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Griffith theory of brittle fracture revisited: merits and drawbacks

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Abstract

A variational reformulation of Griffith's theory of brittle fracture is proposed. At the expense of a slight departure from the classical theory, the new formulation decisively addresses three major issues: crack initiation, crack path, and smoothness of the crack evolution. The new formulation is amenable to numerical implementation; a simulation qualitatively in agreement with experimental data is presented. That computation is well beyond the scope of the classical theory and demonstrates the flexibility of the approach. Extensions of the formulation that cure the weaknesses of the proposed model are suggested.

In this short paper, we first briefly describe in Section 1 the variational formulation that we propose for quasistatic brittle fracture evolution. Our formulation is very close to the spirit of Griffith's approach to brittle fracture [11], and only slightly departs from the classical theory. In Section 2, we present a computation that illustrates the reach of the model. Finally, Section 3 zeroes in on the drawbacks of the formulation and on possible remedies; it focusses on the use of other surface energies.

1 The formulation

Throughout the paper, Ω denotes a smooth bounded connected open domain of $\mathbb{R}^N$, $1 \leq N \leq 3$. As such, Ω represents the crack-free reference configuration of an elastic body. In classical fracture mechanics, the family of possible cracked configurations is in general made up of a small number of smooth surfaces; our family is however very large, since it is the set of all closed subdomains of $\overline{\Omega}$, independently of their shape, subject only to the condition that their dimension is no greater than $N-1$. Note that the crack may become a boundary crack, that is that our setting incorporates potential debonding of the sample from a hard device. An energy is assigned to each member of the family. The surface energy associated to the crack $\Gamma \subset \overline{\Omega}$ is given by

$$E_s(\Gamma) = k\mathcal{H}^{N-1}(\Gamma), \tag{1}$$

where $\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdorff measure, that is a "surface" measure which amounts to the customary surface measure for smooth hypersurfaces.

Only hard devices are considered in the first two sections: the displacements are prescribed (at a value U) on a part $\partial_d\Omega$ of the boundary, and the remaining part $\partial_f\Omega$ of the boundary and

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the lips of the cracks are free (of forces), while the body loads vanish. This is not a whim on our part but an essential restriction of the model, at least in the setting of linearized elasticity. The removal of that restriction will be discussed in Section 3 below. The reader should also refer to [13] for the case of finite elasticity where that restriction is less significant.

Away from the crack Γ, the body is assumed to undergo purely elastic infinitesimal transformations, so that its bulk energy is given, for any kinematically admissible field v with U and Γ, that is all v such that

$$v = U \text{ on } \partial_d\Omega \setminus \Gamma \tag{2}$$

by

$$E_d(U;\Gamma;v) := \int_{\Omega\setminus\Gamma} 1/2Ae(v)\cdot e(v)\, dx, \tag{3}$$

where A is the stiffness matrix of the material.

The total energy of the body for a given crack Γ and a given loading U is then given by

$$E(U;\Gamma;v) = E_d(U;\Gamma;v) + E_s(\Gamma). \tag{4}$$

We now propose to follow the response of the body to a time-dependent loading. To this effect a time-parameterized loading $U(t)$ is applied to $\partial_d\Omega$. Our goal is to determine the evolution of the crack(s) during the loading, *i.e.*, to obtain the time-parameterized map $t \mapsto \Gamma(t)$ (and also that of the accompanying displacement field $u(t)$).

The basic idea is as follows. At a given time t, the pair $(u(t),\Gamma(t))$ is a minimizer for

$$E(t;\Gamma;v) := E(U(t);\Gamma;v) \tag{5}$$

among all cracks Γ which contain all previous $\Gamma(s)$, $s<t$, and all displacement fields v that are kinematically admissible with $U(t)$ and Γ in the sense of Eq.(2).

There are two important features in this formulation. First, the driving principle is global energy minimization. For a fixed crack Γ, this is nothing new; the minimality of $u(t)$ merely expresses quasistatic equilibrium. A rereading of Griffith's classical formulation would easily establish that Griffith's laws of crack propagation, together with quasistatic equilibrium, are formally equivalent to a first order necessary condition for local minimality of E when evaluated at $(u(t),\Gamma(t))$. Our "contribution" is to replace that mathematically untractable condition by a postulate of absolute minimality; the latter is more easily managed in the current environment of the calculus of variations.

On the other hand, the geometry and size of the crack is limited by its predecessors, which encodes both irreversibility of the cracking process and absence of any kind of healing.

