www.lajss.org
Latin American Journal of Solids and Structures 2 (2005) 17–27

A nonlocal formulation applied to ductile damage

E. Lorentz* and V. Cano

EDF-R&D, 1 av. du Général de Gaulle, 92141 Clamart cedex, France

Abstract

This paper presents an approach to predict ductile fracture of real-life structures. It relies on Rousselier's constitutive model to describe plastic void growth, a specific finite strain formulation that preserves energetic properties and a nonlocal theory to deal with strain localisation. It is applied to compute the fracture of a notched specimen.

1 Introduction

This work aims at predicting the ductile fracture of real-life steel structures, from the inception of damaged zones to their ultimate propagation. Three main points have to be studied to achieve this purpose: the theory of finite transformations, the constitutive law to model plastic void growth and the nonlocal formulation to control the localisation phenomenon. We aim at building the whole modelling strategy around a principle of energy minimisation which gives rise to interesting properties ranging from the physics up to numerical solutions. It leads us to apply a new finite strain theory, Rousselier's law and a nonlocal formulation based on the gradient of internal variables. Finally, the numerical simulation of a notched specimen demonstrates the potency of the combination of the three components.

2 Finite strain formulation

2.1 A Maximal dissipation-based finite strain theory

To gain the energetic properties mentioned in the introduction, the plastic finite strain theory has to rely on the principle of maximal dissipation. Besides, we restrict our attention to isotropic constitutive laws under isothermal loading conditions. In this context, a theory has been proposed in [3] that fulfils the demand. It is inspired by [5] and it relies actually on three assumptions.

* Corresponding author E-mail: eric.lorentz@edf.fr
Received 28 October 2004
From *Recent Developments in the Modelling of Rupture in Solids Conference*, ed. A. Benallal & S.P.B. Proença.

Notation

A	driving forces
a	internal variables
c	nonlocal parameter (N)
D	global dissipation potential (J)
D	*yield surface parameter* (dimensionless)
$\mathbf{D}$	eulerian strain rate (s^{-1})
$\mathbf{D^p}$	plastic strain rate (s^{-1})
$\mathbf{e}$	elastic strain (dimensionless)
$\mathbf{F}$	deformation (dimensionless)
f	porosity (dimensionless)
f_0	initial porosity (dimensionless)
F	global free energy (J)
$\mathbf{F^e}$	elastic deformation (dimensionless)
$\mathbf{F^p}$	plastic deformation (dimensionless)
G	yield function (MPa)
$\mathbf{G^p}$	plastic strain (dimensionless)
I_K	indicator function (dimensionless)
l_c	characteristic length (mm)
p	hardening variable (dimensionless)
$R(p)$	hardening stress function (MPa)
$\mathbf{s}$	driving force associated to $\mathbf{e}$ (MPa)
Φ	local free energy (MPa)
Φ^{el}	elastic energy (MPa)
Φ^{st}	(hardening) stored energy (MPa)
λ	plastic multiplier (dimensionless)
Ω	body domain
$\partial\Omega$	domain boundary
Ψ	local dissipation potential (MPa)
σ^y	yield stress (MPa)
σ_1	yield surface parameter (MPa)

Superscript

E	elastic trial

Multiplicative split of the total deformation. The deformation $\mathbf{F}$ is split in a plastic and an elastic part through the introduction of a relaxed configuration (where the stress is zero), leading to the classical definition of the plastic deformation $\mathbf{F^p}$ and the elastic deformation $\mathbf{F^e}$:

$$\mathbf{F} = \mathbf{F^e F^p} \tag{1}$$

Partition of the free energy. The free energy Φ is split in two terms: the stored energy Φ^{st} which depends on the set of hardening internal variables a and the elastic energy Φ^{el} that depends only on the elastic deformation. Actually, the latter can be expressed without loss of generality as a function of an eulerian strain measure $\mathbf{e}$ since the elasticity is isotropic. The partition reads:

$$\Phi(\mathbf{e}, a) = \Phi^{el}(\mathbf{e}) + \Phi^{\mathrm{st}}(a) \quad \text{with} \quad \mathbf{e} = \frac{1}{2}\left(\mathbf{Id} - \mathbf{F^e}\,\mathbf{F^{e^{\mathrm{T}}}}\right) \tag{2}$$

