Latin American Journal of Solids and Structures
www.lajss.org

GFEM STABILIZATION TECHNIQUES APPLIED TO DYNAMIC ANALYSIS OF NON-UNIFORM SECTION BARS

Abstract
The Finite Element Method (FEM), although widely used as an approximate solution method, has some limitations when applied in dynamic analysis. As the loads excite the high frequency and modes, the method may lose precision and accuracy. To improve the representation of these high-frequency modes, we can use the Generalized Finite Element Method (GFEM) to enrich the approach space with appropriate functions according to the problem under study. However, there are still some aspects that limit the GFEM applicability in problems of dynamics of structures, as numerical instability associated with the process of enrichment. Due to numerical instability, the GFEM may lose precision and even result in numerically singular matrices. In this context, this paper presents the application of two proposals to minimize the problem of sensitivity of the GFEM: an adaptation of the Stable Generalized Finite Element Method for dynamic analysis and a stabilization strategy based on preconditioning of enrichment. Examples of one-dimensional modal and transient analysis are presented as bars with cross section area variation. Numerical results obtained are discussed analyzing the effects of the adoption of preconditioning techniques on the approximation and the stability of GFEM in dynamic analysis.

Keywords
GFEM; Dynamic Analysis; Numerical Stability

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http://dx.doi.org/10.1590/1679-78254265

Received: July 15, 2017
In Revised Form: December 13, 2018
Accepted: January 19, 2018
Available Online: February 05, 2018

1 INTRODUCTION

The Finite Element Method (FEM) is the numerical method commonly used in vibration analysis. The free vibration analysis by the FEM gives good results for the lowest frequencies but demands great computational cost to work up the accuracy for the higher frequencies. Thus, the improvement of Finite Element models applied to the structural dynamics is part of a vast field of dynamic analysis of structures, constituting an area of current research and industrial applications.

Using particular information to improve the approximation characteristics is the core of the enriched methods, such as the Generalized Finite Element Method (GFEM), where enriching functions are used to improve the approximation space (Babuška et al., 1994; Melenk, 1995; Duarte and Oden, 1996a; Duarte and Oden, 1996b; Oden and Duarte, 1997; Babuška and Melenk, 1997; Babuška et al., 2004; Belytschko and Black, 1999; Möes et al., 1999). In this context, the GFEM has applications in several areas, such as fracture mechanics (Yazid et al., 2009; Gupta et al., 2015), flow of biphasic fluids (Sauerland and Fries, 2013), electromagnetism (Lu and Shanker, 2007), heat transfer with high gradients (O'Hara et al., 2009; Aragon et al., 2010) and, as approached in this work, in dynamics of structures (Arndt, 2009; Torii, 2012; Torii and Machado 2012; Shang, 2014; Torii et al., 2016; Hsu, 2016; Weinhardt et al., 2016; Piedade Neto and Proença, 2016). Despite the excellent properties of the GFEM, there are aspects that still limit its practical applicability and its efficiency.

One of these limiting factors is the numerical instability associated with the present enrichment process even in well-defined boundary value problems. Several studies have been directed to the treatment of this problem in recent years, mainly in the context of fracture mechanics.

Menk and Bordas (2011) proposed a preconditioning strategy based on the FETI (Finite Element Tearing and Interconnecting) domain decomposition technique considering that linear dependency problems tend to worsen

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in domains with many enriched elements. The idea is to divide the domain so as to limit the rate of enriched elements and avoid conditions where enrichment is ineffective and results in poor conditioning. The method involves the application of Cholesky and LQ decomposition in the stiffness matrices of the subdomains and presents conditioning close to MEF systems. However, the computational effort is heavily dependent on how the domain is partitioned (Menk and Bordas, 2011).

Sauerland and Fries (2013) have pioneered the ideas of Stable-XFEM for modeling fluid interface problems. Since the iterative solvers have their performance intrinsically related to the conditioning of the matrices, the application of Stable-XFEM presented a significant advantage over the classical XFEM approach.

Despite the positive results, the Stable-XFEM was not able to capture with high accuracy the jump in the pressure gradient, leaving a topic open on the issue of reproducibility of a priori known effects in the enrichment functions.

In this context, Shibanuma et al. (2014) observed that stabilized XFEM's accuracy problems could result from its untying of PUFEM. In this way, they proposed a reconceptualization taking into account the basis of the unit partition, naming the PU-XFEM proposal.

The PU-XFEM was applied to two-dimensional linear fracture mechanics in numerical examples in infinite plates. The results of the PU-XFEM are relatively better than those of the classical XFEM (see Belytschko and Black, 1999) and the weighted XFEM (see Fries, 2008), although the results of the weighted XFEM are effective for actual use. In particular, the numerical results showed that a priori knowledge of the solution can be reproduced exactly in the PU-XFEM. This feature practically solves the stability issue of the blending elements.

While Stable-XFEM approaches seek to provoke a quasi-orthogonality of enriched functions, Sillem et al. (2015) present Orthonormalized Generalized Finite Element Method (OGFEM), which explicitly uses a process of orthonormalization in the basis of approximation functions.

The OGFEM can be seen as a process of constructing an enriched orthonormal basis whose stiffness matrix is optimally well conditioned. The examples presented in Sillem's work (2015) show an increase in the convergence rate for the one-dimensional cases of the modified Helmholtz and Poisson equations.

Despite the consistent analytical approach of OGFEM, Sillem et al. (2015) do not present a study of the numerical sensitivity associated with the orthonormalization process itself. That is, the gain of conditioning, which improves the conditions of resolution of the system of equations, appears to come at the cost of adopting a numerical process of orthonormalization with its own numerical instabilities. In addition, it should be emphasized that the method becomes prohibitively costly for two-dimensional and three-dimensional cases.

In the context of GFEM, the Stable Generalized Finite Element Method (SGFEM) was firstly proposed to address numeric conditioning issues of GFEM (Babuška and Banerjee, 2012). This promising methodology has been object of study in several works in recent years, due to its generality and versatility. The stabilization strategy is well grounded in the concept of partition of unity and has a good theoretical foundation.

Addressing numerical sensitivity problems that arise in the application of enrichment proposals, two alternatives are adopted to stabilize the method: application of the concepts of the Stable Generalized Finite Element Method and application of preconditioning changes in enrichment functions, called Heuristic Modification Stabilization, such as presented in Weinhardt et al. (2015), Weinhardt et al. (2016) and Weinhardt et al. (2017).

Following the idea that small changes in enrichment functions can change the numerical conditioning of the problem without significant loss of accuracy properties of the GFEM, this article presents the effects of a subtle change in the functions used for one-dimensional dynamic analysis such as presented by Arndt (2009). The stabilization methods are applied in two examples of bar with cross section area variation, and this proposal has been proved to be very effective in improving the stability for the modal and transient analysis.

The strategies of stabilization of the enriched methods are based on the most varied assumptions, from the most analytical to the more empirical ones. However, most approaches to the assessment of conditioning are focused only on the stiffness matrix. Nevertheless, in problems of dynamic analysis the mass matrix is also used and its characteristics are closely related to the nature of the problem and its numerical sensitivity.

Thus, it is noted that there is a need for stabilization proposals that take into account the inherent characteristics of the field of application of the dynamic analysis and the specific types of enrichment used for this purpose. Thus, it is important to note that there is a wide-open field of research, especially regarding applications to dynamic analysis, both modal and transient. Therefore, it is in this context of innovation that the present work is inserted, seeking to address issues raised by previous studies about the stability and sensitivity of the application of GFEM in the dynamic analysis, as well as aiming to motivate correlated studies.

An outline of the present paper is as follows. In Section 2, it is discussed the bar model under study followed by some basic concepts of the Finite Element Method applied to dynamic analysis. In Section 3, it is presented the numerical results regarding modal and transient analysis of two bar models with polynomial and non-polynomial

section variation. Finally, in Section 4, some considerations about the results obtained, contributions of the work and directions of new researches are presented.

2 METHODOLOGY

2.1 Bar model formulation

Consider straight bar of length L and variable cross-sectional area A, according to Fig. 1. For simplicity, it is considered that the cross section does not vary abruptly and maintains a constant shape, only by varying its area as a function of the horizontal relative position x. No damping effects are incorporated into this model and the constituent material is considered of linear behavior with modulus of elasticity E being able to vary as a function of x and has specific mass ρ (Craig, 1981; Arndt et al., 2010).

