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A numerical method for free vibration analysis of beams

Abstract

In this paper, a numerical method for solution of the free vibration of beams governed by a set of second-order ordinary differential equations of variable coefficients, with arbitrary boundary conditions, is presented. The method is based on numerical integration rather than the numerical differentiation since the highest derivatives of governing functions are chosen as the basic unknown quantities. The kernels of integral equations turn out to be Green's function of corresponding equation with homogeneous boundary conditions. The accuracy of the proposed method is demonstrated by comparing the calculated results with those available in the literature. It is shown that good accuracy can be obtained even with a relatively small number of nodes.

Keywords

Numerical method, Integral equations, Green's function, Vibration, Timoshenko beam.

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1 INTRODUCTION

The vibration analysis for structures is a very important field in engineering and computational mechanics. These dynamic problems are classically described by a system of ordinary differential equation associated with a set of boundary conditions. Mostly, these problems cannot be solved or are difficult to solve analytically. Alternatively, the numerical methods can provide approximate solutions. Numerical methods such as finite element (Thomas and Abbas (1975); Jang and Bert (1989); Klasztorny (1992); Cleghorn and Tabarrok (1992); Friedman and Kosmatka (1993); Boukhalfa and Hadjoui (2010); Hamedani et al. (2012)), finite difference (Popplewell and Chang (1996); Laura and Gutierrez (1993); Fu-le and Zhi-zhong (2007)), differential quadrature (Liu and Wu (2001); Bert et al. (1988); Bert and Malik (1996); Janghorban (2011); Mirtalaie et al. (2012); Rajasekaren (2013)), the dynamic stiffness matrix (Banerjee (1998, 2001); Li et al. (2004); Hashemi and Richard (1999)) and some other methods have been used in solving free vibration problems of structures.

The present paper follows the approach that was first proposed by Hajdin (1958), and later also used in further works by Krajcinovic and Herrmann (1970) and Hajdin and Krajcinovic (1972). The main point of proposed method for numerical solutions of the ordinary differential equations contains in choosing of the highest derivative in these equations for the basic unknown. The unknowns are determined through the corresponding integral equations which kernels are influence function for bending moment of a beam.

In the present paper, the above approach is applied to a set of second-order ordinary differential equations of variable coefficients, with arbitrary boundary conditions, and their application to the free vibration of beams. In the formation of the system of linear algebraic equations from which can be obtained a characteristic equation of natural frequencies, we use in this paper statically-kinematic analogy that exists between the generalized displacements and cross section forces in the appropriate fictitious beam. In that case, the generalized displacements can be determined by known methods of structural analysis used to determine the cross section forces. In contrast to FEM where the displacement field is prescribed, in the present method, as already mentioned, the field of highest derivative of function is assumed, so the solution is obtained through numerical integration. With a relatively small number of points it can be achieved satisfactory accuracy of the results.

2 THEORETICAL CONSIDERATIONS

Consider the problem governed by a system of homogeneous ordinary differentitial equations of second order, with suitable boundary conditions at both ends, $z = 0$ and $z = L$. This system of equations can be written in the matrix form as follow

$$\mathbf{A}\mathbf{q}'' + \mathbf{B}\mathbf{q}' + \left(\mathbf{C} + \omega^2\mathbf{D}\right)\mathbf{q} = \mathbf{0} \tag{1}$$

where **A**, **B**, **C** and **D** are known N x N matrix function of z, **q** is a vector whose elements are N unknown functions (displacement parameters)

$$\mathbf{q} = \begin{bmatrix} q_1(z) \\ q_2(z) \\ \vdots \\ q_i(z) \\ \vdots \\ q_N(z) \end{bmatrix} \tag{2}$$

and ω is circular frequency, which corresponds to the solution for the non-trivial case. Primes in equation (1) denote differentiation with respect to z.