A straightforward calculation shows that it results in the following expression for the intrinsic dissipation

$$\begin{aligned} &Diss = (\tau - \mathbf{s}(\mathbf{Id} - 2\,\mathbf{e})) \cdot \mathbf{D} - \frac{1}{2}\mathbf{s} \cdot \left(\mathbf{F}\,\dot{\mathbf{G}}^{\mathbf{p}}\,\mathbf{F}^T\right) + A\,\dot{a} \\ &\text{with} \quad \mathbf{G^p} \underset{def}{=} \left(\mathbf{F^{p^{\mathrm{T}}}}\,\mathbf{F^p}\right)^{-1} \quad \mathbf{s} \underset{def}{=} -\frac{\partial \Phi}{\partial \mathbf{e}} \quad A \underset{def}{=} -\frac{\partial \Phi}{\partial a} \end{aligned} \tag{3}$$

where τ and $\mathbf{D}$ denote respectively Kirchhoff's stress and the Eulerian strain rate. We can notice that a plastic strain measure $\mathbf{G^P}$ and driving forces $\mathbf{s}$ and A associated to $\mathbf{e}$ and a are naturally defined in the process. As the dissipation is required to be zero for elastic evolutions, the following stress - strain relation is obtained:

$$\tau = \mathbf{s}\,(\mathbf{Id} - 2\mathbf{e}) \tag{4}$$

Definition of the yield surface. The yield surface is defined by $G(\mathbf{s}, A) = 0$. It should be noticed that the convex yield function G depends of the driving forces $\mathbf{s}$ and A defined in Eq.(3). This is a difference with other theories where the stress measure which is involved in the definition of the yield surface is either Cauchy's stress or Kirchhoff's stress. However, for small elastic strain $\mathbf{e}$, the driving force $\mathbf{s}$ is close to Kirchhoff's stress, thanks to Eq.(4). Now, applying the principle of maximal dissipation with respect to the yield surface $G(\mathbf{s}, A) = 0$ leads to the evolution laws:

$$\begin{cases} -\frac{1}{2}\,\mathbf{F}\,\dot{\mathbf{G}}^{\mathbf{p}}\,\mathbf{F}^T = \lambda\,\frac{\partial G}{\partial \mathbf{s}} \\ \dot{p} = \lambda \dfrac{\partial G}{\partial A} \\ \lambda \geq 0 \quad G \leq 0 \quad \lambda G = 0 \end{cases} \tag{5}$$

2.2 Temporal integration: minimisation of an energy

The temporal integration relies on the classical procedure proposed in [6] when making use of Euler's backward scheme: first, compute the elastic trial (incremental elasticity without updating the internal variables $\mathbf{G^P}$ and a) then correct the internal variables if the threshold is crossed (correction phase). More precisely, for a given time-step, we denote q^E, q and $\Delta^E q = q - q^E$ the elastic trial of a quantity q (explicitly computed), its value at the end of the time-step and its plastic correction. Then, the temporal integration of the flow rules and the yield condition Eq.5 results in the following non linear system for the correction phase:

$$\left\{\begin{array}{l} \mathbf{s} = -\dfrac{\partial \Phi}{\partial \mathbf{e}}(\mathbf{e}, a) \\ A = -\dfrac{\partial \Phi}{\partial a}(\mathbf{e}, a) \end{array}\right. \qquad \left\{\begin{array}{l} \Delta^E \mathbf{e} = \lambda \dfrac{\partial G}{\partial \mathbf{s}}(\mathbf{s}, A) \\ \Delta^E a = \lambda \dfrac{\partial G}{\partial A}(\mathbf{s}, A) \\ \lambda \geq 0 \quad G(\mathbf{s}, A) \leq 0 \quad \lambda G(\mathbf{s}, A) = 0 \end{array}\right. \tag{6}$$

