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Modeling and analysis of waves in a heat conducting thermo-elastic plate of elliptical shape

Abstract

Wave propagation in heat conducting thermo elastic plate of elliptical cross-section is studied using the Fourier expansion collocation method based on Suhubi's generalized theory. The equations of motion based on two-dimensional theory of elasticity is applied under the plane strain assumption of generalized thermo elastic plate of elliptical cross-sections composed of homogeneous isotropic material. The frequency equations are obtained by using the boundary conditions along outer and inner surface of elliptical cross-sectional plate using Fourier expansion collocation method. The computed non-dimensional frequency, velocity and quality factor are plotted in dispersion curves for longitudinal and flexural (symmetric and antisymmetric) modes of vibrations.

Keywords

Wave propagation in plate; vibration of thermal plate; Fourier collocation method; generalized thermo elastic plate; thermal relaxation times.

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1 INTRODUCTION

The plates of elliptical cross-sections are frequently used as structural components and their vibration characteristics are important for practical design. The propagation of waves in thermoelastic material has many applications in various fields of science and technology, namely, atomic physics, industrial engineering, thermal power plants, submarine structures, pressure vessel, aerospace, chemical pipes, and metallurgy. The importance of thermal stresses in causing structural damages and changes in functioning of the structure is well recognized whenever thermal stress environments are involved.

A method, for solving wave propagation in doubly connected arbitrary and polygonal cross-sectional plates and to find out the phase velocities in different modes of vibrations namely longitudinal, torsional and flexural, by constructing frequency equations was devised by Nagaya (1981a; 981b; 1983a; 1983b; 1983c). He formulated the Fourier expansion collocation method for this purpose, the same method is adopted in this problem. The generalized theory of thermo elasticity was developed by Lord and Shulman (1967) involving one relaxation time for isotropic homogeneous media, which is called the first generalization to the coupled theory of elasticity. These equations determine the finite speeds of propagation of heat and displacement distributions, the corresponding equations for an isotropic case were obtained by Dhaliwal and Sherief (1980). The second generalization to the coupled theory of elasticity is what is known as the theory of thermo elasticity, with two relaxation times or the theory of temperature-dependent thermoelectricity. A generalization of this inequality was proposed by Green and Laws (1972). Green and Lindsay (1972) obtained an explicit version of the constitutive equations. This theory contains two constants that act as relaxation times and modify not only the heat equations, but also all the equations of the coupled theory. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Suhubi (1986) studied the longitudinal wave propagation in a generalized thermoplastic infinite cylinder and obtained the dispersion relation for a constant surface temperature of the cylinder.

Sharma and Pathania (2005) investigated the generalized wave propagation in circumferential curved plates. Asymptotic of wave motion in transversely isotropic plates was analyzed by Sharma and Kumar (2012). Tso and Hansen (1995) have studied the wave propagation through cylinder/plate junctions. Heyliger and Ramirez (2000) analyzed the free vibration characteristics of laminated circular piezoelectric plates and disc by using a discrete-layer model of the weak form of the equations of periodic motion. Thermal deflection of an inverse thermo elastic problem in a thin isotropic circular plate was presented by Gaikward and Deshmukh (2005). Verma and Hasebe (2001) investigated the wave propagation in plates of general anisotropic media in generalized thermo elasticity. Later, Verma (2002) has presented the propagation of waves in layered anisotropic media in generalized thermo elasticity in an arbitrary layered plate. The free vibration of non-homogeneous transversely isotropic magneto-electro-elastic plates was studied by Chen et al. [19]. Kumar and Partap (2007) presented the free vibration of microstretch thermoelastic plate with one relaxation time. Ponnusamy and Selvamani (2012) have studied the dispersion analysis of generalized magneto-thermo elastic waves in a transversely isotropic cylindrical panel using the wave propagation approach. Later, Selvamani (2012) performed mathematical modeling and analysis for damping of generalized thermoelastic waves in a homogeneous isotropic plate.

In this paper, the in-plane vibration of heat conducting thermo elastic elliptical cross-sectional plate of homogeneous isotropic material is studied. The solutions to the equations of motion for an isotropic medium is obtained by using the two dimensional theory of elasticity. To satisfy the boundary conditions, the Fourier expansion collocation method is performed to the equations of the boundary conditions and the frequency equations are obtained for longitudinal and flexural (symmetric and antisymmetric) modes of vibrations. The computed non-dimensional frequency, velocity and quality factor are plotted in dispersion curves for longitudinal and flexural (symmetric and antisymmetric) modes of vibrations.

2 FORMULATION OF THE PROBLEM

We consider a homogeneous, isotropic, thermally conducting elastic plate of elliptical cross - sections with uniform temperature T_0 in the undisturbed state initially. The system displacements and stresses are defined in polar coordinates r and θ in an arbitrary point inside the plate and denote the displacements u_r in the direction of r and u_θ in the tangential direction θ.

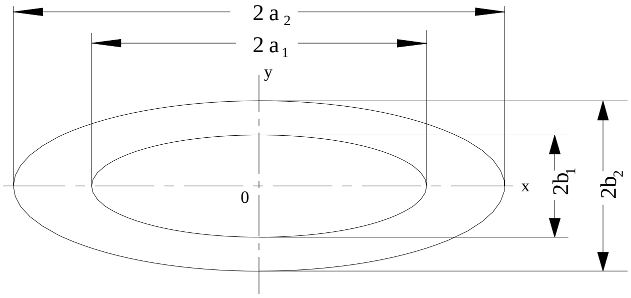

Figure 1: Geometry of ring-shaped elliptical plate.

The two dimensional stress equations of motion, and strain –displacement relations and heat conduction equation in the absence of body force for a linearly elastic medium are

$$\begin{aligned} &\sigma_{rr,r} + r^{-1}\sigma_{r\theta,\theta} + r^{-1}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = \rho u_{r,tt} \\ &\sigma_{r\theta,r} + r^{-1}\sigma_{\theta\theta,\theta} + 2r^{-1}\sigma_{r\theta} = \rho u_{\theta,tt} \\ &K\rho c_\nu\left(T_{,rr} + r^{-1}T_{,r} + r^{-2}T_{,\theta\theta}\right) = \rho c_\nu T_{,t} + \rho\tau T_{,tt} + \beta T_0\left[u_{r,rt} + r^{-1}(u_{r,t} + u_{\theta,\theta t})\right] \end{aligned} \tag{1}$$

and

$$\begin{aligned} \sigma_{rr} &= \lambda\left(e_{rr} + e_{\theta\theta}\right) + 2\mu e_{rr} - \beta\left(T + \eta T_{,t}\right) \\ \sigma_{\theta\theta} &= \lambda\left(e_{rr} + e_{\theta\theta}\right) + 2\mu e_{\theta\theta} - \beta\left(T + \eta T_{,t}\right) \\ \sigma_{r\theta} &= 2\mu e_{r\theta} \end{aligned} \tag{2}$$

where σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ are the stress components, e_{rr}, $e_{\theta\theta}$, $e_{r\theta}$ are the strain components, T is the temperature change about the equilibrium temperature T_0, ρ is the mass density, c_v is the specific heat capacity, β is a coupling factor that couples the heat conduction and elastic field equations, K is the thermal conductivity , η, τ is the thermal relaxation times, t is the time, λ and μ are Lame' constants.
The strain e_{ij} related to the displacements are given by

$$e_{rr} = u_{r,r}\,,\; e_{\theta\theta} = r^{-1}\left(u_r + u_{\theta,\theta}\right),\; e_{r\theta} = u_{\theta,r} - r^{-1}\left(u_\theta - u_{r,\theta}\right) \tag{3}$$

in which u_r and u_θ are the displacement components along radial and circumferential directions, respectively. The comma in the subscripts denotes the partial differentiation with respect to the variables.
Substituting Eqs. (3) and (2) in Eq. (1), the following displacement equations of motions are obtained