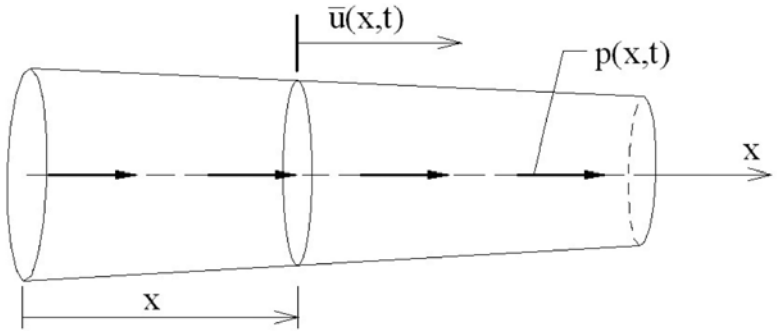


Figure 1: *Bar model.*

The vibration of the bar is a time dependent problem, and the equation of motion governing this problem is a partial differential equation. The problem is to find the axial displacement $\bar{u}$ satisfying:

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial \bar{u}}{\partial x} \right) = p(x,t) \tag{1}$$

where p is the externally applied axial force per unit length and t is the time. The solution $\bar{u} = \bar{u}(x,t)$ must satisfy the boundary and initial conditions defined in the problem.

According to Carey and Oden (1984), in order to obtain the variational form of a time dependent problem, test functions $w = w(x)$, independent of t are selected, and the weighted residual method is applied. Then, this procedure results in the following system of differential equations:

$$M\ddot{u} + Ku = F \tag{2}$$

where K is the stiffness matrix, u is the vector of displacements, M is the mass matrix, $\ddot{u}$ is the vector of accelerations and F is the vector of applied forces. The response of the system of equations can be given through the modal coordinates, natural frequencies and vibration modes, or through transient analysis, applying a direct integration method.

To obtain the fundamental vibration modes and frequencies of the problem it is solved the generalized eigenvalues problem given by (Bathe, 1996; Hughes, 1987; Zienkiewicz and Taylor, 2000):

$$K\Phi = \omega^2 M\Phi \tag{3}$$

where ω are the vibration frequencies and Φ are the vibration modes of the problem.

The stiffness and the mass matrices can be assembled as:

$$K_{ij} = \int_0^L EA \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx \qquad \text{i},j = 1,2,3\ldots N \tag{4}$$

$$M_{ij} = \int_0^L \rho A \phi_i \phi_j dx \qquad \text{i},j = 1,2,3...N \tag{5}$$

where $\phi(x)$ are the basis functions of the approximation subspace, the indices "i" and "j" respectively refer to the "i-th" and "j-th" basis function relative to the calculation of the element (i, j) of the corresponding matrix and "N" is the dimension of the approximation space. Different subspaces of approximation are proposed for different methods, such as the Finite Element Method, and the Generalized Finite Element Method.

2.2 Finite Element Method

The standard Finite Element Method uses polynomials shape functions in the approximated solution which can be expressed generally in matrix form as (Bathe, 1996; Hughes, 1987):

$$u_h^{\,e}(\xi) = N^T q \tag{6}$$

where N is the matrix of shape functions and q is the displacement vector. The polynomial functions may be of any order, the simplest being the linear ones. Taking the uniform bar element (Figure 1) with one degree of freedom per node, the terms of the approximated solution (Equation (6)) using linear Lagrangian polynomials as local shape functions are defined in the master element domain $\xi \in [0,1]$ as

$$N^T = [1-\xi \quad \xi], \tag{7}$$

$$q^T = [\mathrm{u}_1 \quad \mathrm{u}_2], \tag{8}$$

$$\xi = \frac{x}{L_e}, \tag{9}$$

where L_e is the element length, and u_1 and u_2 are the nodal displacements.

2.3 Generalized Finite Element Method

The GFEM is an enriched method of which the main goal is the construction of a finite dimensional subspace of approximating functions using local knowledge about the solution that ensures accurate local and global results. The local enrichment in the approximation subspace is incorporated by the partition of unity approach, which is shown below (Melenk and Babuška, 1996; Babuška et al., 2004; Duarte et al., 2000; Piedade Neto and Proença, 2016).

The approximated solution proposed by the GFEM in the master element domain may be written as the sum of two components:

$$u_h^e(\xi) = u^e{}_{FEM} + u^e{}_{ENRICHED} \tag{10}$$

where $u^e{}_{FEM}$ is the Finite Element Method component based on nodal degrees of freedom and $u^e{}_{ENRICHED}$ is the enriched component by the partition of unity approach based on field degrees of freedom. In this sense, the bar approximated solution on a master element is:

$$u_h^e(\xi) = \sum_{i=1}^{2} \eta_i(\xi) u_i + \sum_{i=1}^{2} \eta_i(\xi) \left[\sum_{j=1}^{n_l} \gamma_{ij}(\xi) a_{ij} + \varphi_{ij}(\xi) b_{ij} \right] \tag{11}$$

where η_i are the partition of unity functions, γ_{ij} and φ_{ij} are the enrichment functions, n_l is the number of enrichment layers, u_i are the nodal displacements and, a_{ij} and b_{ij} are the field degrees of freedom related to the enrichment functions γ_{ij} and φ_{ij}, respectively. The term "layer" is adopted herein as a reference to the use of sets of enrichment functions which must be applied together. Thus, each new level of enrichment works as a new enrichment layer applied at once. For the linear bar element (Figure 1), the partition of unity functions are those given by Equation (7).

In this study, the enrichment functions proposed by Arndt (2009) and Arndt et al. (2011) are written as a group of functions that consists of building a couple of clouds, a sine and a cosine, subordinated to the cover of the

master element. These clouds are written in the element domain as two pairs of sine and cosine functions. The element domain is considered as $\xi \in [0,1]$.

Sine cloud:

$$\begin{aligned} \gamma_{1j} &= \sin(\beta_j \xi) \\ \gamma_{2j} &= \sin(\beta_j(\xi - 1)) \end{aligned} \tag{12}$$

Cosine cloud:

$$\begin{aligned} \varphi_{1j} &= \cos(\beta_j \xi) - 1 \\ \varphi_{2j} &= \cos(\beta_j(\xi - 1)) - 1 \end{aligned} \tag{13}$$

where L_e is the element length and $\beta_j = j\alpha\pi$ is a hierarchical enrichment parameter proposed by Arndt (2009) with $\alpha = 1$ and with j varying from 1 to the number of enrichment layers (n_l), such as $j = 1, 2, \ldots n_l$.

2.4 Stabilization Strategies

2.4.1 SGFEM-based Stabilization

Stable Generalized Finite Element Method (SGFEM) was firstly proposed to address numeric conditioning issues of GFEM (Babuška and Banerjee, 2012). This method consists in the application of a subtle modification of enrichment function prior to its inclusion in GFEM approximation space (Gupta et al., 2013; Li, 2014; Lins et al. 2015).

In the SGFEM, the enrichment functions are modified as described in Equation (14), as presented by Babuška and Banerjee (2012):

$$\tilde{\varphi}_i(x) = \varphi_i(x) - I_\mu(\varphi_i(x)) \tag{14}$$

where $\tilde{\varphi}_i$ as the i-th stabilized enrichment function, φ_i as the i-th enrichment function and $I_\mu(\varphi_i(x))$ as the linear interpolant of the i-th enrichment function subordinated to support μ.

The proposed stabilization of the first layer of enrichment is shown in Figure 2.

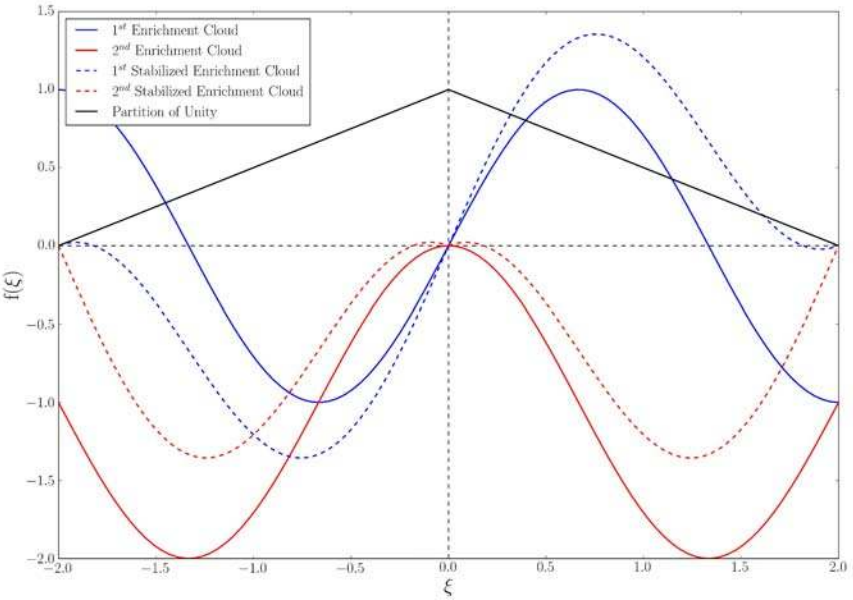


Figure 2: *Stabilization process for the first layer of enrichment functions.*

2.4.2 Heuristic Modification Stabilization

Note that the variation in the enrichment parameter β_j implies different characteristics of approaching, as already pointed out by Arndt (2009) and Torii (2012). However, the gain in accuracy for certain frequencies does not appear to be associated with numerical stability gain. In fact, apparently there is a certain trade-off between accuracy and numerical stability, regarding the choice of the parameter β_j.

