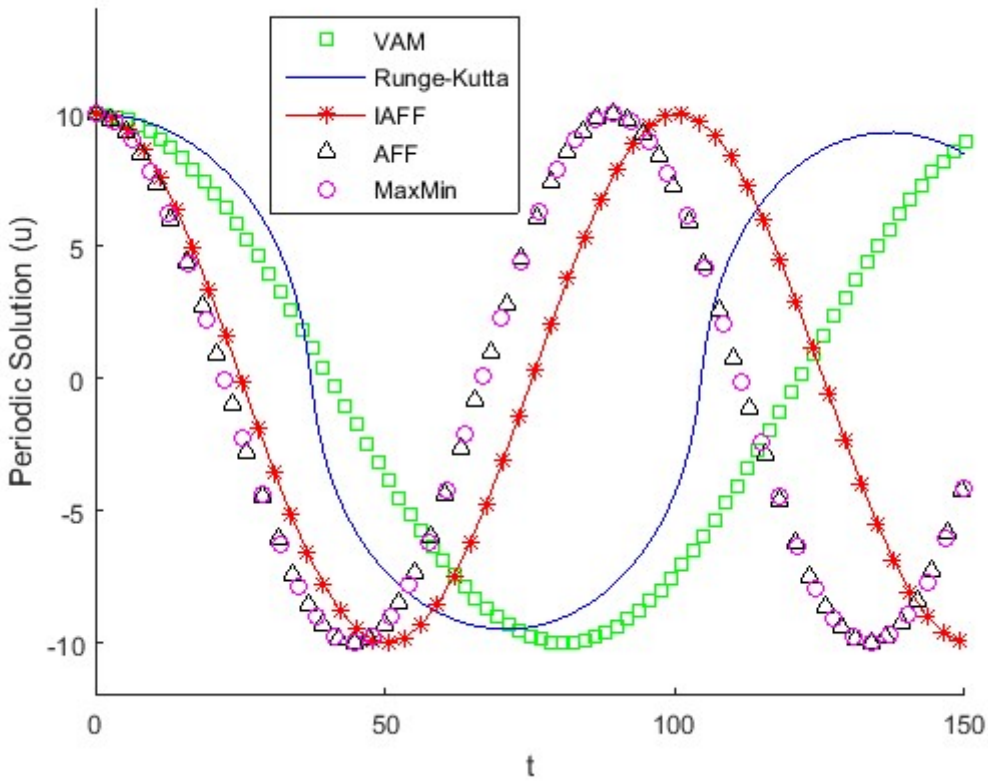

Figure 5: *Response comparison for* $a = 10$ *and* $\beta = 1$.

Obtained solutions using AFF, IAFF, MMA and Runge-Kutta are compared to the results of variational Iteration Method (VIM) (Taghipour et al., 2014), Hamiltonian Approach (HA) (Taghipour et al., 2014) and Variational approach method (VAM) which is reproduced using the derived equations in (Omran et al., 2013) in Table 1 and Table 2.

Table 1: *Response comparison when* $t = 0.5(s)$ *and* $\beta = 1$.

a	VIM (Taghipour et al., 2014)	HA (Taghipour et al., 2014)	VAM (Omran et al., 2013)	IAFF	AFF	MMA	Runge-Kutta
3	2.7467	3.0299	2.9605	2.9608	2.9501	2.9501	2.9759
4	3.8690	4.0171	3.9783	3.9743	3.9673	3.9673	3.9878
5	4.9267	5.0102	4.9874	4.9823	4.9775	4.9775	4.9931
10	9.9691	10.0016	9.9981	9.9951	9.9938	9.9938	9.9990

Table 2: *Response comparison when* $t = 1(s)$ *and* $\beta = 2$.

a	VIM (Taghipour et al., 2014)	HA (Taghipour et al., 2014)	VAM (Omran et al., 2013)	IAFF	AFF	MMA	Runge-Kutta
3	2.6120	3.2420	2.6891	1.6209	2.6089	2.6089	2.8034
4	3.4277	4.1377	3.8278	2.1612	3.7412	3.7412	3.9014
5	4.7012	5.0816	4.8993	2.7015	4.8210	4.8210	4.9452
10	9.5103	10.0130	9.9849	5.4030	9.9503	9.9503	9.9923

Comparing the results in tables 1 and 2, the computed response of the cable under the large amplitude vibration using VAM, AFF and MMA are near eachother, while those of the IAFF is far apart from other methods. Moreover, the obtained results by MMA and AFF are the same for this problem.

3.2 Period parameter

The obtained period parameter $\sqrt{\beta}T$ from AFF, IAFF and MMA are compared to that of VAM (Omran et al., 2013), second-order analytical approximation constructed by harmonic balance method (Gottlieb, 1990), First-order, second-order and third-order analytical approximation developed by coupling Newton's method with the harmonic balance method (NHB) (Lai et al., 2008) and the exact solution (Lai et al., 2008).

Table 3: Comparison of the period parameter $\sqrt{\beta}T$.

a	$\sqrt{\beta}T_{exact}$ (Lai et al., 2008)	$\sqrt{\beta}T_{HB2}$ (Gottlieb, 1990)	$\sqrt{\beta}T_1$ (Lai et al., 2008)	$\sqrt{\beta}T_2$ (Lai et al., 2008)	$\sqrt{\beta}T_3$ (Lai et al., 2008)
0.1	6.294977	6.294980	6.294977	6.294977	6.294977
0.2	6.33047	6.33052	6.33047	6.33047	6.33047
0.5	6.58377	6.58576	6.58377	6.58379	6.58379
5	42.173	48.345	42.173	41.875	41.918
10	152.9	179.9	152.9	149.4	148.8
a	$\sqrt{\beta}T_{VAM}$ (Omran et al., 2013)	$\sqrt{\beta}T_{IAFF}$		$\sqrt{\beta}T_{AFF}$	$\sqrt{\beta}T_{MMA}$
0.1	6.294975	7.103118		6.294975	6.294975
0.2	6.33047	7.14314		6.33044	6.33044
0.5	6.58372	7.42763		6.58257	6.58257
5	44.2076	37.3552		33.1052	33.1052
10	161.9	100.5		89.1	89.1

Comparing the results in table 3, one can conclude that for small values of *a*, obtained results by MMA and AFF is in agreement with the exact solution, and by increasing the value of *a*, the results deviate from the exact solution. Moreover, utilizing IAFF for this problem is not appropriate since its error is more than other mentioned methods.

4 CONCLUSION

In this paper, AFF, IAFF and MMA are employed to derive the analytical approximate solution for large amplitude nonlinear vibration of a string. The obtained response from these methods were compared to other methods such as VAM, VIM, HA and HB. It is shown that the obtained response from MMA and AFF are the same for this problem. On the other hand, IAFF is not appropriate method for this problem since it is not accurate enough in comparison to other mentioned methods.

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