$$\begin{aligned}
&(\lambda+2\mu)\left(u_{r,rr}+r^{-1}u_{r,r}-r^{-2}u_r\right)+\mu r^{-2}u_{r,\theta\theta}+r^{-1}(\lambda+\mu)u_{\theta,r\theta}+r^{-2}(\lambda+3\mu)u_{\theta,\theta}-\beta\left(T+\eta T_{,t}\right)_{,r}=\rho u_{r,tt}\\
&\mu\left(u_{\theta,rr}+r^{-1}u_{\theta,r}-r^{-2}u_\theta\right)+r^{-2}(\lambda+2\mu)u_{\theta,\theta\theta}+r^{-2}(\lambda+3\mu)u_{r,\theta}+r^{-1}(\lambda+\mu)u_{r,r\theta}--\beta\left(T+\eta T_{,t}\right)_{,\theta}=\rho u_{\theta,tt}\\
&K\rho c_\nu\left(T_{,rr}+r^{-1}T_{,r}+r^{-2}T_{,\theta\theta}\right)=\rho c_\nu T_{,t}+\rho\tau T_{,tt}+\beta T_0\left[u_{r,tr}+r^{-1}(u_{r,t}+u_{\theta,t\theta})\right]
\end{aligned} \tag{4}$$

3 SOLUTIONS OF THE PROBLEM

Eq. (4) is a coupled partial differential equation with two displacements and heat conduction components. To uncouple Eq.(4), we follow Mirsky (1964) by assuming the vibration and displacements along the axial direction z equal to zero. Hence assuming the solutions of Eq. (4) in the form

$$\begin{aligned}
u_r(r,\theta,z,t)&=\sum_{n=0}^{\infty}\varepsilon_n\left[\left(\phi_{n,r}+r^{-1}\psi_{n,\theta}\right)+\left(\overline{\phi}_{n,r}+r^{-1}\overline{\psi}_{n,\theta}\right)\right]e^{i\omega t}\\
u_\theta(r,\theta,z,t)&=\sum_{n=0}^{\infty}\varepsilon_n\left[\left(r^{-1}\phi_{n,\theta}-\psi_{n,r}\right)+\left(r^{-1}\overline{\phi}_{n,\theta}-\overline{\psi}_{n,r}\right)\right]e^{i\omega t}\\
T(r,\theta,z,t)&=\left(\lambda+2\mu/\beta a^2\right)\sum_{n=0}^{\infty}\varepsilon_n\left(T_n+\overline{T}_n\right)e^{i\omega t}
\end{aligned} \tag{5}$$

where $\varepsilon_n=1/2$ for $n=0$, $\varepsilon_n=1$ for $n\geq 1$, $i=\sqrt{-1}$, ω is the frequency, $\phi_n(r,\theta)$, $\psi_n(r,\theta)$, $T_n(r,\theta)$, $\overline{\phi}_n(r,\theta)$, $\overline{\psi}_n(r,\theta)$ and $\overline{T}_n(r,\theta)$ are the displacement potentials.
Introducing the dimensionless quantities such as $T_a=t\sqrt{\mu/\rho}/a$, $x=r/a$, $c_1^2=(\lambda+2\mu)/\rho$, $\alpha'=c_1a/K$, $\Omega^2=\omega^2a^2/c_1^2$, $\varepsilon_1=T_0a\beta^2/\left(\rho^2c_vc_1K\right)$, $\varepsilon_2=c_1^2\tau/\left(c_\nu K\right)$, and $\varepsilon_3=c_1\eta/a$ and using Eq. (5) in Eq. (4), we obtain

$$\left(A\nabla^4+B\nabla^2+C\right)\phi=0 \tag{6}$$

where

$$A=1,\ B=\Omega^2\left(1+\varepsilon_1\varepsilon_3\right)+\varepsilon_2-i\Omega\left(\alpha'+\varepsilon_1\right),\ c=\Omega^2\left(\varepsilon_2-i\alpha'\Omega\right) \tag{7}$$

and

$$\left(\nabla^2+\left(2+\overline{\lambda}\right)\Omega^2-\varsigma^2\right)\quad\psi_n=0 \tag{8}$$

where $\nabla^2\equiv\partial^2/\partial x^2+x^{-1}\,\partial/\partial x+x^{-2}\,\partial^2/\partial\theta^2$

The parameters defined in Eq. (7) namely, ε_1 couples the equations corresponding to the elastic wave propagation and the heat conduction which is called the coupling factor; the coefficient ε_2, which is introduced by the theory of generalized thermo elasticity, may render the governing system of equations hyperbolic. The parameter ε_3 is the coefficient of the term indicating the difference between empirical and thermodynamic temperatures. Solving the partial differential equation (6), the solutions for symmetric mode is obtained as

$$\phi_n = \sum_{i=1}^{2}\left[A_{in}J_n\left(\alpha_i ax\right) + B_{in}Y_n\left(\alpha_i ax\right)\right]\cos n\theta \tag{9.a}$$

$$T_n = \sum_{i=1}^{2} d_i\left[A_{in}J_n\left(\alpha_i ax\right) + B_{in}Y_n\left(\alpha_i ax\right)\right]\cos n\theta \tag{9.b}$$

and the solution for the antisymmetric mode $\bar{\phi}_n$ is obtained by replacing $\cos n\theta$ by $\sin n\theta$ in Eqs. (9a) and (9b), we get

$$\bar{\phi}_n = \sum_{i=1}^{2}\left[\bar{A}_{in}J_n\left(\alpha_i ax\right) + \bar{B}_{in}Y_n\left(\alpha_i ax\right)\right]\sin n\theta \tag{10.a}$$

$$\bar{T}_n = \sum_{i=1}^{2} d_i\left[\bar{A}_{in}J_n\left(\alpha_i ax\right) + \bar{B}_{in}J_n\left(\alpha_i ax\right)\right]\sin n\theta \tag{10.b}$$

where J_n is the Bessel function of first kind of order n and Y_n is the Bessel function of second kind of order n. Solving Eq. (8), we obtain

$$\psi_n = \left[A_{3n}J_n\left(\alpha_3 ax\right) + B_{3n}Y_n\left(\alpha_3 ax\right)\right]\sin n\theta \tag{11.a}$$

for symmetric mode, and for the antisymmetric mode $\bar{\psi}_n$ is obtained from Eq. (11a) by replacing $\sin n\theta$ by $\cos n\theta$.

$$\bar{\psi}_n = \left[\bar{A}_{3n}J_n\left(\alpha_3 ax\right) + \bar{B}_{3n}J_n\left(\alpha_3 ax\right)\right]\cos n\theta \tag{11.b}$$

where $\left(\alpha_3 a\right)^2 = \left(2+\bar{\lambda}\right)\Omega^2 - \varsigma^2$. If $\left(\alpha_i a\right)^2 < 0$ $(i = 1,2,3)$, then the Bessel functions J_n and Y_n are replaced by the modified Bessel function I_n and K_n respectively .