Since the enrichment parameters β_j are intrinsically related to the properties of the generated approximation space, another aspect to be considered relative to these parameters is their construction rule. Thus, a search for better numerical approximation properties, such as stability, should contemplate a study of this aspect. In this context, the proposed modification presented in this paper simply consists of the change in the formation rule of this enrichment parameter.

The choice of this new rule follows an extensive sequence of tests performed for several enrichment parameters and growth rates of these parameters, where it was possible to empirically observe a pattern of minimization of stability problems while maintaining accuracy. This part of the development of the methodology did not result in a general form of representation and therefore the proposal converged to an empirical and heuristic approach that will be explained below.

Recalling that parameter β_j is calculated by $\beta_j = j\alpha\pi$, it was proposed a modification, creating new stabilized parameters $\bar{\beta}_j$ given by:

$$\bar{\beta}_j = \left[2(j-1) + \frac{\beta_1}{\pi}\right]\pi \qquad j = 1, 2, 3..., \text{nl} \tag{15}$$

It is interesting to note that $\bar{\beta}_1 = \beta_1$, since:

$$\bar{\beta}_1 = \left[2(1-1) + \frac{\beta_1}{\pi}\right]\pi = \left[0 + \frac{\beta_1}{\pi}\right]\pi = \beta_1 \tag{16}$$

This implies that there is no difference between approximation taken by this approach and the standard trigonometric one for the first layer of enrichment.

3 NUMERICAL RESULTS

In this section modal and transient analysis of bars with polynomial and sinusoidal cross area section variation are presented.

In order to clarify the results presentation, it is necessary to explain some preliminary points.

Firstly, the condition numbers of the matrices are calculated in a classical manner (see Wilkinson, 1965) multiplying the norm of the matrix by the norm of its inverse, that is, $\kappa(M) = \left\|M^{-1}\right\|\left\|M\right\|$. It is equivalent to divide the greatest eigenvalue by the smallest eigenvalue of the matrix. Conditioning results for stiffness matrices are analogous and mass matrices are adopted for such analysis for its dominant behavior in sensitivity analysis, as highlighted by Petroli et al. (2017).

As for the enrichment parameter, the $\beta_1 = \frac{3\pi}{4}$ value was adopted based on the experience of other works. This value was first used in the Torii's thesis (2012). Later, already in the stabilization context, the value of the parameter was again tested in Weinhardt's dissertation (Weinhardt, 2016) according to accuracy parameters, resulting in the choice of the $\beta_1 = \frac{3\pi}{4}$ value for the following tests. Thus, this enrichment parameter value is adopted for every test result presented in this paper.

3.1 Bar with polynomial cross section area variation

One of the examples of this paper is a bar with polynomial cross area variation modeled as:

$$A(x) = (a + bx)^4 \tag{17}$$

where a and b are polynomial coefficients.

The results for a bar with sinusoidal cross section area variation are presented in the items 3.1.1 and 3.1.2.

3.1.1 Modal Analysis

In order to validate the numerical results, we present the comparison of the numerical values obtained through the exposed methods. Kumar and Sujith (1997) verified that the lower natural frequencies are more affected by the variation of the cross section, and the highest ones are very close to the frequencies of the equivalent uniform bar. Therefore, the results obtained for the first six frequencies of a double-clamped bar with chosen area variation are exposed in the Table 1.

Table 1: *Prior Modal Results - Dimensionless Eigenvalues*

Vibration mode	Analytical Solution	FEM - 100 elements 99 dofs	Error	GFEM - 20 elements 1 enrichment layer 99 dofs	Error	GFEM - 20 elements 2 enrichment layers 179 dofs	Error
1	3.286007	3.286175	1.680000E-04	3.286008	1.000000E-06	3.286007	0.000000E+00
2	6.360678	6.361800	1.122000E-03	6.360690	1.200000E-05	6.360678	0.000000E+00
3	9.477196	9.480820	3.624000E-03	9.477233	3.700000E-05	9.477196	0.000000E+00
4	12.605890	12.614341	8.451000E-03	12.605973	8.300000E-05	12.605890	0.000000E+00
5	15.739656	15.756038	1.638200E-02	15.739809	1.530000E-04	15.739656	0.000000E+00
6	18.876001	18.904194	2.819300E-02	18.876251	2.500000E-04	18.876001	0.000000E+00

As one may observe in the Table 1, GFEM solution achieve better approximation than FEM using less elements and only one enrichment layer. Applying the second enrichment layer improves even more the accuracy, as shown in the last column of the Table 1. However, the approximation of higher frequencies may require more enrichment layers, as discussed in Weinhardt et al. (2015) and Weinhardt et al. (2017). This enrichment process may result in numerical instabilities arising from ill-conditioned matrices. The Figure 3 presents the evolution of the condition number of the mass matrix related to a 20-element uniform mesh successively enriched.

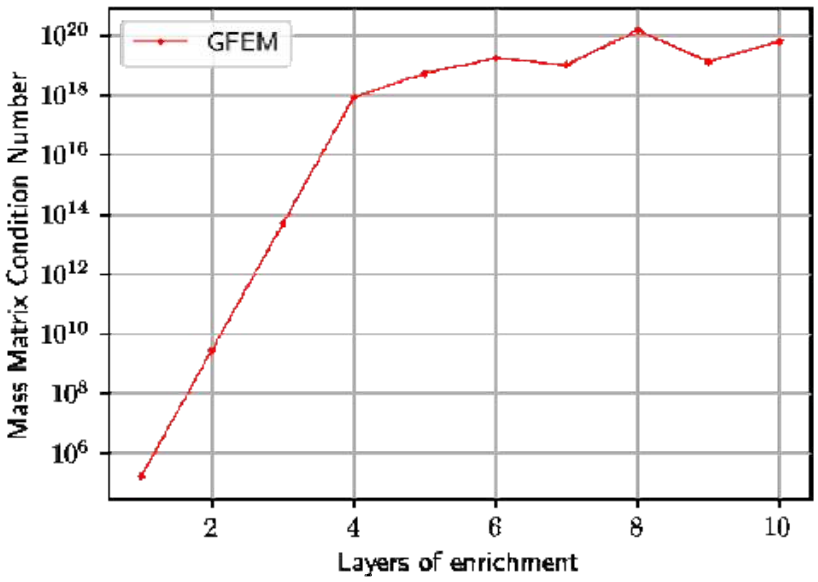


Figure 3: *Evolution of Mass Matrix Condition Number (Numerically Calculated) – First example.*

3.1.2 Transient Analysis

The trigonometric enrichment of GFEM applied to one-dimensional transient analysis was discussed earlier by Torii (2012) and Shang (2014) covering several examples. This work will continue the discussion by presenting the application of refinement with stabilization proposals in an enlightening example. Stabilization alternatives in transient analysis, such as HHT used by Shang (2014) and Hsu et al. (2016), were avoided in order to keep focus on the analysis of the interactions of the enrichment process and the stabilization proposals.

For the following examples, transient analysis arises in the application of the Newmark Method, as described in Bathe (1996), using mass and stiffness matrices generated by the application of GFEM with different approaches. Parameters were set such as $\sqrt{\frac{E}{\rho}} = c = 1$, neglecting damping, and adopting an uniform mesh of 20 finite elements. For the time discretization, the 20 seconds analysis interval was divided in 2000 steps of 10^{-2} seconds.

The model considered for the examples consists of a bar with a clamped end and the other free end, where the load is applied. Trigonometric enrichment was adopted using $\beta_1 = \frac{3\pi}{4}$ due to its performance in modal analysis as presented by Weinhardt et al. (2015).

Firstly it is compared an approximation performed using 20-element uniform mesh with: no enrichment (FEM); standard one-layer trigonometric enrichment (GFEM); and Stable one-layer trigonometric enrichment (SGFEM). The results are presented in the Figs. 4 and 5, and the external load is applied with a value of 1N in the time range from 0 to 10^{-2} seconds. It should be noted that, for the first level of enrichment, the Heuristic Modification coincides with the standard GFEM.