In addition, the first law of thermodynamics must be satisfied throughout the evolution. In the current context, that translates into:

$$E(t;\Gamma(t);u(t)) = \int_{\partial\Omega_d\setminus\Gamma(t)} Ae(u(t))n.\dot{U}(t)d\mathcal{H}^{N-1}. \tag{6}$$

One should then think of $E(t;\Gamma(t);u(t))$ as the sum of the internal (elastic) energy of the sample and of the energy dissipated through crack opening. Remark that, in the absence of cracks, that condition is automatically satisfied by a quasistatic elastic evolution.

In the case of a monotone hard device, that is when

$$U(x,t) = tU_0(x), t \geq 0, \tag{7}$$

the model boils down to the following

Evolution Law *The displacement–crack pair must satisfy*
i/ $\Gamma(t) \nearrow$ *with* t*;*
ii/ $E(t;\Gamma(t);u(t)) \leq E(t;\Gamma;v)$*, for all cracks* $\Gamma \supset \cup_{s<t}\Gamma(s)$ *and displacement fields* v *kinematically admissible with* $U(t)$ *and* Γ*; and*
iii/ $E(t;\Gamma(t)) \leq E(t,\Gamma)$*, for all* $\Gamma \subset \Gamma(t)$*.*

The mechanical implications of the model are discussed at length in [10]. Most notably, crack initiation *always* occurs in finite time. The interplay between the strength of the singularity of the elastic field at a given point and the type of crack that can develop or extend from that point is also investigated in that paper; this provides a bridge between our formulation and the classical analysis of stress intensity factors originating in Irwin's work [12].

Mathematically, the model has been shown to be well–posed, that is to admit at least one pair–solution $(u(t),\Gamma(t))$, in the following cases:

1. two–dimensional linearized elasticity with connected cracks [7];
2. three–dimensional antiplane shear with no restrictions on the possible connectedness of the cracks [9];
3. three–dimensional finite elasticity with no restrictions on the possible connectedness of the craks and a mandatory presence of body and surface loads of a particular type [13].

The case of three–dimensional linearized elasticity remains open at this time.

The formulation is computation–friendly. At least two methods have been suggested. The first is based on a 'smearing" of the crack into a kind of "damaged" zone of size ϵ; the crack becomes a diffuse auxiliary variable $v(x,t) \in [0,1]$, and the sharp crack is recovered upon letting ϵ tend to 0. The underlying mathematics are discussed in [1] and [4]. The second method is drastically different. It consists in adapting the triangulation, so as to follow the crack in its evolution. The mesh becomes an unknown and mesh optimization is performed [5].

We show in the next section an example of such a computation which uses the first method outlined in the previous paragraph.

2 Numerical experiment

The computed experiment is a traction experiment on a fiber reinforced matrix. A square elastic matrix is reinforced by a rigid circular fiber as shown in the figure below. The fiber remains fixed, while a uniform displacement field te_2 is imposed on the upper side of the square; the remaining sides are traction-free.

The computed evolution shown in Figure 1 is observed as t grows.

Figure 2 shows how both bulk and surface energies change as the loading parameter t increases. Note that the total energy, that is the sum of the bulk and surface energies does not experience a jump, as predicted by the theory.

In spite of the lack of exact solution for this example, qualitative comparison with theoretical results proves fruitful; it is pursued in details in [6]. We merely point out here that the brutal onset of the cracking process is in agreement with a result obtained in [10]. Indeed, according to Subsection 4.4 in [10], if the crack initiation point x_0 is a non–singular point for the purely elastic solution, the crack must appear with finite length. Also note that the energy is (nearly) conserved during phases 2, 4, 6 of brutal growth, as theoretically expected (the total energy should in particular be continuous in time; see [9, 11]).

Finally, the Griffith law of crack evolution is satisfied during the phases 3 and 5 of progressive growth of the crack. Figure 3 below is an attempt at checking this. Indeed, in our setting that law reads as

$$t^2 \frac{d\mathcal{E}}{dl}(l(t)) + k = 0, \tag{8}$$

where $l(t)$ is the length of the crack $\Gamma(t)$ and, recalling Eq.(3),

$$\mathcal{E}(l) := \min_{v=U \text{ on upper edge}} E_d(1; \Gamma(t); v). \tag{9}$$

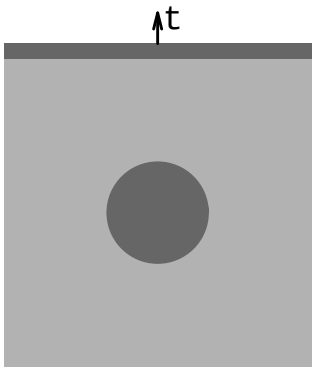
Thus it is easily deduced from Eq.(8) that

$$\frac{dE}{dt}(t; \Gamma(t); u(t)) - \frac{2E_d(U(t); \Gamma(t); u(t))}{t} = 0. \tag{10}$$