4 BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

In this problem, the vibration of thick arbitrary cross-sectional plate is considered. Since the boundary is irregular in shape, it is difficult to satisfy the boundary conditions along both outer and inner surface of the plate directly. Hence, the Fourier expansion collocation method is applied to satisfy the boundary conditions. For the plate, the normal stress $\sigma_{xx}^{'}$, shearing stress $\sigma_{xy}^{'}$ and thermal field $T^{'}$ along the inner surface of the plate is equal to zero. Similarly, normal stress σ_{xx}, shearing stress σ_{xy} and thermal field T along the outer surface of the plate is equal to zero. Thus, the boundary conditions along the outer boundary of the plate are

$$\left(\sigma_{xx}\right)_i = \left(\sigma_{xy}\right)_i = \left(T\right)_i = 0 \tag{12}$$

and for the inner boundary, the boundary conditions are

$$\left(\sigma'_{xx}\right)_i = \left(\sigma'_{xy}\right)_i = \left(T'\right)_i = 0 \tag{13}$$

where x is the coordinate normal to the boundary and y is the coordinate tangential to the boundary, σ_{xx}, σ'_{xx} are the normal stresses, σ_{xy}, σ'_{xy} are the shearing stresses, T, T' are the thermal fields and $(\ )_i$ is the value at the $i-th$ segment of the outer and inner boundary respectively. Since the cross-section of the plate is irregular, it is difficult to find the transformed expression of the stresses in the thick arbitrary cross-sectional plates because the coordinate x and y are vary with the angle θ. Therefore, the inner and outer boundary of the thick arbitrary cross-sections is divided into small segments such that the variations of the stresses are assumed to be constant. Assuming the angle γ_i, between the normal to the segment and the reference axis to be constant, the transformed expressions for the stresses are followed by Nagaya (1983b) as

$$\begin{aligned}\sigma'_{xx} &= \lambda\left(u_{r,r} + r^{-1}\left(u_r + u_{\theta,\theta}\right)\right) + 2\mu[u_{r,r}\cos^2\left(\theta-\gamma_i\right) + r^{-1}\left(u_r + u_{\theta,\theta}\right)\sin^2\left(\theta-\gamma_i\right) \\ &\quad + 0.5\left(r^{-1}\left(u_\theta - u_{r,\theta}\right) - u_{\theta,r}\right)\sin 2\left(\theta-\gamma_i\right)] - \beta\left(T+\eta T_{,t}\right) \\ \sigma'_{xy} &= \mu[u_{r,r} - r^{-1}\left(u_{\theta,\theta} + u_r\right)\sin 2\left(\theta-\gamma_i\right) + \left(r^{-1}\left(u_{r,\theta} - u_\theta\right) + u_{\theta,r}\right)\sin 2\left(\theta-\gamma_i\right)]\end{aligned} \tag{14}$$

Substituting Eqs. (9)-(11)in Eqs. (12) and (13), the boundary conditions are transformed as follows:

$$\begin{aligned}\left[\left(S'_{xx}\right)_i + \left(\bar{S}'_{xx}\right)_i\right]e^{i\Omega T_a} &= 0 \\ \left[\left(S'_{xy}\right)_i + \left(\bar{S}'_{xy}\right)_i\right]e^{i\Omega T_a} &= 0 \\ \left[\left(S'_{t}\right)_i + \left(\bar{S}'_{t}\right)_i\right]e^{i\Omega T_a} &= 0\end{aligned} \tag{15}$$

for the inner surface and

$$\begin{aligned}\left[\left(S_{xx}\right)_i + \left(\bar{S}_{xx}\right)_i\right]e^{i\Omega T_a} &= 0 \\ \left[\left(S_{xy}\right)_i + \left(\bar{S}_{xy}\right)_i\right]e^{i\Omega T_a} &= 0 \\ \left[\left(S_{t}\right)_i + \left(\bar{S}_{t}\right)_i\right]e^{i\Omega T_a} &= 0\end{aligned} \tag{16}$$

for the outer surface, where

$$\begin{aligned}S'_{xx} &= 0.5\left(\hat{e}^1_0 A_{10} + \hat{e}^2_0 B_{10} + \hat{e}^3_0 A_{20} + \hat{e}^4_0 B_{20}\right) + \sum_{n=1}^{\infty}\left(\hat{e}^1_n A_{1n} + \hat{e}^2_n B_{1n} + \hat{e}^3_n A_{2n} + \hat{e}^4_n B_{2n} + \hat{e}^5_n A_{3n} + \hat{e}^6_n B_{3n}\right) \\ S'_{xy} &= 0.5\left(\hat{f}^1_0 A_{10} + \hat{f}^2_0 B_{10} + \hat{f}^3_0 A_{20} + \hat{f}^4_0 B_{20}\right) + \sum_{n=1}^{\infty}\left(\hat{f}^1_n A_{1n} + \hat{f}^2_n B_{1n} + \hat{f}^3_n A_{2n} + \hat{f}^4_n B_{2n} + \hat{f}^5_n A_{3n} + \hat{f}^6_n B_{3n}\right) \\ S'_{t} &= 0.5\left(\hat{g}^1_0 A_{10} + \hat{g}^2_0 A_{10} + \hat{g}^3_0 A_{20} + \hat{g}^2_0 A_{20}\right) + \sum_{n=1}^{\infty}\left(\hat{g}^1_n A_{1n} + \hat{g}^2_n B_{1n} + \hat{g}^3_n A_{2n} + \hat{g}^4_n B_{2n} + \hat{g}^5_n A_{3n} + \hat{g}^6_n B_{3n}\right)\end{aligned} \tag{17.a}$$

$$
\begin{aligned}
S_{xx} &= 0.5\left(e_0^1 A_{10} + e_0^2 B_{10} + e_0^3 A_{20} + e_0^4 B_{20}\right) + \sum_{n=1}^{\infty}\left(e_n^1 A_{1n} + e_n^2 B_{1n} + e_n^3 A_{2n} + e_n^4 B_{2n} + e_n^5 A_{3n} + e_n^6 B_{3n}\right)\\
S_{xy} &= 0.5\left(f_0^1 A_{10} + f_0^2 B_{10} + f_0^3 A_{20} + f_0^4 B_{20}\right) + \sum_{n=1}^{\infty}\left(f_n^1 A_{1n} + f_n^2 B_{1n} + f_n^3 A_{2n} + f_n^4 B_{2n} + f_n^5 A_{3n} + f_n^6 B_{3n}\right)\\
S_{t} &= 0.5\left(g_0^1 A_{10} + g_0^2 B_{10} + g_0^3 A_{20} + g_0^4 B_{20}\right) + \sum_{n=1}^{\infty}\left(g_n^1 A_{1n} + g_n^2 B_{1n} + g_n^3 A_{2n} + g_n^4 B_{2n} + g_n^5 A_{3n} + g_n^6 B_{3n}\right)
\end{aligned}
\tag{17.b}
$$

$$
\begin{aligned}
\bar{S}'_{xx} &= 0.5\left(\hat{e}_0^{3}\bar{A}_{20} + \hat{e}_0^{4}\bar{B}_{20}\right) + \sum_{n=1}^{\infty}\left(\hat{e}_n^{1}\bar{A}_{1n} + \hat{e}_n^{2}\bar{B}_{1n} + \hat{e}_n^{3}\bar{A}_{2n} + \hat{e}_n^{4}\bar{B}_{2n} + \hat{e}_n^{5}\bar{A}_{3n} + \hat{e}_n^{6}\bar{B}_{3n}\right)\\
\bar{S}'_{xy} &= 0.5\left(\hat{f}_0^{3}\bar{A}_{20} + \hat{f}_0^{4}\bar{B}_{20}\right) + \sum_{n=1}^{\infty}\left(\hat{f}_n^{1}\bar{A}_{1n} + \hat{f}_n^{2}\bar{B}_{1n} + \hat{f}_n^{3}\bar{A}_{2n} + \hat{f}_n^{4}\bar{B}_{2n} + \hat{f}_n^{5}\bar{A}_{3n} + \hat{f}_n^{6}\bar{B}_{3n}\right)\\
\bar{S}'_{t} &= 0.5\left(\hat{g}_0^{3}\bar{A}_{20} + \hat{g}_0^{4}\bar{B}_{20}\right) + \sum_{n=1}^{\infty}\left(\hat{g}_n^{1}\bar{A}_{1n} + \hat{g}_n^{2}\bar{B}_{1n} + \hat{g}_n^{3}\bar{A}_{2n} + \hat{g}_n^{4}\bar{B}_{2n} + \hat{g}_n^{5}\bar{A}_{3n} + \hat{g}_n^{6}\bar{B}_{3n}\right)
\end{aligned}
\tag{18.a}
$$