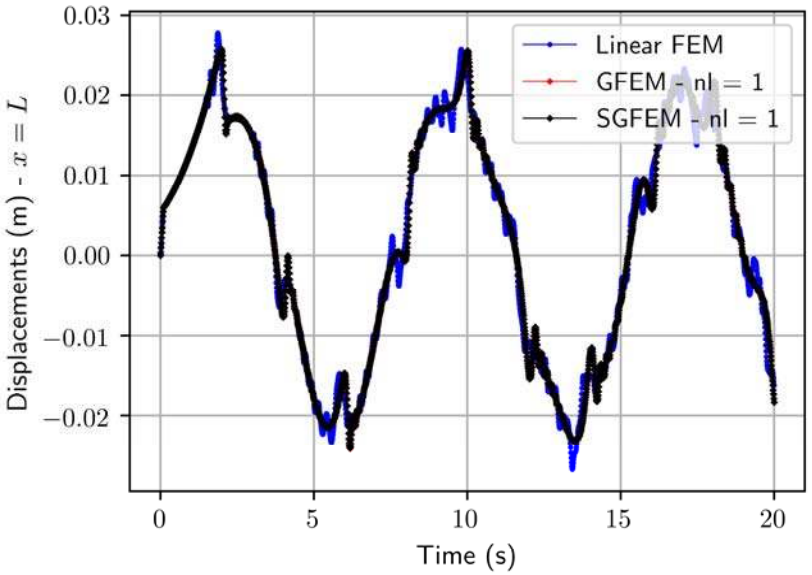


Figure 4: *Displacements over time.*

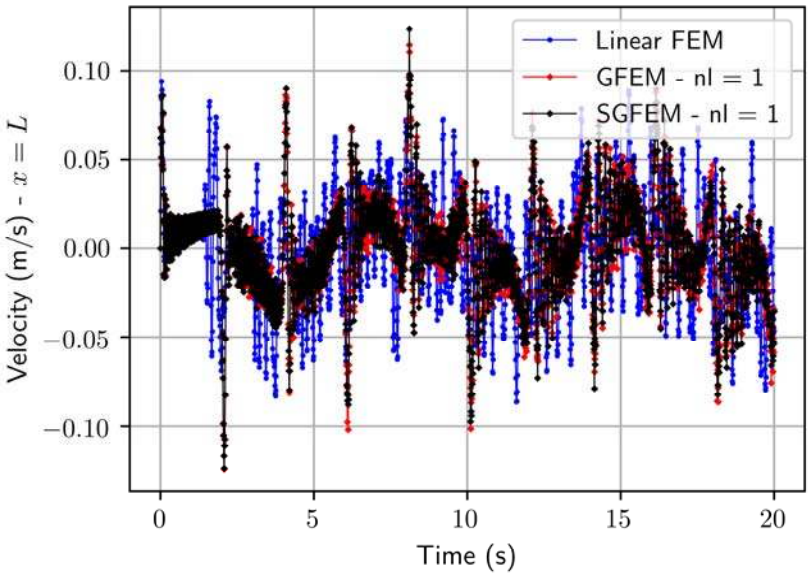


Figure 5: *Velocities over time.*

Raising enrichment layer, the results obtained are presented in Figs. 6 and 7 for: GFEM; SGFEM; and Heuristically Modified GFEM.

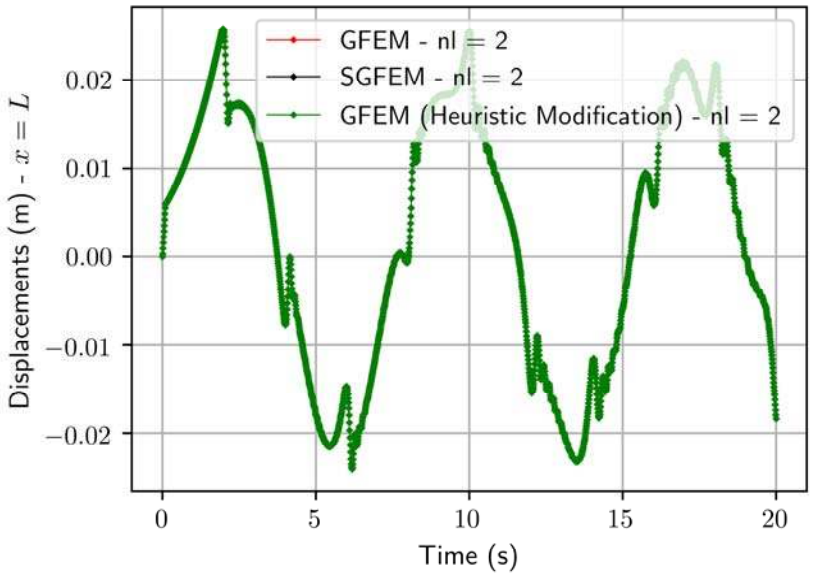


Figure 6: *Displacements over time.*

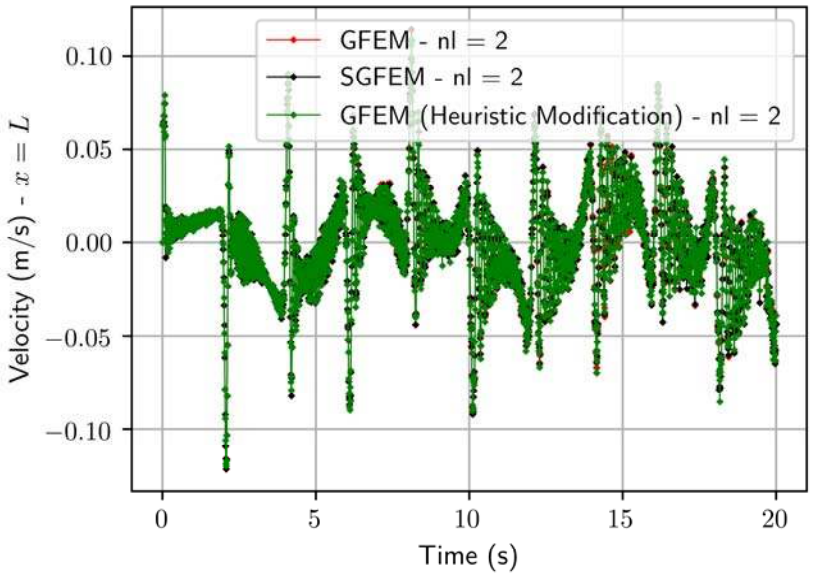


Figure 7: *Velocities over time.*

Continuing the process of successive enrichment, Figs. 8 and 9 present the respective results.

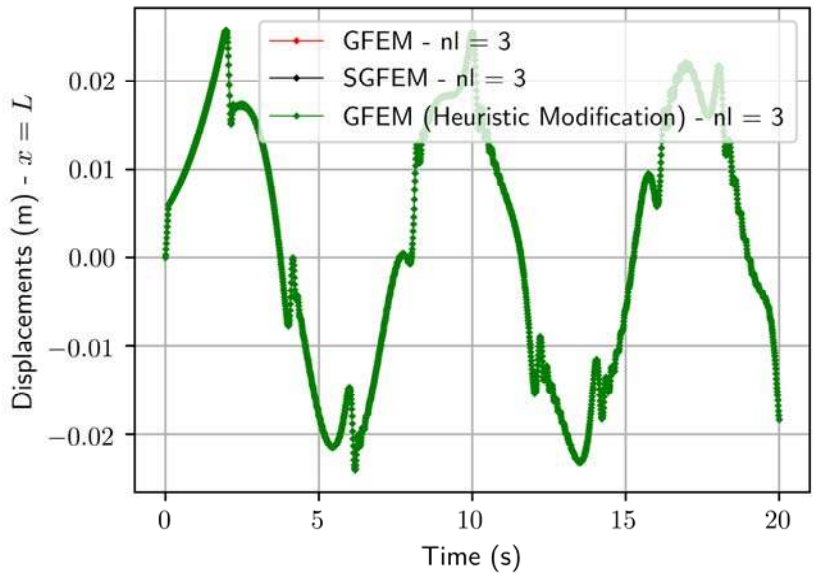


Figure 8: *Displacements over time.*

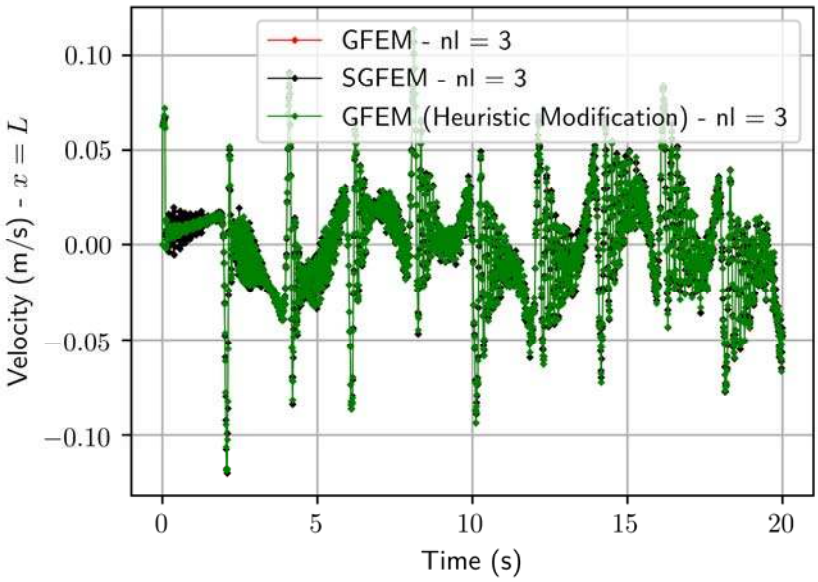


Figure 9: *Velocities over time.*

For 4 enrichment layers results are shown in Figs. 10 and 11.