The closeness with the temporal integration for small strain plasticity is obvious: the specificities of the finite transformation are confined in the elastic trial and the final computation of Cauchy's stress. That's why the solution of this non linear system admits a variational characterisation, in the same way as small strain generalised standard materials:

$$\begin{array}{c} \Delta^E \mathbf{e},\ \Delta^E a \text{ are solutions of:} \\ \min\limits_{\Delta^E \mathbf{e},\, \Delta^E a} \left[\Phi\left(\mathbf{e}^E + \Delta^E \mathbf{e}, p^E + \Delta^E p\right) + \Psi\left(\Delta^E \mathbf{e}, \Delta^E p\right)\right] \end{array} \tag{7}$$

where the dissipation potential Ψ is defined through a Legendre transform of the yield function G:

$$\Psi(\mathbf{D^p}, \dot{a}) \underset{def.}{=} \max_{\substack{s,A \\ G(s,\,A) \leq 0}} (\mathbf{s} \cdot \mathbf{D^p} + A\,\dot{a}) \tag{8}$$

3 Application to Rousselier's law

In our opinion, the essential feature of Rousselier's law [4] to model the physics of plastic void growth lies in its yield criterion which constraints as well von Mises stress τ_{eq} as the stress trace:

$$\tau_{eq} + \sigma_1\, D\, f \exp\left(\frac{tr\tau}{3\,\sigma_1}\right) - \sigma^y - R(p) \leq 0 \tag{9}$$

An isotropic hardening effect is introduced through the hardening variable p, while the shape of the yield surface also evolves with the porosity f. σ^y, σ_1 and D are material parameters; $R(p)$ denotes the hardening curve. Finally, the plastic flow is given by a normality rule with respect to the yield surface. Therefore, for a given porosity, Rousselier's law may be cast into the finite

strain theory of sub-section 2.1 with the following (usual) free energy and yield function (where Kirchhoff's stress has been replaced by the driving force $\mathbf{s}$):

$$\Phi\left(\mathbf{e},\ p\right) = \underbrace{\frac{1}{2}\left[K\left(tr\mathbf{e}\right)^2 + 2\mu\,\mathbf{e}^D\cdot\mathbf{e}^D\right]}_{\Phi^{el}(\mathbf{e})} + \underbrace{\int_0^p R\left(s\right)ds}_{\Phi^{\mathrm{st}}(p)} \tag{10}$$

$$G\left(\mathbf{s}, A\ ;\ f\right) = s_{eq} + \sigma_1\, D\, f \exp\left(\frac{tr\mathbf{s}}{3\,\sigma_1}\right) + A - \sigma^y \tag{11}$$

If needed, the dissipation potential is expressed through a Legendre transform of G, where I_K denotes the indicator function of a convex K ($+\infty$ outside K, 0 inside):

$$\begin{aligned}\Psi\left(\mathbf{D^p}, \dot{p}\ ;\ f\right) = {} & \sigma^y\dot{p} + \sigma_1 tr\mathbf{D^p}\left(\ln\frac{tr\mathbf{D^p}}{D\,f\,\dot{p}} - 1\right) \\ & + I_{\mathrm{IR}^+}\left(tr\mathbf{D^p}\right) + I_{\mathrm{IR}^+}\left(\dot{p} - \tfrac{2}{3}D^p_{eq}\right)\end{aligned} \tag{12}$$

The question of the evolution of the porosity has not yet been raised. In fact, the microstructural interpretation relies generally on an assumption of incompressibility for the matrix, including its elastic part. In that case, the porosity is expressed as a function of the macroscopic deformation and the initial porosity f_0:

$$\dot{f} = (1-f)\,tr\mathbf{D} \Leftrightarrow \det\mathbf{F} = \frac{1-f_0}{1-f} \tag{13}$$