$$
\begin{aligned}
\bar{S}_{xx} &= 0.5\left(\bar{e}_0^{3}\bar{A}_{20} + \bar{e}_0^{4}\bar{B}_{20}\right) + \sum_{n=1}^{\infty}\left(\bar{e}_n^{1}\bar{A}_{1n} + \bar{e}_n^{2}\bar{B}_{1n} + \bar{e}_n^{3}\bar{A}_{2n} + \bar{e}_n^{4}\bar{B}_{2n} + \bar{e}_n^{5}\bar{A}_{3n} + \bar{e}_n^{6}\bar{B}_{3n}\right)\\
\bar{S}_{xy} &= 0.5\left(\bar{f}_0^{3}\bar{A}_{20} + \bar{f}_0^{4}\bar{B}_{20}\right) + \sum_{n=1}^{\infty}\left(\bar{f}_n^{1}\bar{A}_{1n} + \bar{f}_n^{2}\bar{B}_{1n} + \bar{f}_n^{3}\bar{A}_{2n} + \bar{f}_n^{4}\bar{B}_{2n} + \bar{f}_n^{5}\bar{A}_{3n} + \bar{f}_n^{6}\bar{B}_{3n}\right)\\
\bar{S}_{t} &= 0.5\left(\bar{g}_0^{3}\bar{A}_{20} + \bar{g}_0^{4}\bar{B}_{20}\right) + \sum_{n=1}^{\infty}\left(\bar{g}_n^{1}\bar{A}_{1n} + \bar{g}_n^{2}\bar{B}_{1n} + \bar{g}_n^{3}\bar{A}_{2n} + \bar{g}_n^{4}\bar{B}_{2n} + \bar{g}_n^{5}\bar{A}_{3n} + \bar{g}_n^{6}\bar{B}_{3n}\right)
\end{aligned}
\tag{18.b}
$$

The coefficients for $\hat{e}_n^{i} - \bar{g}_n^{i}$ are given in the Appendix A.

Performing the Fourier series expansion to Eqs. (12) and (13) along the boundary, the boundary conditions along the inner and outer surfaces are expanded in the form of double Fourier series. When a plate is symmetric about more than one axis, the boundary conditions, in the case of symmetric mode can be written in the form of a matrix as given below:

$$
\begin{bmatrix}
\hat{E}^1_{00} & \hat{E}^2_{00} & \hat{E}^3_{00} & \hat{E}^4_{00} & \hat{E}^1_{01} & \cdots & \hat{E}^1_{0N} & \hat{E}^2_{01} & \cdots & \hat{E}^2_{0N} & \hat{E}^3_{01} & \cdots & \hat{E}^3_{0N} & \hat{E}^4_{01} & \cdots & \hat{E}^4_{0N} & \hat{E}^5_{01} & \cdots & \hat{E}^5_{0N} & \hat{E}^6_{01} & \cdots & \hat{E}^6_{0N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
\hat{E}^1_{N0} & \hat{E}^2_{N0} & \hat{E}^3_{N0} & \hat{E}^4_{N0} & \hat{E}^1_{N1} & \cdots & \hat{E}^1_{NN} & \hat{E}^2_{N1} & \cdots & \hat{E}^2_{NN} & \hat{E}^3_{N1} & \cdots & \hat{E}^3_{NN} & \hat{E}^4_{N1} & \cdots & \hat{E}^4_{NN} & \hat{E}^5_{N1} & \cdots & \hat{E}^5_{NN} & \hat{E}^6_{N1} & \cdots & \hat{E}^6_{NN}\\
\hat{F}^1_{10} & \hat{F}^2_{10} & \hat{F}^3_{10} & \hat{F}^4_{10} & \hat{F}^1_{11} & \cdots & \hat{F}^1_{1N} & \hat{F}^2_{11} & \cdots & \hat{F}^2_{1N} & \hat{F}^3_{11} & \cdots & \hat{F}^3_{1N} & \hat{F}^4_{11} & \cdots & \hat{F}^4_{1N} & \hat{F}^5_{11} & \cdots & \hat{F}^5_{1N} & \hat{F}^6_{11} & \cdots & \hat{F}^6_{1N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
\hat{F}^1_{N0} & \hat{F}^2_{N0} & \hat{F}^3_{N0} & \hat{F}^4_{N0} & \hat{F}^1_{N1} & \cdots & \hat{F}^1_{NN} & \hat{F}^2_{N1} & \cdots & \hat{F}^2_{NN} & \hat{F}^3_{N1} & \cdots & \hat{F}^3_{NN} & \hat{F}^4_{N1} & \cdots & \hat{F}^4_{NN} & \hat{F}^5_{N1} & \cdots & \hat{F}^5_{NN} & \hat{F}^6_{N1} & \cdots & \hat{F}^6_{NN}\\
\hat{G}^1_{00} & \hat{G}^2_{00} & \hat{G}^3_{00} & \hat{G}^4_{00} & \hat{G}^1_{01} & \cdots & \hat{G}^1_{0N} & \hat{G}^2_{01} & \cdots & \hat{G}^2_{0N} & \hat{G}^3_{01} & \cdots & \hat{G}^3_{0N} & \hat{G}^4_{01} & \cdots & \hat{G}^4_{0N} & \hat{G}^5_{01} & \cdots & \hat{G}^5_{0N} & \hat{G}^6_{01} & \cdots & \hat{G}^6_{0N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
\hat{G}^1_{N0} & \hat{G}^2_{N0} & \hat{G}^3_{N0} & \hat{G}^4_{N0} & \hat{G}^1_{N1} & \cdots & \hat{G}^1_{NN} & \hat{G}^2_{N1} & \cdots & \hat{G}^2_{NN} & \hat{G}^3_{N1} & \cdots & \hat{G}^3_{NN} & \hat{G}^4_{N1} & \cdots & \hat{G}^4_{NN} & \hat{G}^5_{N1} & \cdots & \hat{G}^5_{NN} & \hat{G}^6_{N1} & \cdots & \hat{G}^6_{NN}\\
E^1_{00} & E^2_{00} & E^3_{00} & E^4_{00} & E^1_{10} & \cdots & E^1_{0N} & E^2_{10} & \cdots & E^2_{0N} & E^3_{10} & \cdots & E^3_{0N} & E^4_{10} & \cdots & E^4_{0N} & E^5_{10} & \cdots & E^5_{0N} & E^6_{10} & \cdots & E^6_{0N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
E^1_{N0} & E^2_{N0} & E^3_{N0} & E^4_{N0} & E^1_{N1} & \cdots & E^1_{NN} & E^2_{N1} & \cdots & E^2_{NN} & E^3_{N1} & \cdots & E^3_{NN} & E^4_{N1} & \cdots & E^4_{NN} & E^5_{NN} & \cdots & E^5_{NN} & E^6_{N1} & \cdots & E^6_{NN}\\
F^1_{10} & F^2_{10} & F^3_{10} & F^4_{10} & F^1_{11} & \cdots & F^1_{1N} & F^2_{11} & \cdots & F^2_{1N} & F^3_{11} & \cdots & F^3_{1N} & F^4_{11} & \cdots & F^4_{1N} & F^5_{11} & \cdots & F^5_{1N} & F^6_{11} & \cdots & F^6_{1N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
F^1_{N0} & F^2_{N0} & F^3_{N0} & F^4_{N0} & F^1_{N1} & \cdots & F^1_{NN} & F^2_{N1} & \cdots & F^2_{NN} & F^3_{N1} & \cdots & F^3_{NN} & F^4_{N1} & \cdots & F^4_{NN} & F^5_{N1} & \cdots & F^5_{NN} & F^6_{N1} & \cdots & F^6_{NN}\\
G^1_{00} & G^2_{00} & G^3_{00} & G^4_{00} & G^1_{00} & \cdots & G^1_{0N} & G^2_{00} & \cdots & G^2_{0N} & G^3_{00} & \cdots & G^3_{0N} & G^4_{00} & \cdots & G^4_{0N} & G^5_{00} & \cdots & G^5_{0N} & G^6_{00} & \cdots & G^6_{0N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
G^1_{N0} & G^2_{N0} & G^3_{N0} & G^4_{N0} & G^1_{N1} & \cdots & G^1_{NN} & G^2_{N1} & \cdots & G^2_{NN} & G^3_{N1} & \cdots & G^3_{NN} & G^4_{N1} & \cdots & G^4_{N1} & G^5_{N1} & \cdots & G^5_{N1} & G^6_{N1} & \cdots & G^6_{NN}
\end{bmatrix}
\begin{bmatrix}
A_{10}\\ B_{10}\\ A_{20}\\ B_{20}\\ A_{11}\\ \vdots\\ A_{1N}\\ B_{11}\\ \vdots\\ B_{1N}\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ A_{31}\\ \vdots\\ A_{3N}\\ B_{31}\\ \vdots\\ B_{3N}
\end{bmatrix}
= 0
\tag{19}
$$