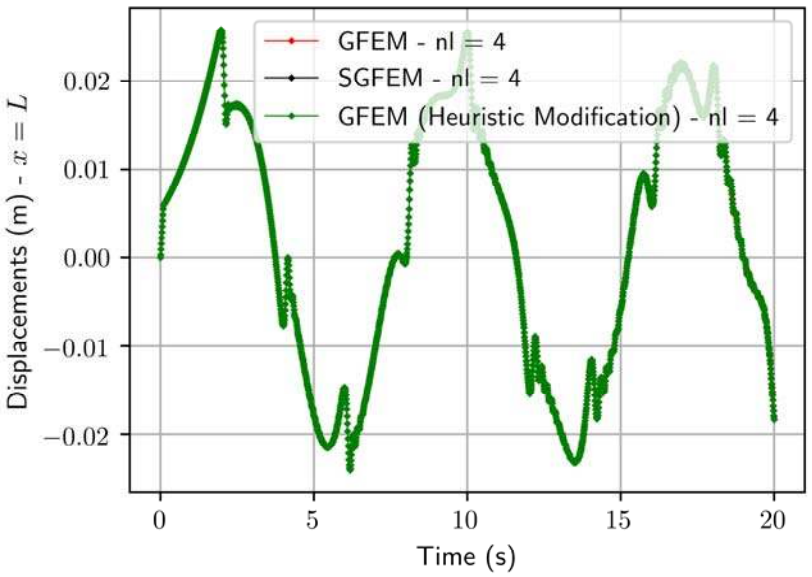


Figure 10: *Displacements over time*

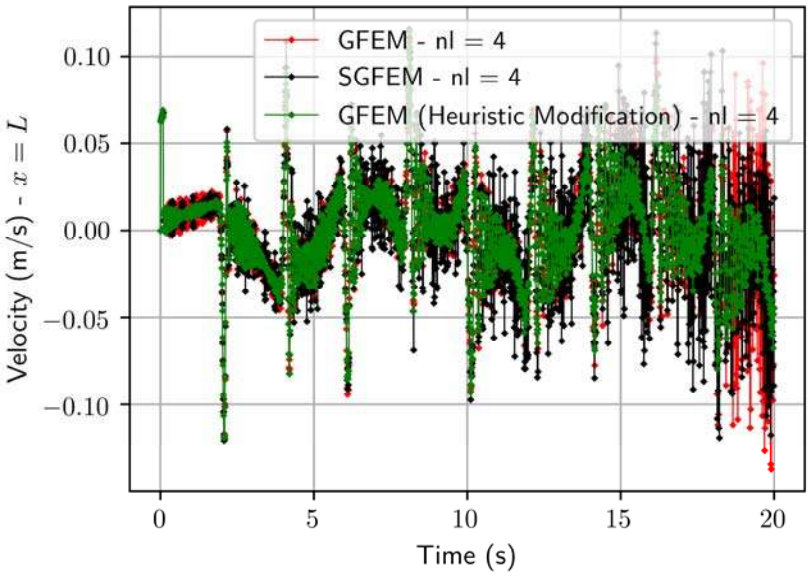


Figure 11: *Velocities over time.*

Results aiming to highlight the stabilization proposal are presented in Fig. 12 and Fig. 13, taking up to refinement to high-order.

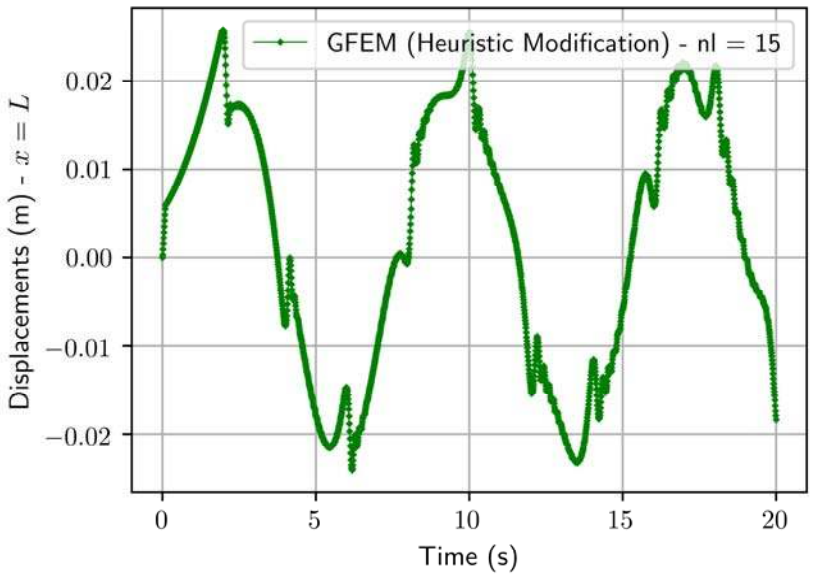


Figure 12: *Displacements over time*

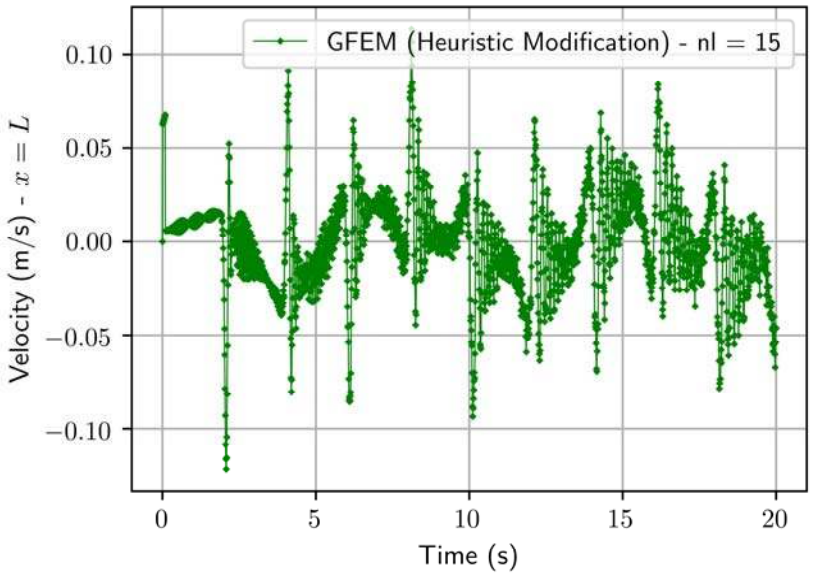


Figure 13: *Velocities over time.*

Fig. 14 presents the evolution of the condition number of the mass matrix related to a 20 element uniform mesh successively enriched for different stabilization strategies, trying to tie the observations of numerical instability in the transient analysis with the condition number of mass matrices.

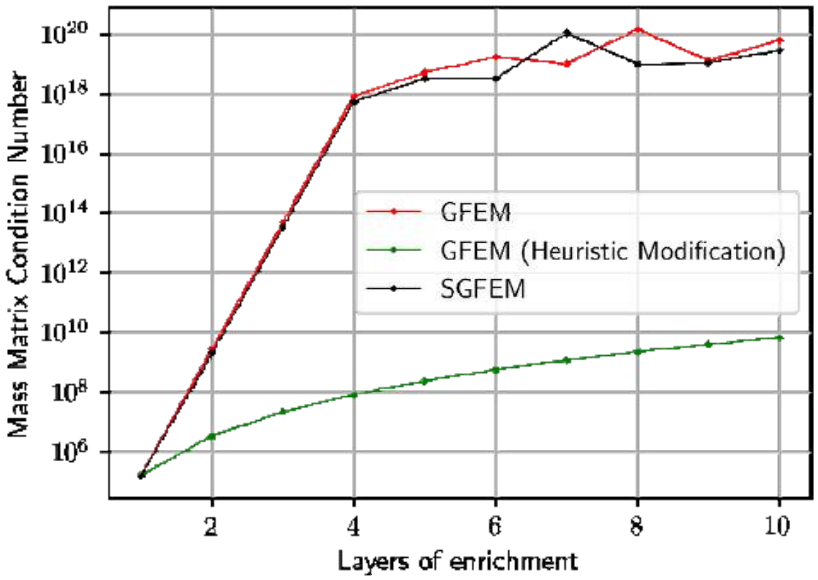


Figure 14: *Evolution of Mass Matrix Condition Number (Numerically Calculated) – First example.*

3.2 Bar with sinusoidal cross section area variation

This paper also studied a bar with sinusoidal cross area variation modeled as:

$$A(x) = A_0 \sin^2(ax + b) \tag{18}$$

where A_0, a and b are parameters that describe the sinusoidal variation.