The second derivatives of any component $q_i \, (i=1, 2,..., N)$ of vector **q**, shall be denoted

$$q_i'' = -p_i(z) \tag{3}$$

In addition to differential equation (3) the following boundary conditions are given

$$for\ z = 0 \begin{cases} q_i = q_{i0} \\ q_i' = q_{i0}' \end{cases} \tag{4}$$

The solution of Eq. (3) may be written as (Byron and Fuller 1992)

$$q_i(z) = \int_0^l s(z,\varsigma)\, p_i(\varsigma)\, d\varsigma + q_{i0}' z + q_{i0} \tag{5}$$

where ς is the integration variable and $s(s,\varsigma)$ is Green's function corresponding to the differential equation (3), which obeys homogeneous boundary conditions (4). Function $s(s,\varsigma)$, known as kernel of integral equation (5), is defined by

$$s\ \ z,\varsigma\ \ =\begin{cases} -\ \ z-\varsigma & \varsigma\leq z \\ 0 & \varsigma>z \end{cases} \tag{6}$$

If $p_i\ \ z$ is understood as external transverse load distributed along the axis of the beam, q_{i0} and q'_{i0} as a bending moment and a transversal force at the left end of the beam, respectively, and $q_i\ \ z$ as bending moment along the beam, than integral equation (5) defines the dependence between the cross section bending moment and external load of the fictitious cantilever beam of length l, fixed at the right end, Fig. 1. From this it follows that Green's function is the influence function for the bending moment of a cantilever beam, well known to civil engineering.

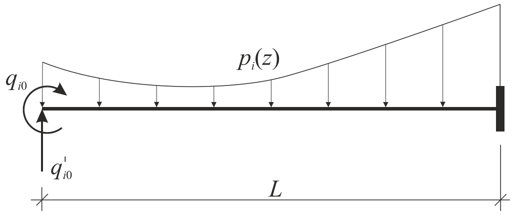

Figure 1: Fictitious cantilever beam.

Differentiating equation (5) we get

$$q'_i\ \ z\ \ =\int_0^l s'\ \ z,\varsigma\ \ p_i\ \ \varsigma\ \ d\varsigma+q'_{i0} \tag{7}$$

where

$$s'\ \ z,\varsigma\ \ =\begin{cases} -1 & \varsigma\leq z \\ 0 & \varsigma>z \end{cases} \tag{8}$$

is influence function for transversal force of the cantilever beam.

3 NUMERICAL SOLUTION

The values of definite integral in equation (5) can be expressed approximately, using method for numerical integration. Unknown function $p_i\ \ z$, which by static analogy with beam in bending represents the load, will be shown in the form of polygonal line with characteristic values p_{ik} at selected points k ($k = 0, 1, 2,.., M$) of the equidistant spacing λ

$$p_i\ \ z\ \ =\sum_{k=0}^{M}\Omega_k\ \ z\ p_{ik} \tag{9}$$

where Ω_k is linear function which exists only along parts between the point k, where it takes the value 1, and adjacent nodes, where it takes the value 0, Fig. 2. The ordinate p_{i0} of function $p_i\ \ z$ at point $k = 0$, is determined by linear extrapolation, so that $p_{i0} = 2p_{i1} - p_{i2}$.

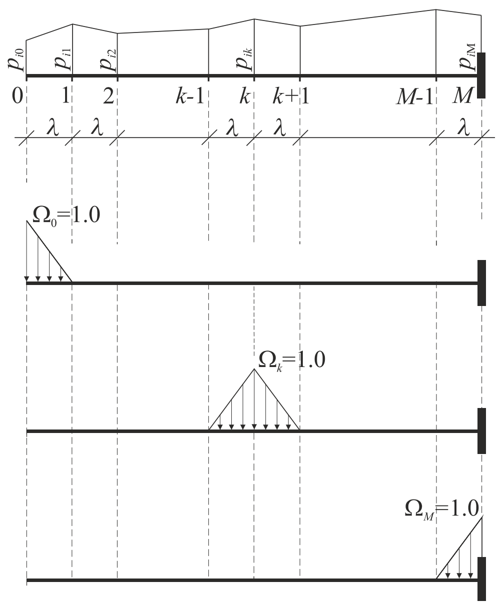

Fig. 2: **Characteristic values p_{ik} and functions Ω_i (i = 1, 2,..., M)**

Substituting Eq. (9) into Eq. (5) we obtain

$$q_i\ \ z\ \ = \sum_{k=0}^{M} S\ \ z,\varsigma_k\ \ p_{ik} + q'_{i0}z + q_{i0} \tag{10}$$

where

$$S\ \ z,\varsigma_k\ \ = \int_0^l s\ \ z,\varsigma\ \ \Omega_k d\varsigma \tag{11}$$

represents influence line for bending moment in section z of cantilever beam, caused by unit triangular load, Fig. 3.