where

$$
\begin{aligned}
\widehat{E}^{i}_{mn} &= \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} e_n^i\left(\widehat{R}_i,\theta\right)\cos m\theta d\theta \\
\widehat{F}^{i}_{mn} &= \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} f_n^i\left(\widehat{R}_i,\theta\right)\sin m\theta d\theta \\
\widehat{G}^{i}_{mn} &= \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} g_n^i\left(\widehat{R}_i,\theta\right)\cos m\theta d\theta
\end{aligned}
\qquad (20.a)
$$

$$
\begin{aligned}
E^{i}_{mn} &= \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} e_n^i\left(R_i,\theta\right)\cos m\theta d\theta \\
F^{i}_{mn} &= \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} f_n^i\left(R_i,\theta\right)\sin m\theta d\theta \\
G^{i}_{mn} &= \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} g_n^i\left(R_i,\theta\right)\cos m\theta d\theta
\end{aligned}
\qquad (20.b)
$$

Similarly, the matrix for the antisymmetric mode is obtained as

$$
\begin{bmatrix}
\overline{\widehat{E}}{}^{3}_{10} & \overline{\widehat{E}}{}^{4}_{10} & \overline{\widehat{E}}{}^{1}_{11} & \cdots & \overline{\widehat{E}}{}^{1}_{1N} & \overline{\widehat{E}}{}^{2}_{11} & \cdots & \overline{\widehat{E}}{}^{2}_{1N} & \overline{\widehat{E}}{}^{3}_{11} & \cdots & \overline{\widehat{E}}{}^{3}_{1N} & \overline{\widehat{E}}{}^{4}_{11} & \cdots & \overline{\widehat{E}}{}^{4}_{1N} & \overline{\widehat{E}}{}^{5}_{11} & \cdots & \overline{\widehat{E}}{}^{5}_{1N} & \overline{\widehat{E}}{}^{6}_{11} & \cdots & \overline{\widehat{E}}{}^{6}_{1N} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{\widehat{E}}{}^{3}_{N0} & \overline{\widehat{E}}{}^{4}_{N0} & \overline{\widehat{E}}{}^{1}_{N1} & \cdots & \overline{\widehat{E}}{}^{1}_{NN} & \overline{\widehat{E}}{}^{2}_{N1} & \cdots & \overline{\widehat{E}}{}^{2}_{NN} & \overline{\widehat{E}}{}^{3}_{N1} & \cdots & \overline{\widehat{E}}{}^{3}_{NN} & \overline{\widehat{E}}{}^{4}_{N1} & \cdots & \overline{\widehat{E}}{}^{4}_{NN} & \overline{\widehat{E}}{}^{5}_{N1} & \cdots & \overline{\widehat{E}}{}^{5}_{NN} & \overline{\widehat{E}}{}^{6}_{N1} & \cdots & \overline{\widehat{E}}{}^{6}_{NN} \\
\overline{\widehat{F}}{}^{3}_{00} & \overline{\widehat{F}}{}^{4}_{00} & \overline{\widehat{F}}{}^{1}_{01} & \cdots & \overline{\widehat{F}}{}^{1}_{0N} & \overline{\widehat{F}}{}^{2}_{01} & \cdots & \overline{\widehat{F}}{}^{2}_{0N} & \overline{\widehat{F}}{}^{3}_{01} & \cdots & \overline{\widehat{F}}{}^{3}_{0N} & \overline{\widehat{F}}{}^{4}_{01} & \cdots & \overline{\widehat{F}}{}^{4}_{0N} & \overline{\widehat{F}}{}^{5}_{01} & \cdots & \overline{\widehat{F}}{}^{5}_{0N} & \overline{\widehat{F}}{}^{6}_{01} & \cdots & \overline{\widehat{F}}{}^{6}_{0N} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{\widehat{F}}{}^{3}_{N0} & \overline{\widehat{F}}{}^{3}_{N0} & \overline{\widehat{F}}{}^{1}_{N1} & \cdots & \overline{\widehat{F}}{}^{1}_{NN} & \overline{\widehat{F}}{}^{2}_{N1} & \cdots & \overline{\widehat{F}}{}^{2}_{NN} & \overline{\widehat{F}}{}^{3}_{N1} & \cdots & \overline{\widehat{F}}{}^{3}_{NN} & \overline{\widehat{F}}{}^{4}_{N1} & \cdots & \overline{\widehat{F}}{}^{4}_{NN} & \overline{\widehat{F}}{}^{5}_{N1} & \cdots & \overline{\widehat{F}}{}^{5}_{NN} & \overline{\widehat{F}}{}^{6}_{N1} & \cdots & \overline{\widehat{F}}{}^{6}_{NN} \\
\overline{\widehat{G}}{}^{3}_{10} & \overline{\widehat{G}}{}^{4}_{10} & \overline{\widehat{G}}{}^{1}_{11} & \cdots & \overline{\widehat{G}}{}^{1}_{1N} & \overline{\widehat{G}}{}^{2}_{11} & \cdots & \overline{\widehat{G}}{}^{2}_{1N} & \overline{\widehat{G}}{}^{3}_{11} & \cdots & \overline{\widehat{G}}{}^{3}_{1N} & \overline{\widehat{G}}{}^{4}_{11} & \cdots & \overline{\widehat{G}}{}^{4}_{1N} & \overline{\widehat{G}}{}^{5}_{11} & \cdots & \overline{\widehat{G}}{}^{5}_{1N} & \overline{\widehat{G}}{}^{6}_{11} & \cdots & \overline{\widehat{G}}{}^{6}_{1N} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{\widehat{G}}{}^{3}_{N0} & \overline{\widehat{G}}{}^{4}_{N0} & \overline{\widehat{G}}{}^{1}_{N1} & \cdots & \overline{\widehat{G}}{}^{1}_{NN} & \overline{\widehat{G}}{}^{2}_{N1} & \cdots & \overline{\widehat{G}}{}^{2}_{NN} & \overline{\widehat{G}}{}^{3}_{N1} & \cdots & \overline{\widehat{G}}{}^{3}_{NN} & \overline{\widehat{G}}{}^{4}_{N1} & \cdots & \overline{\widehat{G}}{}^{4}_{NN} & \overline{\widehat{G}}{}^{5}_{N1} & \cdots & \overline{\widehat{G}}{}^{5}_{NN} & \overline{\widehat{G}}{}^{6}_{N1} & \cdots & \overline{\widehat{G}}{}^{6}_{NN} \\
\overline{E}^{3}_{10} & \overline{E}^{4}_{10} & \overline{E}^{1}_{11} & \cdots & \overline{E}^{1}_{1N} & \overline{E}^{2}_{11} & \cdots & \overline{E}^{2}_{1N} & \overline{E}^{3}_{11} & \cdots & \overline{E}^{3}_{1N} & \overline{E}^{4}_{11} & \cdots & \overline{E}^{4}_{1N} & \overline{E}^{5}_{11} & \cdots & \overline{E}^{5}_{1N} & \overline{E}^{6}_{11} & \cdots & \overline{E}^{6}_{1N} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{E}^{3}_{N0} & \overline{E}^{4}_{N0} & \overline{E}^{1}_{N1} & \cdots & \overline{E}^{1}_{NN} & \overline{E}^{2}_{N1} & \cdots & \overline{E}^{2}_{NN} & \overline{E}^{3}_{N1} & \cdots & \overline{E}^{3}_{NN} & \overline{E}^{4}_{N1} & \cdots & \overline{E}^{4}_{NN} & \overline{E}^{5}_{N1} & \cdots & \overline{E}^{5}_{NN} & \overline{E}^{6}_{N1} & \cdots & \overline{E}^{6}_{NN} \\
\overline{F}^{3}_{00} & \overline{F}^{4}_{00} & \overline{F}^{1}_{01} & \cdots & \overline{F}^{1}_{0N} & \overline{F}^{2}_{01} & \cdots & \overline{F}^{2}_{0N} & \overline{F}^{3}_{01} & \cdots & \overline{F}^{3}_{0N} & \overline{F}^{4}_{01} & \cdots & \overline{F}^{4}_{0N} & \overline{F}^{5}_{01} & \cdots & \overline{F}^{5}_{0N} & \overline{F}^{6}_{01} & \cdots & \overline{F}^{6}_{0N} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{F}^{3}_{N0} & \overline{F}^{4}_{10} & \overline{F}^{1}_{N1} & \cdots & \overline{F}^{1}_{NN} & \overline{F}^{2}_{N1} & \cdots & \overline{F}^{2}_{NN} & \overline{F}^{3}_{N1} & \cdots & \overline{F}^{3}_{NN} & \overline{F}^{4}_{N1} & \cdots & \overline{F}^{4}_{NN} & \overline{F}^{5}_{N1} & \cdots & \overline{F}^{5}_{NN} & \overline{F}^{6}_{N1} & \cdots & \overline{F}^{6}_{NN} \\
\overline{G}^{3}_{10} & \overline{G}^{4}_{10} & \overline{G}^{1}_{11} & \cdots & \overline{G}^{1}_{1N} & \overline{G}^{2}_{11} & \cdots & \overline{G}^{2}_{1N} & \overline{G}^{3}_{11} & \cdots & \overline{G}^{3}_{1N} & \overline{G}^{4}_{11} & \cdots & \overline{G}^{4}_{1N} & \overline{G}^{5}_{11} & \cdots & \overline{G}^{5}_{1N} & \overline{G}^{6}_{11} & \cdots & \overline{G}^{6}_{11} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{G}^{3}_{N0} & \overline{G}^{4}_{N0} & \overline{G}^{1}_{N1} & \cdots & \overline{G}^{1}_{NN} & \overline{G}^{2}_{N1} & \cdots & \overline{G}^{2}_{NN} & \overline{G}^{3}_{N1} & \cdots & \overline{G}^{3}_{NN} & \overline{G}^{4}_{N1} & \cdots & \overline{G}^{4}_{NN} & \overline{G}^{5}_{N1} & \cdots & \overline{G}^{5}_{NN} & \overline{G}^{6}_{N1} & \cdots & \overline{G}^{6}_{NN}
\end{bmatrix}
\begin{bmatrix}
A_{30} \\ B_{30} \\ A_{11} \\ \vdots \\ A_{1N} \\ B_{11} \\ \vdots \\ B_{1N} \\ A_{21} \\ \vdots \\ A_{2N} \\ B_{21} \\ \vdots \\ B_{2N} \\ A_{31} \\ \vdots \\ A_{3N} \\ B_{31} \\ \vdots \\ B_{3N}
\end{bmatrix}
= 0 \qquad (21)
$$