The results for a bar with sinusoidal cross section area variation are presented in the items 3.2.1 and 3.2.2.

3.2.1 Modal Analysis

As in the example 3.1, in order to validate the numerical results, we present the comparison of the numerical solutions obtained through the exposed methods. The results obtained for the first six frequencies of a double-clamped bar with area variation as chosen are exposed in Table 2.

Table 2: *Prior Modal Results - Dimensionless Eigenvalues*

Vibration mode	Analytical Solution	FEM - 100 elements 99 dofs	Error	GFEM - 20 elements 1 enrichment layer 99 dofs	Error	GFEM - 20 elements 2 enrichment layers 179 dofs	Error
1	2.978189	2.978330	1.410000E-04	2.978190	1.000000E-06	2.978188	-1.000000E-06
2	6.203097	6.204151	1.054000E-03	6.203108	1.100000E-05	6.203097	0.000000E+00
3	9.371576	9.375094	3.518000E-03	9.371612	3.600000E-05	9.371576	0.000000E+00
4	12.526519	12.534827	8.308000E-03	12.526600	8.100000E-05	12.526519	0.000000E+00
5	15.676100	15.692301	1.620100E-02	15.676252	1.520000E-04	15.676100	0.000000E+00
6	18.823011	18.850986	2.797500E-02	18.823259	2.480000E-04	18.823011	0.000000E+00

As one may observe in the Table 2, GFEM solution achieve better approximation than FEM using less elements and only one enrichment layer. Applying the second enrichment layer improves even more the accuracy, as shown in the last column of the Table 2. This enrichment process may result in numerical instabilities arising from ill-conditioned matrices. The Figure 15 presents the evolution of the condition number of the mass matrix related to a 20-element uniform mesh successively enriched.

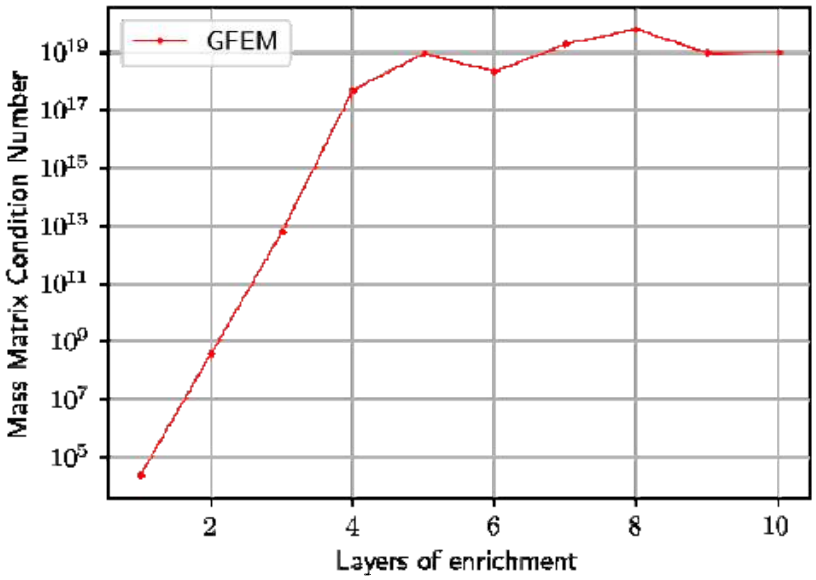


Figure 15: *Evolution of Mass Matrix Condition Number (Numerically Calculated) – Second example.*

3.2.2 Transient Analysis

As in the previous example, transient analysis arises in the application of the Newmark Method, as described in Bathe (1996), using mass and stiffness matrices generated by the application of GFEM with different approaches. Parameters were set such as $\sqrt{\frac{E}{\rho}} = c = 1$, neglecting damping, and adopting a uniform mesh of 20 finite elements. For the time discretization, the 20 seconds analysis interval was divided in 2000 steps of 10^{-2} seconds.

The model considered for the example consists of a bar with a clamped end and the other free end, where the load is applied. Trigonometric enrichment was adopted using $\beta_1 = \frac{3\pi}{4}$ due to its performance in modal analysis as presented by Weinhardt et al. (2015).

Firstly, it is compared an approximation performed using 20-element uniform mesh with: no enrichment (FEM); standard one-layer trigonometric enrichment (GFEM); and Stable one-layer trigonometric enrichment (SGFEM). The results are presented in the Figs. 16 and 17, and the external load is applied with a value of 1N in the time range from 0 to 10^{-2} seconds. It should be noted that for the first level of enrichment, the heuristic modification coincides with the standard GFEM.

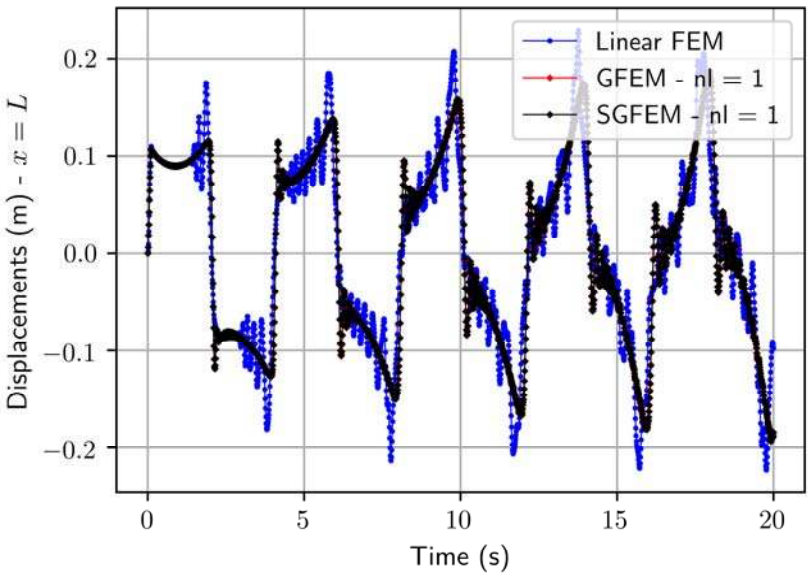


Figure 16: *Displacements over time.*

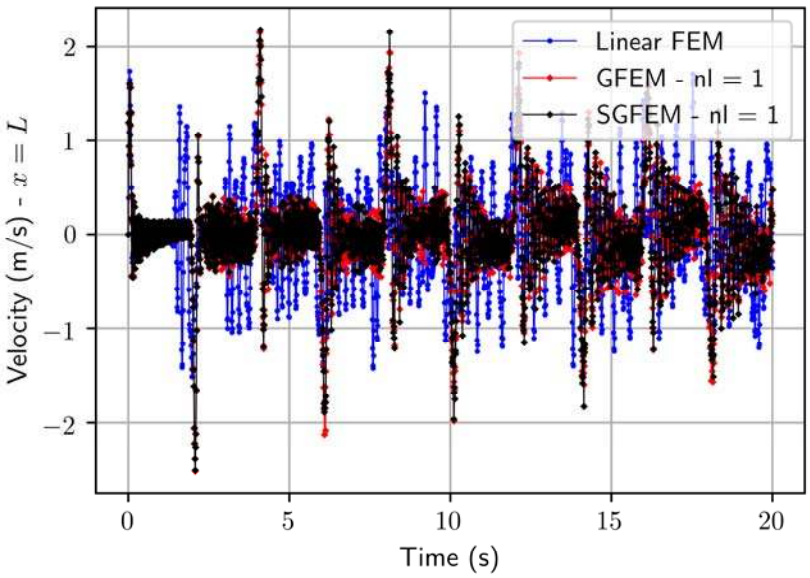


Figure 17: *Velocities over time.*

Raising enrichment layer, the results obtained are presented in Figs. 18 and 19 for: GFEM; SGFEM; and Heuristically Modified GFEM.