Therefore, the porosity is known when integrating the internal variables through Eq.(6). However, the elastic compressibility has to be taken into account to avoid locking phenomena in hydrostatic compression. Most authors then introduce an incremental expression based on the hydrostatic plastic flow (measured in the relaxed configuration):

$$\dot{f} = (1-f)\,tr\mathbf{D^p} \tag{14}$$

Therefore, this evolution equation should be coupled to those which govern the evolution of the internal variables, thus breaking the energetic property Eq.(7). But we think that Eq.(14) is only another approximation of the real porosity evolution, not better than Eq.(13). That's why we keep the interesting part of each of them: the porosity is considered set when integrating the internal variables (leading to an uncoupled treatment), and it is finally up-dated through the following expression which involves the eulerian hydrostatic plastic flow:

$$\dot{f} = (1-f)\ tr\left(-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^{\mathbf{p}}\mathbf{F}^T\right) = (1-f)\ tr\left(\mathbf{F^e}\mathbf{D^p}\mathbf{F^{e^{\mathrm{T}}}}\right) \tag{15}$$

4 Nonlocal formulation

4.1 Variational principle

To control the high spatial variations of the mechanical fields resulting from localisation, a gradient law is derived from Rousselier's local law. We choose to introduce the gradient of the hardening variable p, which proves sufficient to stabilise the localisation of all the mechanical fields, including the porosity. On the basis of [1], the gradient is introduced through a quadratic term at the structural level, resulting in new definitions of a global free energy and a global dissipation potential which depend on the field of state variables:

$$\begin{aligned} F\left(\mathbf{e},p\right) &= \int_{\Omega} \Phi\left(\mathbf{e}(x),p(x)\right) + \frac{1}{2}c\,\nabla p(x)\cdot\nabla p(x) \quad dx \\ D\left(\mathbf{D^p},\dot{p}\right) &= \int_{\Omega} \Psi\left(\mathbf{D^p}(x),\dot{p}(x)\right) \quad dx \end{aligned} \tag{16}$$

where Ω denotes the body domain in the initial configuration and c a new material parameter. The minimisation property Eq.(7) remains applicable, even though it is now expressed in function spaces since the variables are fields and no more pointwise values. It results in the following minimisation problem, while the evolution of the porosity which has been treated as a parameter is still up-dated through Eq.(15):

$$\begin{array}{c} \Delta^E\mathbf{e},\Delta^E p \text{ are solutions of:} \\ \min\limits_{\Delta^E\mathbf{e},\,\Delta^E p} \left[F\left(\mathbf{e}^E+\Delta^E\mathbf{e},p^E+\Delta^E p\right)+D\left(\Delta^E\mathbf{e},\Delta^E p\right)\right] \end{array} \tag{17}$$

4.2 Pointwise Interpretation

The characterisation of the minimum Eq.(17) is given by the following variational expression, where the subgradient ∂D has to be introduced instead of the usual derivative because the dissipation potential is not differentiable:

$$\begin{aligned} &-\left(\frac{\partial F}{\partial \mathbf{e}},\ \frac{\partial F}{\partial p}\right) \in \partial D\left(\Delta^E\mathbf{e},\ \Delta^E p\right) \\ \text{with}\quad &\left\langle \frac{\partial F}{\partial p}\ \middle|\ \delta p\right\rangle = \int_{\partial\Omega} c\,\nabla p\cdot\mathbf{n}\,\delta p \quad + \int_{\Omega}\left(\frac{\partial\Phi}{\partial p}-c\,\Delta p\right)\,\delta p \end{aligned} \tag{18}$$

with $\partial\Omega$ the boundary of the body domain and $\mathbf{n}$ its outer normal. Actually, the only nonlocal term results from the variation of the global free energy with respect to the hardening variable, since the dissipation potential depends only on local contributions. As shown in Eq.(18), an integration by parts expresses the nonlocality in terms of a surface integral that should be zero (boundary condition), as well as any interface integral, and a bulk integral in which appear a

corrected local driving force: the laplacian of the hardening variable appears in its expression. Therefore, a pointwise characterisation of the minimum Eq.(17) simply reads, outside of the singular point of the yield surface:

$$\left\{\begin{array}{c} \mathbf{s} = -K\left(tr\mathbf{e}\right)\mathbf{Id} - 2\mu\,\mathbf{e}^D \\ A = -R\left(p\right) + c\,\Delta p \end{array}\right. \qquad \left\{\begin{array}{c} \Delta^E\mathbf{e} = \lambda\frac{\partial G}{\partial s}\left(\mathbf{s}, A\ ;\ f^E\right) \\ \Delta^E p = \lambda \end{array}\right. \tag{19}$$
$$\lambda \geq 0 \quad G\left(\mathbf{s}, A\ ;\ f^E\right) \leq 0 \quad \lambda G\left(\mathbf{s}, A\ ;\ f^E\right) = 0$$