where

$$\overline{\widehat{E}}_{mn}^{j} = \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} \bar{e}_n^{j}\left(\widehat{R}_i,\theta\right)\sin m\theta d\theta$$
$$\overline{\widehat{F}}_{mn}^{j} = \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} \bar{f}_n^{j}\left(\widehat{R}_i,\theta\right)\cos m\theta d\theta \tag{22.a}$$
$$\overline{\widehat{G}}_{mn}^{j} = \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} \bar{g}_n^{j}\left(\widehat{R}_i,\theta\right)\sin m\theta d\theta$$

$$\overline{E}_{mn}^{j} = \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} \bar{e}_n^{j}\left(R_i,\theta\right)\sin m\theta d\theta$$
$$\overline{F}_{mn}^{j} = \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} \bar{f}_n^{j}\left(R_i,\theta\right)\cos m\theta d\theta \tag{22.b}$$
$$\overline{G}_{mn}^{j} = \left(2\varepsilon_n/\pi\right)\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} \bar{g}_n^{j}\left(R_i,\theta\right)\sin m\theta d\theta$$

and where $j = 1,2,3,4,5$ and 6, I is the number of segments, $\widehat{R}_i$ is the coordinate $\hat{r}$ at the inner boundary, R_i is the coordinate r at the outer boundary and N is the number of truncation of the Fourier series. For the nontrivial solution of the system of equations given in Eqs. (19) and (21), the determinant of the coefficient matrix must vanish and these determinants give the frequencies of symmetric and antisymmetric modes of vibrations respectively.

4.1 Elliptic cross-sectional plate

The geometry of elliptic cross-sectional ring shaped plate is shown in Figure 1.The geometrical relations of an elliptic ring shaped plate given by Nagaya (1981b) are used for numerical calculation and are given below:

$$R_i/b_1 = \left(a_2/b_1\right)\Big/\left[\cos^2\theta + \left(a_2/b_2\right)^2\sin^2\theta\right]^{1/2}$$
$$\gamma_i = \pi/2 - \tan^{-1}\left[\left(b_2/a_2\right)^2\Big/\tan\theta_i^*\right], \quad \text{for } \theta_i^* < \pi/2$$
$$\gamma_i = \pi/2,\ \theta_i^* = \pi/2 \tag{23.a}$$
$$\gamma_i = \pi/2 + \tan^{-1}\left[\left(b_2/a_2\right)^2\Big/\left|\tan\theta_i^*\right|\right], \quad \text{for } \theta_i^* > \pi/2$$

for the outer surface and

$$\widehat{R}_i/b_1 = \left(a_1/b_1\right)\Big/\left[\cos^2\theta + \left(a_1/b_1\right)^2\sin^2\theta\right]^{1/2}$$
$$\gamma_i = \pi/2 - \tan^{-1}\left[\left(b_1/a_1\right)^2\Big/\tan\theta_i^*\right], \quad \text{for } \theta_i^* < \pi/2$$
$$\gamma_i = \pi/2,\ \theta_i^* = \pi/2 \tag{23.b}$$
$$\gamma_i = \pi/2 + \tan^{-1}\left[\left(b_1/a_1\right)^2\Big/\left|\tan\theta_i^*\right|\right], \quad \text{for } \theta_i^* > \pi/2$$

for the inner surface, where a_1 and a_2 are the length of inner and outer semi major axis, and b_1 and b_2 are the length of semi minor axis of an elliptic cross-section. Also $\theta_l^* = \left(\theta_l + \theta_{l-1}\right)/2$ and R_i is the coordinate r at the $i-th$ boundary, γ_i is the angle between the reference axis and the normal to the segment. For in-plane vibration problem, there exist two types of vibrations; one of which is generated mainly by flexural motion and the other by longitudinal motion.