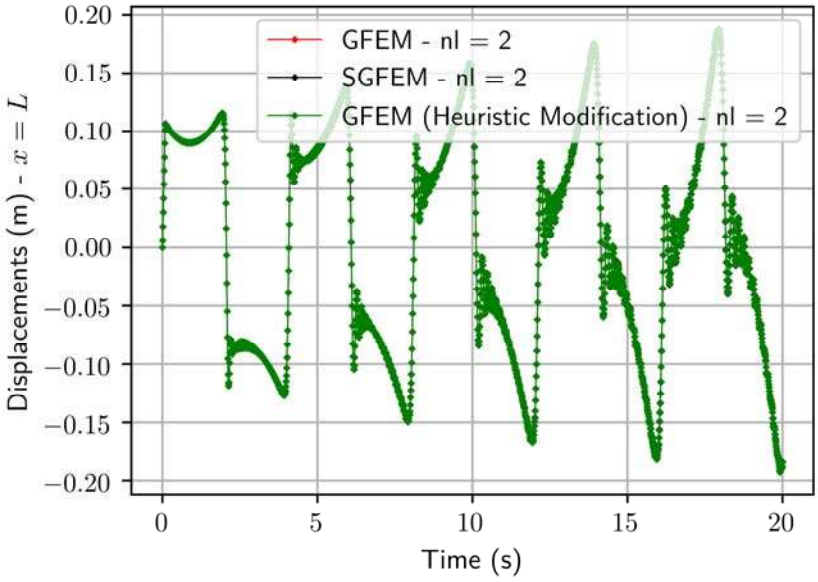


Figure 18: *Displacements over time.*

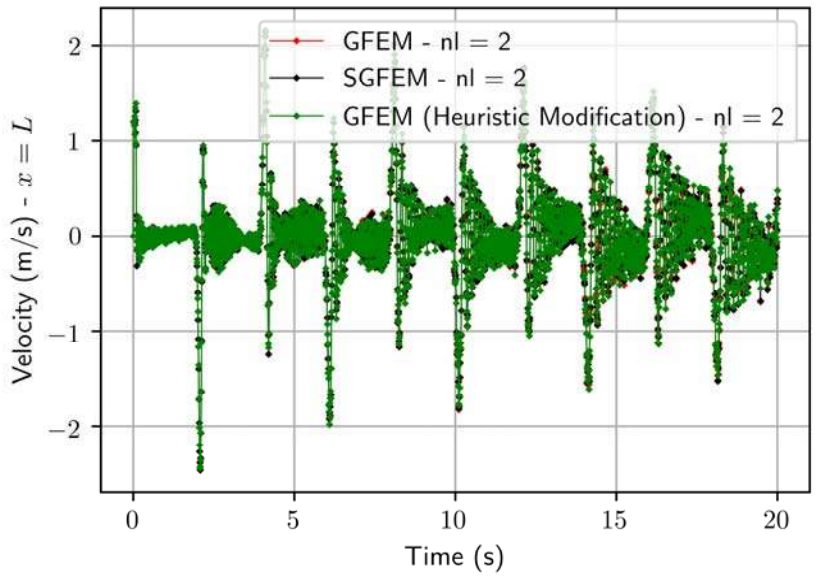


Figure 19: *Velocities over time.*

Continuing the process of successive enrichment, Figs. 20 and 21 present the respective results.

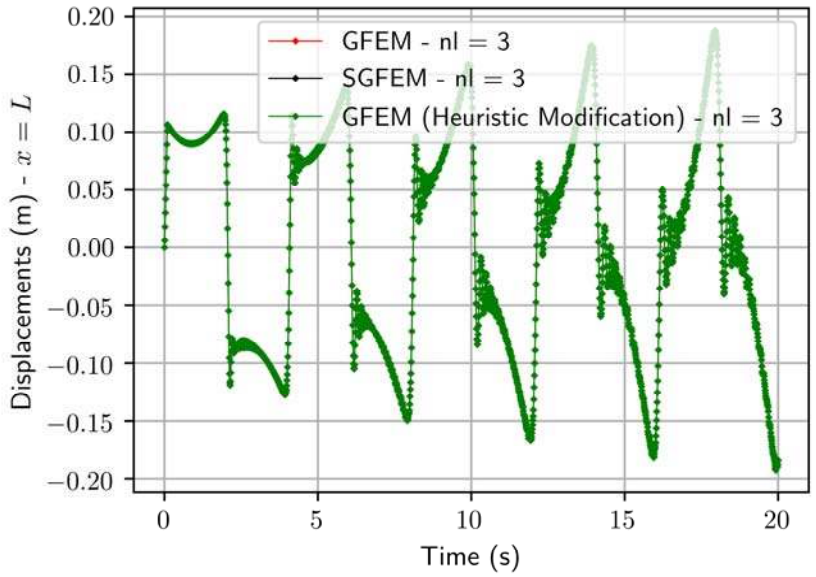


Figure 20: *Displacements over time.*

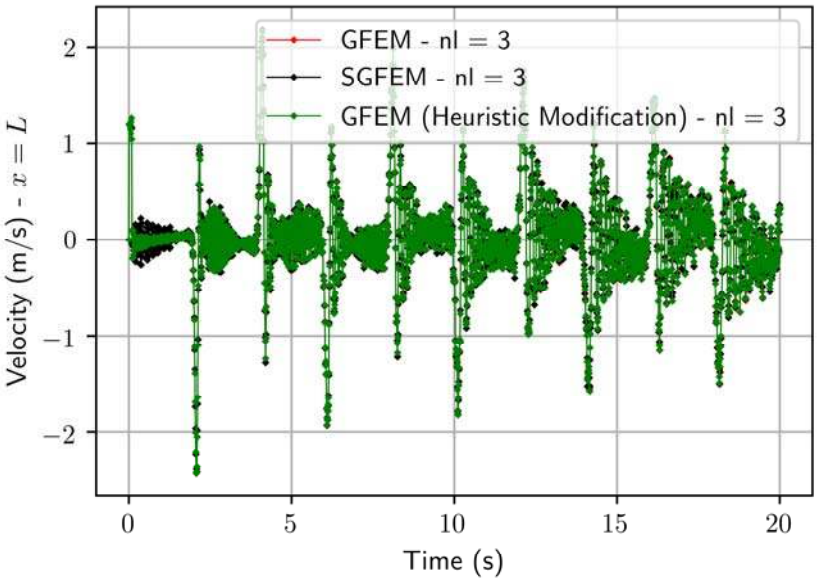


Figure 21: *Velocities over time.*

For 4 enrichment layers results are shown in Figs. 22 and 23.

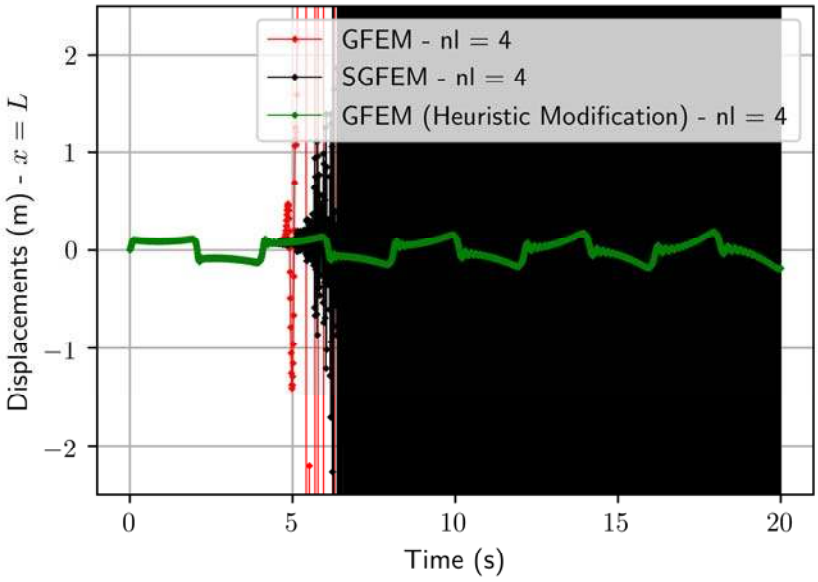


Figure 22: *Displacements over time*

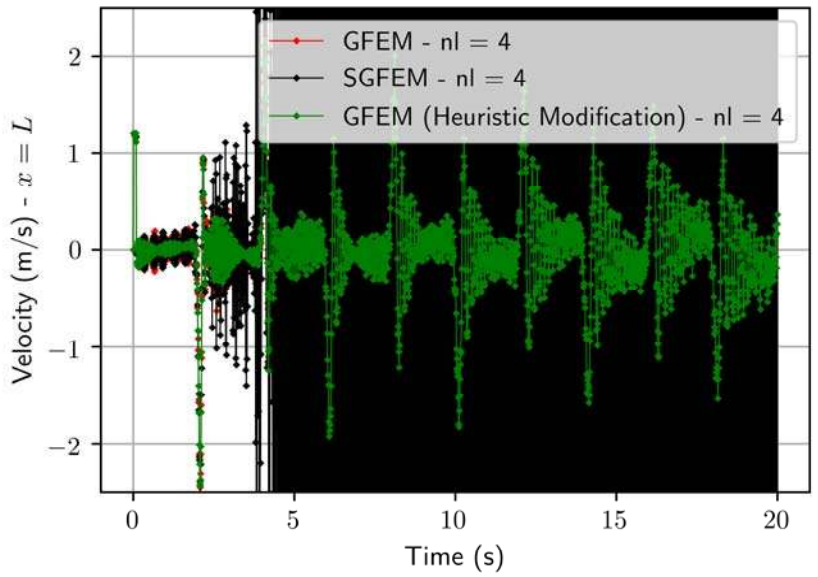


Figure 23: *Velocities over time.*

Results aiming to highlight the stabilization proposal are presented in Fig. 24 and Fig. 25, taking up to refinement to high-order.