In addition, the boundary condition and the natural interface condition across any surface I of normal ν are:

$$\nabla p \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ and } \left[c\,\nabla p \cdot \nu\right]_I = 0 \tag{20}$$

Moreover, the quadratic term in ∇p hints at seeking the field p in the function space $H^1\left(\Omega\right)$, so that the essential interface condition is:

$$\left[p\right]_I = 0 \tag{21}$$

5 Numerical application

To analyse the characteristics of such a gradient law, a numerical simulation is carried out. A notched specimen is submitted to tension, see Figure 1 for the geometry, the loading and the material parameters. The computations benefit from the axial and the plane symmetries. They are led with two meshes, one with 0.25 mm element size in the localisation zone (coarse mesh), and the other with 0.1 mm element size. The minimisation of Eq.(17) is handled by the algorithm presented in [2], which has already proven its robustness in the context of gradient brittle damage simulations.

The numerical results are presented in terms of the global force – displacement response (Figure 2 and the distribution of the porosity during the propagation of a damaged zone (Figure 3). They are also compared to purely local computations. The global response shows the good independence of the results with respect to the mesh refinement, on the contrary of the local model. This is highlighted when observing the distribution of the porosity: the localisation band spreads over several finite elements (Gauss point visualisation to avoid any artefact related to a graphical extrapolation) whose width is not set by the mesh refinement, on the contrary again of the local computations.

These results seem promising regarding the potencies of this gradient law. Other computations are at stage to evaluate the physics of this model through comparisons with experimental results.