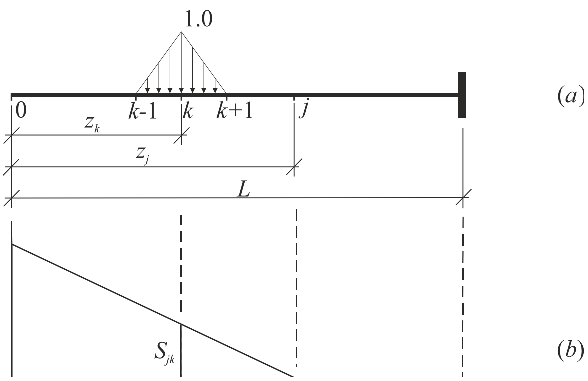

Fig. 3: (a) Unit triangular 'load' and (b) influence line for bending moment of fictitious cantilever beam.

The functional equation (10) may be transformed into a system of algebraic equations if the argument z is assigned at discrete points $j \quad j = 1, 2,..., \mathrm{M}$, in which the integration is performed

$$q_{ij} = \sum_{k=0}^{M} S_{jk} p_{ik} + q'_{i0} z_j + q_{i0} \tag{12}$$

where

$$\begin{aligned} q_{ij} &= q_i \quad z_j \\ S_{jk} &= S \quad z_j, \varsigma_k \end{aligned} \tag{13}$$

and

$$S_{jk} = \begin{cases} -\lambda^2/6 & \mathrm{j = k} \\ 0 & \mathrm{j < k} \\ - \quad j-k \quad \lambda^2 & \mathrm{j > k} \end{cases} \tag{14}$$

Taking into account that $p_{i0} = 2p_{i1} - p_{i2}$ we have the additional members

$$\bar{S}_{jk} = \begin{cases} -\left(j - \dfrac{1}{3}\right)\lambda^2 & \mathrm{k = 1} \\ \dfrac{1}{2}\left(j - \dfrac{1}{3}\right)\lambda^2 & k = 2 \end{cases} \tag{15}$$

Differentiating equation (10) with respect to z, and applying the previous transformations, in a similar way we obtain

$$q'_{ij} = \sum_{k=0}^{M} S'_{jk} p_{ik} + q'_{i0} \tag{16}$$

where

$$S'_{jk} = \begin{cases} -\lambda/2 & \mathrm{j = k} \\ 0 & \mathrm{j < k} \\ -\lambda & \mathrm{j > k} \end{cases} \tag{17}$$

$$\bar{S}'_{jk} = \begin{cases} -\lambda & k = 1 \\ \lambda/2 & k = 2 \end{cases} \tag{18}$$