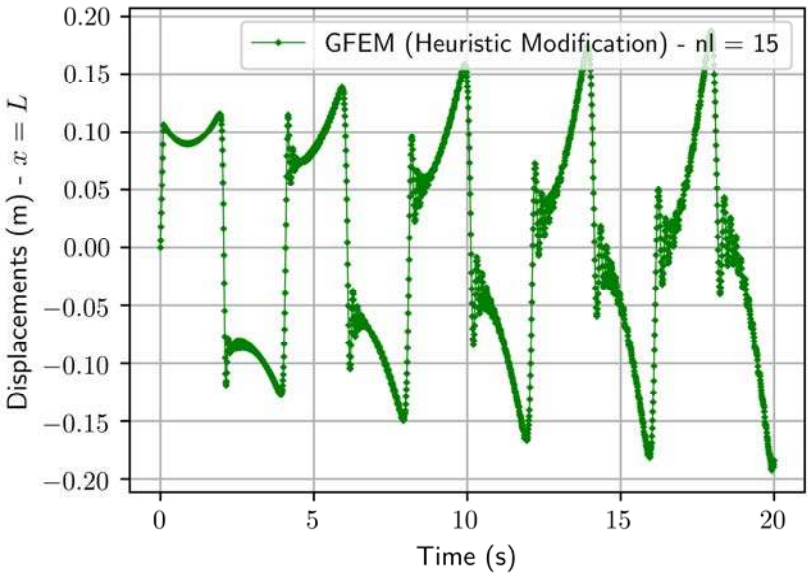


Figure 24: *Displacements over time*

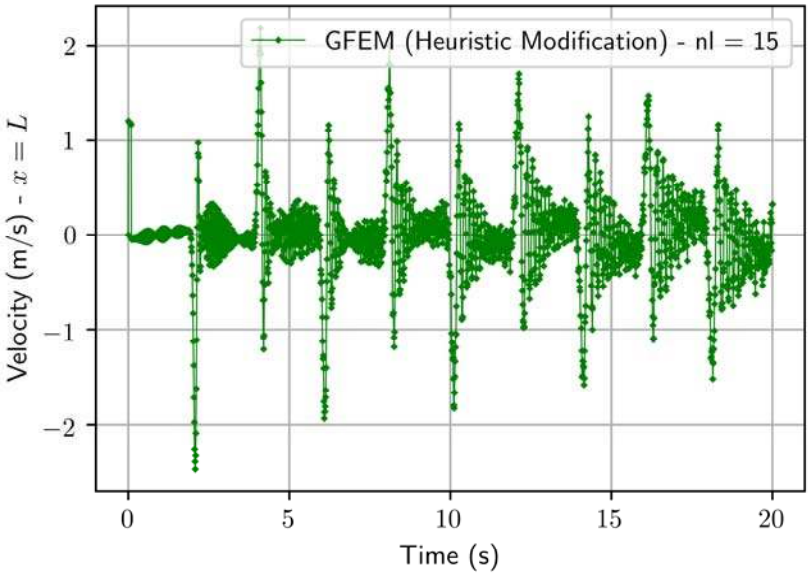


Figure 25: *Velocities over time.*

The Fig. 26 presents the evolution of the condition number of the mass matrix related to a 20-element uniform mesh successively enriched for different stabilization strategies, trying to tie the observations of numerical instability in the transient analysis with the condition number of mass matrices.

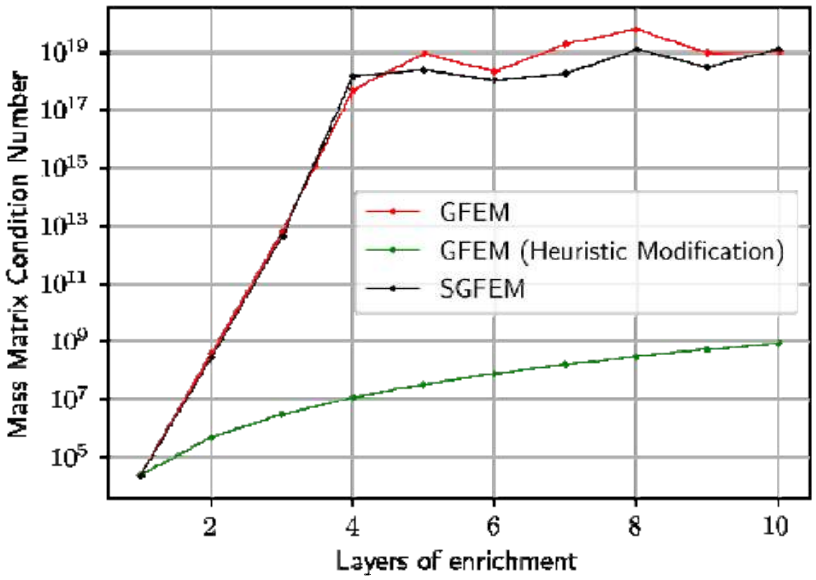


Figure 26: *Evolution of Mass Matrix Condition Number (Numerically Calculated) – Second example.*

3.3 Behavior Analysis

The same methodology of numerical experimentation was adopted for both bar examples with variation of polynomial and sinusoidal cross section and the results analysis were described. A central point of this analysis is the observation that although the two examples have different responses and characteristics, the results in terms of the stabilization procedure were similar, which highlights the technique's robustness in this family of problems.

The impulse loading case was studied and responses were presented for displacements and velocities. Acceleration was omitted since it is poorly approximated using continuous functions. Firstly, the approximation was made by one layer of enrichment, concerning GFEM, SGFEM, and FEM. As shown in Figs. 4, 5, 16 and 17, displacements response has large oscillations for Linear FEM and a more accurate behavior for all enriched approaches. On the other hand, the answer in terms of velocities presents great disturbance for the three alternative approaches.

Results for displacements and velocities tend to present more accurate behavior as analytical refinement is performed, as shown in Figs. 6, 7, 18 and 19 for 2 layers and Figs. 8, 9, 20 and 21 for 3 layers of enrichment.

Applying 4 layers of enrichment, GFEM, SGFEM and the Heuristic Modification were compared, as shown in Figs. 10, 11, 22 and 23. The corresponding approximations for displacements are considerably close and more accurate than the previous results, and the Heuristic Modification resulted in a fairly accurate approximation for the first time steps. However, the velocities response for SGFEM presented a pretty deteriorated behavior over time.

Testing the stability of the proposed Heuristic Modification, Figs. 12, 13, 24 and 25 show the results of application of 15 layers of enrichment. In these examples the high order refinement did not result in significant gains in accuracy for both displacements and velocities. However, it is worth noting that the application of many enrichment layers did not compromised the stability of the Heuristic Modification numerical approach, following the trend shown previously in modal analysis as presented by Weinhardt et al. (2015), Weinhardt et al. (2016) and Weinhardt et al. (2017).

4. CONCLUDING REMARKS

This paper discussed issues which are relevant to the stability of the Generalized Finite Element Method applied to dynamic analysis. One-dimensional bars with non-uniform cross area were presented contemplating modal and transient analysis. GFEM formulation with trigonometric enrichment was based on proposals of Arndt (2009) and Torii (2012).

Seeking to address the stability issue, two stabilization alternatives were analyzed. The first was based on an adaptation of the Stabilized Generalized Finite Element Method, initially proposed for problems that lie in the

solution of equation systems by Babuška and Banerjee (2012). The second proposal was the modification of the parameter βj present in GFEM trigonometric functions enrichment proposed by Arndt (2009) and Torii (2012).

For transient analysis it was used the Newmark method reusing the already calculated mass and stiffness matrices generated by GFEM. Although the results presented inherent disturbances of Newmark method, it was possible to compare the proposals for approximation among each other, and the second proposal stabilization stood out in most examples. The possibility of making high-order return fines with consequent improvement in response without affecting the CFL stability condition (Courant-Friedrichs-Lewy, De Moura and Kubrusly (2012)) was presented.

The results of this work point out that there are ways to overcome instability problems in GFEM applied to dynamic analysis, since simple proposals were able to positively impact the approaches.

Additionally, it was possible to observe that the Heuristic Modification stood out in stabilizing the problem, despite the non-uniform cross section. Therefore, it is possible to suppose that stabilization strategies similar to those presented in this paper may be extended to other applications.

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