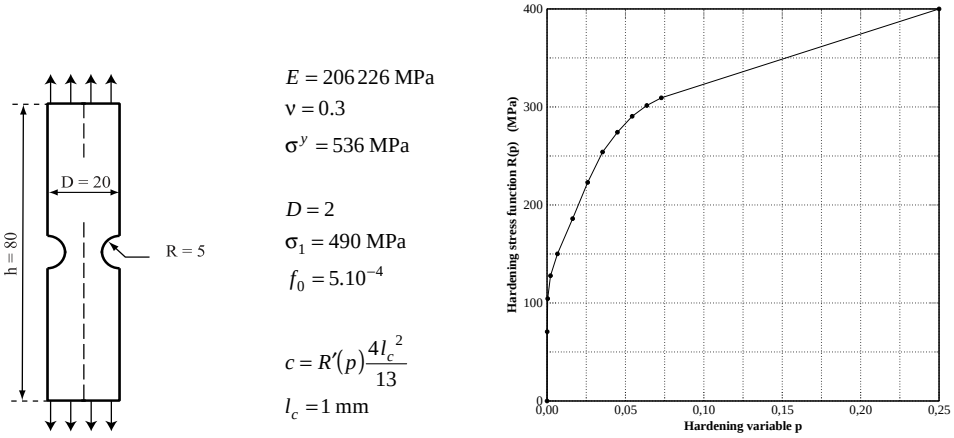


Figure 1: Geometry, loading and material parameters

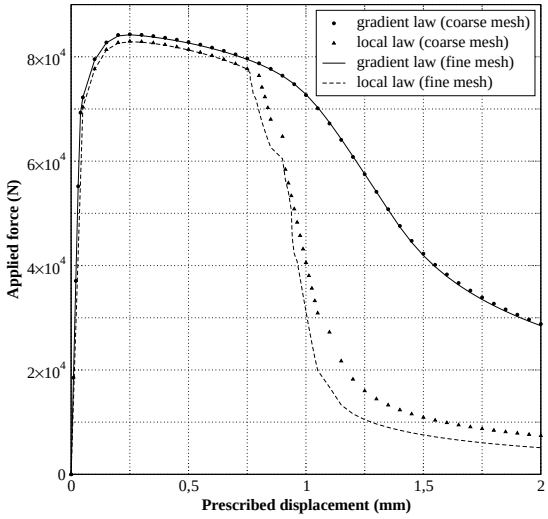
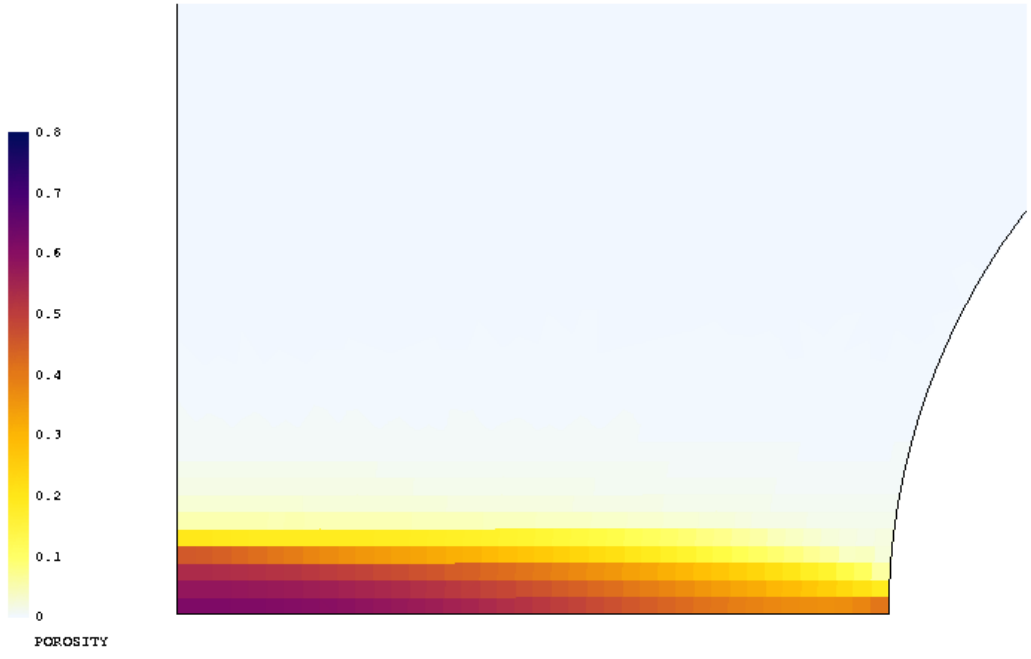


Figure 2: Force – displacement response





(cont.)

Figure 3: Porosity field (zoom) corresponding to an applied force of 40000 N





Figure 3: Porosity field (zoom) corresponding to an applied force of 40000 N (continued)

References

[1] E. Lorentz and S. Andrieux. A variational formulation for nonlocal damage models. *Int. J. Plas.*, 15:119–138, 1999.

[2] E. Lorentz and A. Benallal. Gradient constitutive relations: Numerical aspects and application to gradient damage. *submitted in Comp. Meth. Appl. Mech. Engng.*

[3] E. Lorentz and V. Cano. A minimization principle for finite strain plasticity: Incremental objectivity and immediate implementation. *Comm. Num. Meth. Engng.*, 18:851–859, 2002.

[4] G. Rousselier. Finite deformation constitutive relation including ductile fracture damage. In Nemat Nasser, editor, *3D Constitutive Relations and Ductile Fracture*, pages 331–355, North Holland, 1981.

[5] J. C. Simo and C. Miehe. Associative coupled thermoplasticity at finite strains: Formulation, numerical analysis and implementation. *Comp. Meth. Appl. Mech. Engng.*, 98:41–104, 1992.

[6] J. C. Simo and R. L. Taylor. Consistent tangent operators for rate-independent elastoplasticity. *Comp. Meth. Appl. Mech. Engng.*, 48:101–118, 1985.