Also, from equation (3) it follows

$$q''_{ij} = -p_{ij} \tag{19}$$

Equations (12), can be represented in matrix form as

$$\mathbf{Q} = \mathbf{SP} + \mathbf{LQ}_0 \tag{20}$$

or explicitly

$$\underbrace{\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \vdots \\ \mathbf{q}_j \\ \vdots \\ \mathbf{q}_M \end{bmatrix}}_{\mathbf{Q}} = \underbrace{\begin{bmatrix} \left(S_{11}+\bar{S}_{11}\right)\mathbf{I} & \bar{S}_{12}\mathbf{I} & & \cdots & & \cdots & \\ \left(S_{21}+\bar{S}_{21}\right)\mathbf{I} & \left(S_{22}+\bar{S}_{22}\right)\mathbf{I} & & \cdots & & \cdots & \\ \left(S_{31}+\bar{S}_{31}\right)\mathbf{I} & \left(S_{32}+\bar{S}_{32}\right)\mathbf{I} & S_{33}\mathbf{I} & \cdots & & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(S_{j1}+\bar{S}_{j1}\right)\mathbf{I} & \left(S_{j2}+\bar{S}_{j2}\right)\mathbf{I} & S_{j3}\mathbf{I} & \cdots & S_{jj}\mathbf{I} & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(S_{M1}+\bar{S}_{M1}\right)\mathbf{I} & \left(S_{M2}+\bar{S}_{M2}\right)\mathbf{I} & S_{M3}\mathbf{I} & \cdots & S_{Mj}\mathbf{I} & \cdots & S_{MM}\mathbf{I} \end{bmatrix}}_{\mathbf{S}} \underbrace{\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \vdots \\ \mathbf{p}_j \\ \vdots \\ \mathbf{p}_M \end{bmatrix}}_{\mathbf{P}} + \underbrace{\begin{bmatrix} z_1\mathbf{I} & \mathbf{I} \\ z_2\mathbf{I} & \mathbf{I} \\ z_3\mathbf{I} & \mathbf{I} \\ \vdots & \vdots \\ z_j\mathbf{I} & \mathbf{I} \\ \vdots & \vdots \\ z_M\mathbf{I} & \mathbf{I} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \mathbf{q}_0' \\ \mathbf{q}_0 \end{bmatrix}}_{\mathbf{Q}_0} \tag{21}$$

where $\mathbf{q}_j$ and $\mathbf{p}_j$ ($j = 1, 2, \ldots, M$) are the vectors of nodal displacements and "load" of the element, all of which, in general, has N components.

The matrix forms of Eqs. (16) and (19) are given by

$$\mathbf{Q}' = \mathbf{S}'\mathbf{P} + \mathbf{L}'\mathbf{Q}_0 \tag{22}$$

$$\mathbf{Q}'' = -\mathbf{P} \tag{23}$$

Conditions that differential equations (1) are satisfied at all discrete points j ($j = 1, 2, \ldots, M$), can be expressed as

$$\mathbf{A}^*\mathbf{Q}'' + \mathbf{B}^*\mathbf{Q}' + \mathbf{C}^* + \omega^2\mathbf{D}^* \mathbf{Q} = \mathbf{0} \tag{24}$$

where

$$\mathbf{A}^* = \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & M \end{array} \\ \begin{bmatrix} \mathbf{A} & & & \\ & \mathbf{A} & & \\ & & \ddots & \\ & & & \mathbf{A} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ M \end{array} \quad \mathbf{B}^* = \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & M \end{array} \\ \begin{bmatrix} \mathbf{B} & & & \\ & \mathbf{B} & & \\ & & \ddots & \\ & & & \mathbf{B} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ M \end{array}$$

$$\mathbf{C}^* = \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & M \end{array} \\ \begin{bmatrix} \mathbf{C} & & & \\ & \mathbf{C} & & \\ & & \ddots & \\ & & & \mathbf{C} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ M \end{array} \quad \mathbf{D}^* = \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & M \end{array} \\ \begin{bmatrix} \mathbf{D} & & & \\ & \mathbf{D} & & \\ & & \ddots & \\ & & & \mathbf{D} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ M \end{array} \tag{25}$$

Substituting eqs. (20), (22) and (23) into Eq. (24) yields

$$\left[-\mathbf{A}^* + \mathbf{B}^*\mathbf{S}' + \ \mathbf{C}^* + \omega^2\mathbf{D}^* \ \ \mathbf{S}\right]\mathbf{P} + \left[\mathbf{B}^*\mathbf{L}' + \ \mathbf{C}^* + \omega^2\mathbf{D}^* \ \ \mathbf{L}\right]\mathbf{Q}_0 = \mathbf{0} \tag{26}$$

from which we obtain

$$\mathbf{P} = -\mathbf{K}^{-1}\bar{\mathbf{L}}\mathbf{Q}_0 \tag{27}$$

where

$$\begin{aligned} \mathbf{K} &= -\mathbf{A}^* + \mathbf{B}^*\mathbf{S}' + \ \mathbf{C}^* + \omega^2\mathbf{D}^* \ \ \mathbf{S} \\ \bar{\mathbf{L}} &= \mathbf{B}^*\mathbf{L}' + \ \mathbf{C}^* + \omega^2\mathbf{D}^* \ \ \mathbf{L} \end{aligned} \tag{28}$$