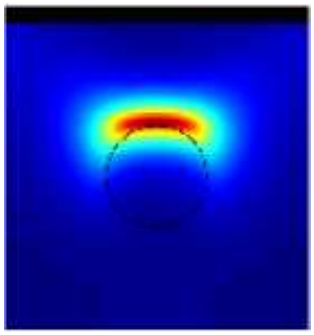
In Figure 3, $\dfrac{dE}{dt} - 2E_d/t$ is plotted as a function of t, and we see that the criterion given by Eq.(10) is indeed met during the progressive phases of the evolution, while it is not satisfied at $t = 0.32$ and at $t = 0.37$, that is when the right and left ligaments break. Note however that it is nearly satisfied at $t = 0.2$.

This numerical experiment is exemplary because of the wide range of associated crack behaviors throughout the evolution: brutal versus progressive fracture, boundary versus bulk crack, symmetric versus asymmetric path, curvilinear crack, ...

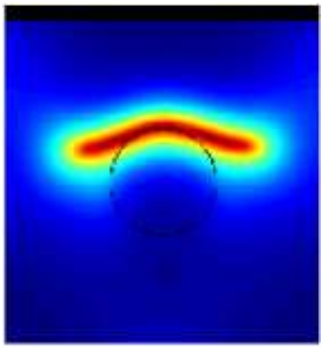
Other experiments are described in [6]; a fascinating decohesion problem is also numerically addressed in [3].



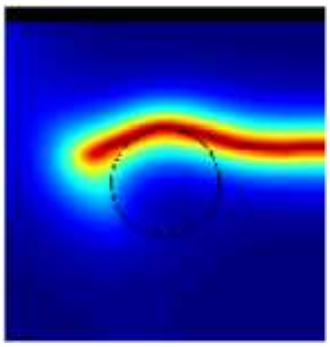
Phase 1 : Elastic response. If $t < 0.2$, the matrix remains purely elastic.



Phase 2 : Brutal onset. At $t \sim 0.2$, a crack of finite length brutally appears near the top of the inclusion. The crack is symmetric with respect to the 2 axis; it is not straight. This is confirmed in Figure 2 with the sudden increase/decrease of the surface/bulk energies at that load.

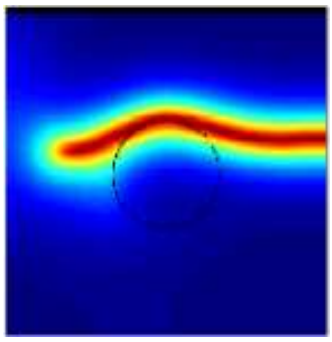


Phase 3 : Progressive and symmetric evolution of the crack. When t varies between 0.2 and 0.32, the crack progressively grows in the matrix. The evolution is smooth as shown in Figure 2 where the surface energy increases smoothly, while the bulk energy is nearly constant. The propagation is symmetric but not straight.

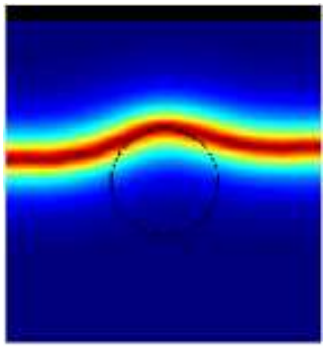


Phase 4 : Rupture of the right ligament. At $t \sim 0.32$, the right hand-side of the matrix is brutally cut. The brutal character of the phenomenon is evident on Figure 2 with a jump discontinuity of both surface and bulk energies. The crack is no longer symmetric.

Figure 1: Computed evolution (continued)



Phase 5 : Progressive propagation of the left branch. When t varies between 0.32 and 0.37, the left part of the crack progressively grows, the surface energy slowly increases, whereas the bulk energy remains constant.



Phase 6 : Rupture of the left ligament. At $t \sim 0.37$, the crack brutally severs the remaining filament of uncracked material. In Figure 2, the corresponding energy jumps are clearly evidenced.

Figure 1: Computed evolution (cont)

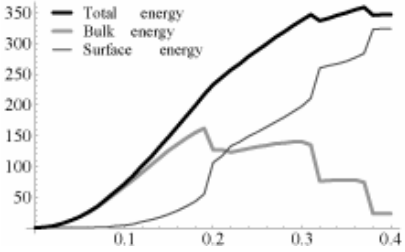


Figure 2: Evolution of the energies with the load.