In the present problem, there are three kinds of basic independent modes of wave propagation have been considered, namely, the longitudinal and two flexural (symmetric and antisymmetric) modes.

5 NUMERICAL ANALYSIS

The numerical analysis of the frequency equation is carried out for heat conducting thermo elastic elliptic cross-sectional plates. The material properties of copper at 42 K are taken approximately as Poisson ratio $\nu = 0.3$, density $\rho = 8.96\text{x}10^3$ kg/m^3, the Young's modulus $E = 2.139\text{x}10^{11}$ N/m^2, $\lambda = 8.20\text{x}10^{11}$ kg/m*s^2, $\mu = 4.20\text{x}10^{10}$ kg/m*s^2, $c_\nu = 9.1\text{x}10^{-2}$ m^2/K*s^2, and $K = 113\text{x}10^{-2}$ kg*m/K*s^2. The thermal parameters such α', ε_1, ε_2, and ε_3 are chosen by the following arguments are given by Erbay and Suhubi (1986).

In the numerical calculation, the angle θ is taken as an independent variable and the coordinate R_i and $\widehat{R}_i$ are at the $i-th$ segment of the boundary is expressed in terms of θ. Substituting R_i, $\widehat{R}_i$ and the angle γ_i, between the reference axis and the normal to the $i-th$ boundary line, the integrations of the Fourier coefficients e_n^i, f_n^i, g_n^i, $\bar{e}_n^i$, $\bar{f}_n^i$, and $\bar{g}_n^i$ can be expressed in terms of the angle θ. Using these coefficients in to equations (20) and (22), the frequencies are obtained for heat conducting thermo elastic elliptical cross-sectional plate.

5.1 Longitudinal mode

The geometrical relations for the elliptic cross-sections given in Eq. (23) are used directly for the numerical calculations, and three kinds of basic independent modes of wave propagation are studied. In case of the longitudinal mode of elliptical cross-section, the cross-section vibrates along the axis of the cylinder, so that the vibration and displacements in the cross-section is symmetrical about both major and minor axes. Hence, the frequency equation is obtained by choosing both terms of n and m as $0,2,4,6,...$ in Eq. (19) for the numerical calculations. Since the boundary of the elliptical cross-sections is irregular in shape, it is difficult to satisfy the boundary conditions along the curved surface, and hence Fourier expansion collocation method is applied. In this method, the curved surface, in the range $\theta = 0$ and $\theta = \pi$ is divided into 20 segments, such that the distance between any two segments is negligible and the integrations is performed for each segment numerically by using the Gauss five point formula .The non-dimensional frequencies are computed for $0 < \Omega \le 1.2$, using the secant method.

5.2 Flexural mode

In the case of flexural mode of elliptical cross-section, the vibration and displacements are antisymmetrical about the major axis and symmetrical about the minor axis. Hence, the frequency equations are obtained from Eq. (21) by choosing $n, m = 1,3,5,...$. Two kinds of flexural (symmetric and antisymmetric) modes are considered. The computed non-dimensional frequencies are presented in the form of dispersion curves.

5.3 Quality factor

The ratio of the total elastic energy and the energy loss in one cycle of a material is defined as the quality factor. In engineering and physics, the quality factor of a material is a dimensionless parameter that compares the time constant for decay of an oscillating physical structure's amplitude to its oscillation period. Equivalently, it compares the frequency at which a structure oscillates to the rate at which it dissipates its energy. The quality factor is defined as

$$Q_F = \frac{1}{2}\left|\frac{\omega}{VQ}\right|$$

where V and Q represents the phase velocity and attenuation coefficient, respectively.

5.4 Dispersion curve

The variation of non-dimensional frequency versus dimensinonless wave number of longitudinal modes of thermo-elastic circular cross sectional plate with and without thermal field for the two values of aspect ratio $a_1/b_1 = a_2/b_2 = 0.5,1.5$ is shown in Figures 2 and 3 respectively. From Figure 2, it is observed that the dispersion linearly increase with respect to the wave number. For without thermal field, the relation between the wave number and frequency in both the cases of aspect ratios are alike, but for the thermal inclusion the behavior is oscillating throughout the full range of wave number. Figures 4 and 5 shows the dispersion characteristic of flexural (symmetric) modes of elliptic cross-sectional plate with and without thermal field. It shows that, the dispersion is common for both the plate with and without thermal inclusion, where the higher aspect ratio attains higher frequency compared with lower aspect ratio.

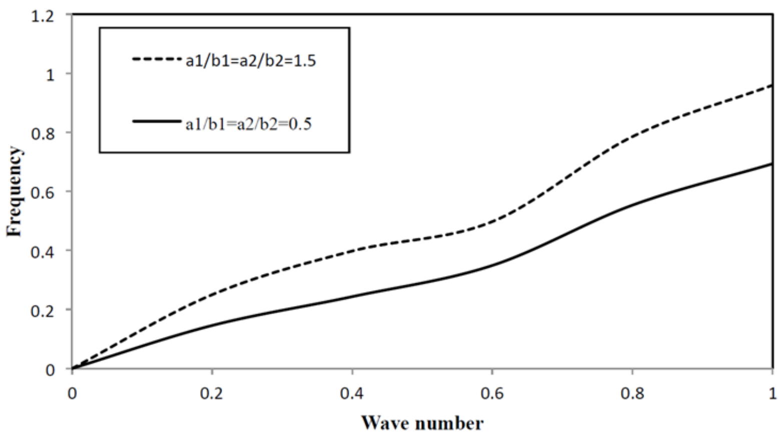

Figure 2: Non-dimensional wave number versus dimensinonless frequency of longitudinal modes of circular cross sectional plate.

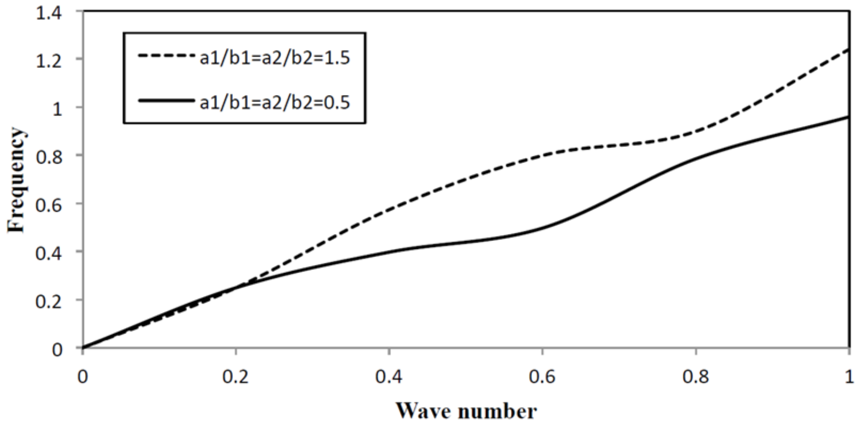

Figure 3: Non-dimensional wave number versus dimensinonless frequency of longitudinal modes of thermo-elastic circular cross sectional plate.