Taking into account equations (21) and (22) we can establish a connection between the values of the vector $\mathbf{q}$ and $\mathbf{q}'$ on the right and left end of the element

$$\underbrace{\begin{bmatrix} \mathbf{q}'_M \\ \mathbf{q}_M \end{bmatrix}}_{\mathbf{Q}_M} = \underbrace{\left[\begin{array}{c:c:c:c:c:c:c} S'_{M1} + \bar{S}'_{M1} \ \ \mathbf{I} & S'_{M2} + \bar{S}'_{M2} \ \ \mathbf{I} & S'_{M3}\mathbf{I} & \cdots & S'_{Mj}\mathbf{I} & \cdots & S'_{MM}\mathbf{I} \\ \hdashline S_{M1} + \bar{S}_{M1} \ \ \mathbf{I} & S_{M2} + \bar{S}_{M2} \ \ \mathbf{I} & S_{M3}\mathbf{I} & \cdots & S_{Mj}\mathbf{I} & \cdots & S_{MM}\mathbf{I} \end{array}\right]}_{\mathbf{S}_M} \underbrace{\left[\begin{array}{c} \mathbf{p}_1 \\ \hdashline \mathbf{p}_2 \\ \hdashline \mathbf{p}_3 \\ \hdashline \vdots \\ \hdashline \mathbf{p}_j \\ \hdashline \vdots \\ \hdashline \mathbf{p}_M \end{array}\right]}_{\mathbf{P}} + \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ l\mathbf{I} & \mathbf{I} \end{bmatrix}}_{\mathbf{T}} \underbrace{\left[\begin{array}{c} \mathbf{q}'_0 \\ \hdashline \mathbf{q}_0 \end{array}\right]}_{\mathbf{Q}_0} \tag{29}$$

or

$$\mathbf{Q}_M = \mathbf{S}_M\mathbf{P} + \mathbf{T}\mathbf{Q}_0 \tag{30}$$

Substituting (27) into (30) we get

$$\mathbf{Q}_M = \ -\mathbf{S}_M\mathbf{K}^{-1}\bar{\mathbf{L}} + \mathbf{T} \ \ \mathbf{Q}_0 \tag{31}$$

At the beam ends, depending on the support conditions, boundary conditions are given by the forces and/or displacements. Consider the most general case of mixed boundary conditions at both ends of the beam. Geometric quantities (displacement parameters), as before, will be marked with $\mathbf{q}$ and static (cross-section forces) with $\mathbf{r}$. For the left and right ends of the beam, we introduce generalized vectors

$$\mathbf{R}_0 = \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{r}_0 \end{bmatrix} \qquad \mathbf{R}_M = \begin{bmatrix} \mathbf{q}_M \\ \mathbf{r}_M \end{bmatrix} \tag{32}$$

Between vectors $\mathbf{R}_0$ and $\mathbf{Q}_0$, and $\mathbf{R}_M$ and $\mathbf{Q}_M$ can be established a connection

$$\mathbf{R}_0 = \mathbf{E}\mathbf{Q}_0 \tag{33}$$

$$\mathbf{R}_M = \mathbf{E}\mathbf{Q}_M \tag{34}$$

Multiplying the left side of the matrix equation (31) with matrix $\mathbf{E}$ and taking into account the equations (33) and (34), we obtain

$$\mathbf{R}_M = \mathbf{E}\left(-\mathbf{S}_M\mathbf{K}^{-1}\overline{\mathbf{L}} + \mathbf{T}\right)\mathbf{E}^{-1}\mathbf{R}_0 \tag{35}$$

or

$$\mathbf{R}_M = \mathbf{F}\mathbf{R}_0 \tag{36}$$

where

$$\mathbf{F} = \mathbf{E}\left(-\mathbf{S}_M\mathbf{K}^{-1}\overline{\mathbf{L}} + \mathbf{T}\right)\mathbf{E}^{-1} \tag{37}$$

This is a system of $2\times N$ linear homogeneous algebraic equations for the $4\times N$ unknowns. If homogeneous boundary conditions at the ends of the beam, which number is $2\times N$, enter in the equation (36) we get the homogeneous system of $2\times N$ equations with $2\times N$ unknowns. This system, representing an algebraic eigenvalue problem, can have a nonzero solution only when the determinant of the equation system vanishes.