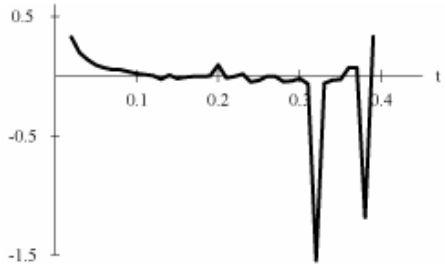


Figure 3: Verification of the Griffith law.

3 Drawbacks and remedies

The generic inability of the formulation to handle body loads or soft devices is the main drawback of the proposed formulation. This is easily understood. If for example a traction load g is applied to a part $\partial\Omega_f$ of the boundary, then a term of the form $-\int_{\partial\Omega_f} g \cdot v d\mathcal{H}^{N-1}$ has to be added to the expression for the bulk energy E_d. But then, the minimization of E in Eq.(4) with respect to Γ, v is straightforward: cut–off $\partial\Omega_f$ from Ω with $\Gamma = \partial\Omega_f$, which creates a surface energy $E_s(\Gamma) = \mathcal{H}^{N-1}(\partial\Omega_f)$, and take $v \equiv \lambda g$ on $\partial\Omega_f$, which will yield a bulk energy $E_d(U;\Gamma;v) = -c\lambda$ for some constant c, hence $E(U;\Gamma;v) \searrow -\infty$ as $\lambda \nearrow \pm\infty$, and there is no global minimizer for E.

In the light of the above, it is tempting to resort to a notion of local minimality in lieu of its

global analogue. Of course, in contrast to global minimizers, local minimizers may depend upon the choice of the distance between admissible test fields. If we choose as measure of locality the total variation, i.e. if we say that

$$\begin{array}{c} u \text{ is close to } v \\ iff \\ 1 \gg \int_\Omega |u-v| dx + \int_\Gamma |(u^+ - u^-) - (v^+ - v^-)| d\mathcal{H}^{N-1} \end{array} \tag{11}$$

(where $u^\pm$ are the values of the displacement field across the crack), then, at least in a one–dimensional setting, it is established in [8] that, in a traction experiment, i.e. when a force tF is applied at one end of the bar, the other end remaining fixed, the elastic response $u(t)$ is a local minimizer for any value of the parameter t. In other words, initiation will *never* occur with such a model, and replacement of a global criterion by a local one, although it may cure the problem raised by soft devices, reintroduces in turn the issue of crack initiation.

As outlined by Griffith himself, the surface energy becomes proportional to the area of the surface of discontinuity when the distance between the lips of the crack is large compared to the characteristic atomic length. Following Barenblatt's approach [2], we may thus assume that the surface energy depends on the value of the jump of the displacement across the crack: it starts at 0 and progressively grows to its effective Griffith value k when the jump becomes large with respect to the characteristic atomic length. The surface energy is then of the following form:

$$E_s(\Gamma; v) = \int_\Gamma \kappa((v^+ - v^-)(x))\, d\mathcal{H}^{N-1}. \tag{12}$$

It has become a function of both the crack Γ and the displacement field v.

Asssuming that

$$\begin{cases} \kappa(0) = 0 \\ \kappa(s) \overset{s}{\nearrow} \text{ is twice differentiable} \\ \kappa(+\infty) = k, \end{cases} \tag{13}$$

it is then shown in [8] that — in a one–dimensional setting and in a traction experiment — the elastic response remains a local minimum for the total variation (in the sense of Eq.(11)) as long as

$$t < t_f := \frac{\kappa'(0) S}{F}, \tag{14}$$

and ceases to be a local minimum provided that $t > t_f$. In Eq(14), S denotes the cross–sectional area of the bar.

The value tF represents the reaction of the wall at its fixed end. It is the maximal tension sustained by the bar. Consequently, the bar must break when the tension at that point is greater than $\kappa'(0)S$; $\kappa(0)$ thus plays the role of a yield stress for the material.

It would thus seem that a "realistic" model should incorporate both an adequate notion of locality — a necessity if soft devices and/or body loads are considered, at least in the setting

of linearized elasticity —- and some kind of adequate cohesive surface energy which seems to provide a notion of yield stress. The mathematical and numerical hurdles in the handling of such a model are however formidable when leaving the one–dimensional setting.

In conclusion, the variational theory of brittle fracture is yet in its infancy. The mechanical, numerical and mathematical results obtained in the restrictive framework of global minimization and of a surface energy à la Griffith are deemed promising. The extension to local minimization and to cohesive energies à la Barenblatt could provide the answer to the criticisms aimed at the initial approach; it accomplishes precisely that in the one–dimensional setting. The extension to higher dimensions is essentially open at present.

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