4 NUMERICAL EXAMPLES

In this section, the introduced method will be employed in analyzing the free vibration of beams with different boundary conditions. Two numerical examples are presented to demonstrate the applicability of the proposed method, and numerical results by the present study are compared to those reported by other researchers.

4.1 Example 1

As the first illustrative problem, consider the transversely vibrating uniform Timoshenko beam. The free harmonic vibrations of beam are defined by homogeneous differential equations of motion

$$\begin{aligned} EI\Psi'' + kGF\left(V' - \Psi\right) + \rho I\omega^2\Psi &= 0 \\ kGF\left(V'' - \Psi'\right) + m\omega^2 V &= 0 \end{aligned}$$

and appropriate end conditions

- The geometric boundary conditions:

$$v = \psi = 0$$

- The natural boundary conditions:

Bending moment (M): $EI\psi' = 0$

Shear force (Q): $kGF\left(v' - \psi\right) = 0$

In above equations $V(z)$ and $\Psi(z)$ denote the amplitudes of the sinusoidally varying transverse displacement and flexural rotation, ω the circular frequency, ρ the material density, F the cross sectional area, I the second area moment of inertia about the neutral axis of the beam cross section, EI and kGF the flexural and shear rigidity of beam, respectively and $m = \rho F$ the mass per unit length. Primes denotes differentiation with respect to coordinate z.

Displacement state vector $\mathbf{q}$, Eq. (2), consisting of 2 displacement parameters

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} V \;\; z \\ \Psi \;\; z \end{bmatrix}$$

and matrix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are as follows

$$\mathbf{A} = \left[\begin{array}{c|c} & EI \\ \hline kGF & \end{array}\right] \quad \mathbf{B} = \left[\begin{array}{c|c} kGF & \\ \hline & -kGF \end{array}\right] \quad \mathbf{C} = \left[\begin{array}{c|c} & -kGF \\ \hline & \end{array}\right] \quad \mathbf{D} = \left[\begin{array}{c|c} & \rho I \\ \hline m & \end{array}\right]$$

$$\mathbf{E} = \left[\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline kFG & & & -kFG \\ & EI & & \end{array}\right]$$

For the cantiliver glass-epoxy composite beam of Han et al (1999), clamped at the left end

$$V_0 = \Psi_0 = 0$$
$$Q_M = M_M = 0$$

and for the structural and material properties as follows:

$E = 200$ GPa
$G = 77.5$ GPa
$\rho = 7830 \ \mathrm{kg/m^3}$
$F = 0.0097389 \ \mathrm{m^2}$
$I = 0.0001171 \ \mathrm{m^4}$
$L = 1.0$ m
$k = 0.53066$

in Table 1 are listed the first six natural frequencies. In the same table, the results of Han et al (1999) are also given for comparison.

Mode	Natural frequency ω (rad/s)				
	k = 5	k = 10	k = 20	k = 40	Han et al (1999)
1	1748.59	1709.61	1699.39	1696.82	1696.03
2	7111.95	6886.33	6798.68	6775.81	6768.24
3	14690.43	14683.53	14387.63	14297.80	14267.26
4	21190.22	20948.34	20604.33	20465.15	20415.37
5	26196.82	25677.57	25369.31	25208.17	25150.52
6	30250.93	30146.82	29641.34	29329.49	29211.86

Table 1: The first six natural frequencies of Timoshenko beam obtained by the present method for various numbers of internal points k and by Han et al (1999)

Close agreements in natural frequencies are obtained between the proposed method and method of Han et al. (1999). For $k = 12$, the differences between the two methods are less than 3%. As can be seen from Table 1, a small number of interior points are required for a good prediction of frequencies.

4.2 Example 2

The governing partial differential equations of harmonic motion, with circular frequency ω, for the laminated composite Timoshenko beam exhibiting coupled flexure–torsion free natural vibration are given by

$$
\begin{aligned}
&EI_y\Psi'' + kGF\;\; V' - \Psi\;\; + K\Phi'' + \rho I_y\omega^2\Psi = 0\\
&kGF\;\; V'' - \Psi'\;\; + m\omega^2 V = 0\\
&K\Psi'' + GJ\Phi'' + I_s\omega^2\Phi = 0
\end{aligned}
$$

where $V\;\; z$, $\Psi\;\; z$ and $\Phi\;\; z$ are transverse deflection, bending rotation and twist angle of the beam, respectively. Beside the above differential equations, geometric and natural boundary conditions must be taken into account

- The geometric boundary conditions:

$$v = \psi = \varphi = 0$$

- The natural boundary conditions:

Bending moment (M): $EI_y\psi' + K\varphi' = 0$

Shear force (Q): $kGF\;\; v' - \psi\;\; = 0$

Torque (T): $K\psi' + GJ\varphi' = 0$

Displacement state vector $\mathbf{q}$, Eq. (2), is given by

$$
\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} V\;\; z \\ \Psi\;\; z \\ \Phi\;\; z \end{bmatrix}
$$

Matrix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are as follows

$$
\mathbf{A} = \left[\begin{array}{c|c|c} & EI_y & K \\ \hline kGF & & \\ \hline & K & GJ \end{array}\right]
\quad
\mathbf{B} = \left[\begin{array}{c|c|c} kGF & & \\ \hline & -kGF & \\ \hline & & \end{array}\right]
\quad
\mathbf{C} = \left[\begin{array}{c|c|c} & -kGF & \\ \hline & & \\ \hline & & \end{array}\right]
\quad
\mathbf{D} = \left[\begin{array}{c|c|c} & \rho I_y & \\ \hline m & & \\ \hline & & I_s \end{array}\right]
$$

$$
\mathbf{E} = \left[\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ \hline kFG & & & & -kFG & \\ & EI_y & K & & & \\ & K & GJ & & & \end{array}\right]
$$

The beam model studied by Mirtalaie et al (2012) is solved in this study as an illustrative example, by using proposed method. It is a cantilever glass-epoxy composite beam with a rectangular cross section with width 12.7 mm and thickness 3.18 mm. Unidirectional plies each having fiber angles of +15 are used in the analysis. The data used for the analysis are as follows:

bending rigidity (EIy) = 0.2865 Nm^2;
torsional rigidity (GJ) = 0.1891 Nm^2;
bending±torsion coupling rigidity (K) = 0.1143 Nm^2;
shear rigidity (kFG) = 6343.3 N;
mass per unit length (m) = 0.0544 kg/m;
mass moment of inertia per unit length (Is) = 0.7770 x10-6 kgm;
length of the beam (L) = 0.1905 m.
$\rho I_y = 4.584 \times 10^{-8}$ kgm

Because of symmetry, only half of the beam is considered.

Mode	Natural frequency ω (rad/s)					
	S-S		C-C		F-F	
	Present study	Exact	Present study	Ref. [13]	Present study	Ref. [13]
1	541.683	540.735	1205.619	1203.500	1222.274	1220.371
2	2132.634	2128.709	3229.957	3223.950	3306.512	3300.886
3	4678.952	4669.661	6112.393	6100.096	6274.497	6262.882
4	8048.008	8030.457	8125.178	8125.010	8184.199	8183.620

Table 2: The first four natural frequencies of composite Timoshenko beam for various boundary conditions

For $k = 60$, the lowest four natural frequencies of the beam are presented in Table II, where they are compared with the exact results obtained by the method of Prokić (2005 and 2006), for the pinned-pinned boundary conditions. Also, in Table 2, for clamped–clamed and free–free boundary conditions, the results of the present method are compared with the results by Mirtalaie et al (202). The numerical results of the present study are in good agreement (the differences are less than 0.2%) with exact and results by Mirtalaie et al (2012).

4 CONCLUSIONS

In this paper a numerical method is proposed which can be applied to a wide range of problems defined by a set of second-order ordinary differential equations with different boundary conditions. The solution is obtained through numerical integration. The basic mathematical operation is simple and can be readily solved by the application of matrix calculus. Numerical verification demonstrates that the proposed method is reasonably accurate, e.i. the numerical approximations, in most cases, is accurate for quite low values of k. Finally, it can be said that proposed method can serve as a convenient alternative to the similar numerical techniques in the analysis of problems defined by a system of second-order ordinary differential equations, with arbitrary boundary conditions.

The problems which are reduced to differential equations with variable coefficients and some other integration schemes will be investigated in next work.

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