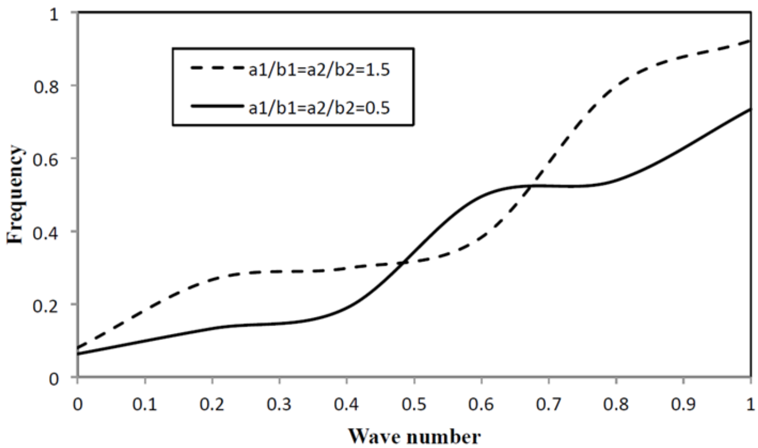

Figure 4: Non-dimensional wave number versus dimensinonless frequency of flexural symmetric modes of elliptical cross sectional plate.

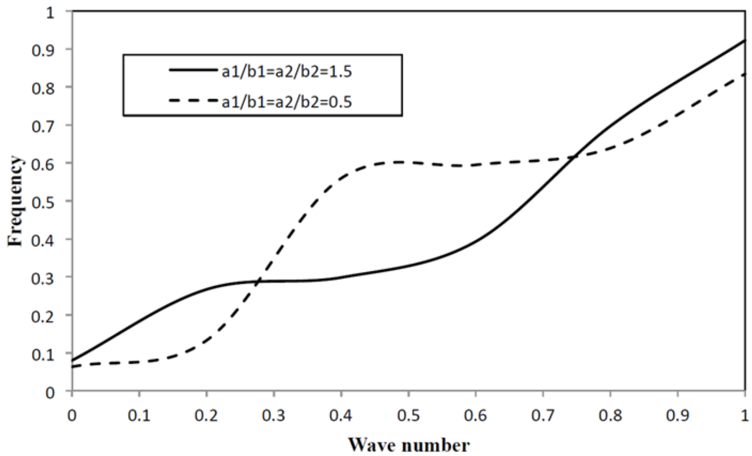

Figure 5: Non-dimensional wave number versus dimensinonless frequency of flexural symmetric modes of thermo-elastic elliptical cross sectional plate.

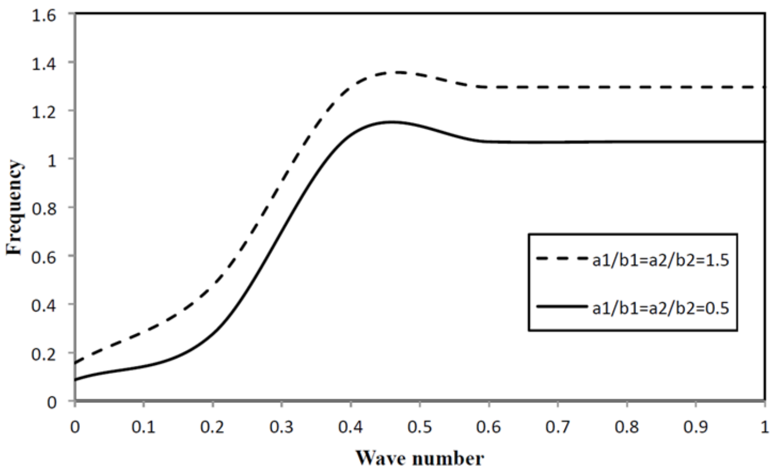

Figure 6: Non-dimensional wave number versus dimensinonless frequency of flexural anti symmetric modes of elliptical cross sectional plate.

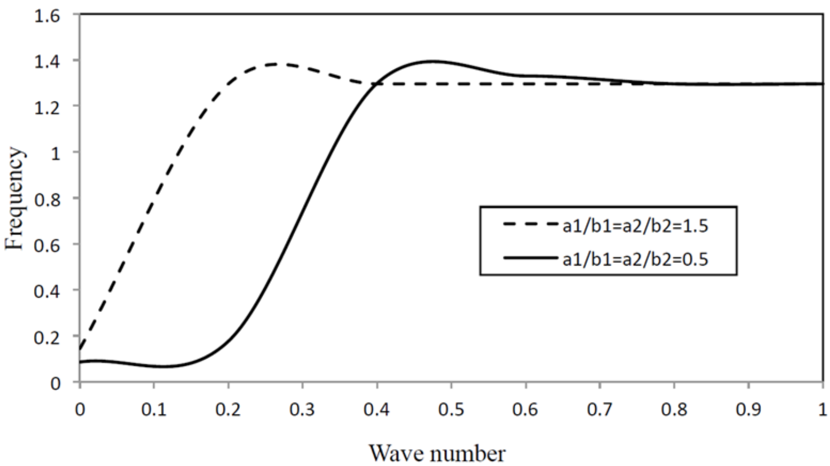

Figure 7: Non-dimensional wave number versus dimensinonless frequency of flexural anti symmetric modes of thermo-elastic elliptical cross sectional plate.

A graph is drawn between non-dimensional wave number versus dimensionless frequency of flexural (antisymmetric) modes of elliptic cross-sectional plate with and without thermal environment are shown in Figures 6 and 7 respectively. In both figures, it is observed that as the wave number increases, the non-dimensional frequencies also increases, also it could be inferred that, the dispersion characteristic are prominent only in lower orders with negligible variations in higher rang. From the figures, it is observed that the non-dimensionless frequency increases with respect to its wave number. It is also observed that, the cross over points between the modes of different aspect ratios of longitudinal and flexural symmetric and antisymmetric modes represents the transportation of energy between the two different vibration medium.

The variation of velocity versus dimensionless frequency of a flexural symmetric modes of elliptic cross-sectional plate with and without thermal environment are shown in Figures 8 and 9, respectively. From the Figures 8 and 9, it is observed that, for the increasing values of the aspect ratio modes are merges and oscilating through entire range of frequency. The merging of thermal modes and oscillation of point between the vibrational modes shows that, there is an energy transportation between the modes of vibrations by the effect of aspect ratio and anisotropic of the material.

Figures 10 and 11 reveals that the variation of quality factor with the dimensionless frequency for the flexural symmetric modes with and without thermal signal. The quality factor is quite high at lower range of frequency and starts to decay with increasing frequency. The quality factor profile are dispersive in trend for flexural symmetrical modes with thermal signal than in flexural symmetrical modes without thermal signal and experience oscillation in the wave number range $0.2 \le |\varsigma| \le 0.6$ for flexural symmetric thermal modes of vibration. The cross over points between the vibration modes represents the transfer of energy between the thermal modes.

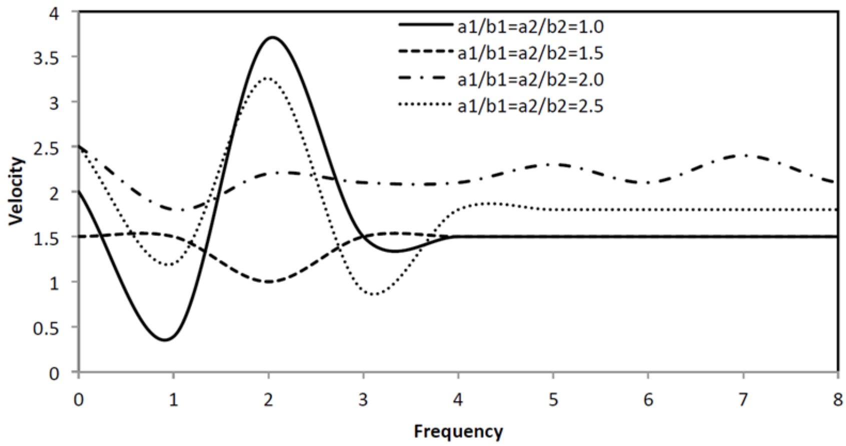

Figure 8: Variation of velocity versus dimensinonless frequency of flexural symmetric modes of elliptical cross sectional plate.

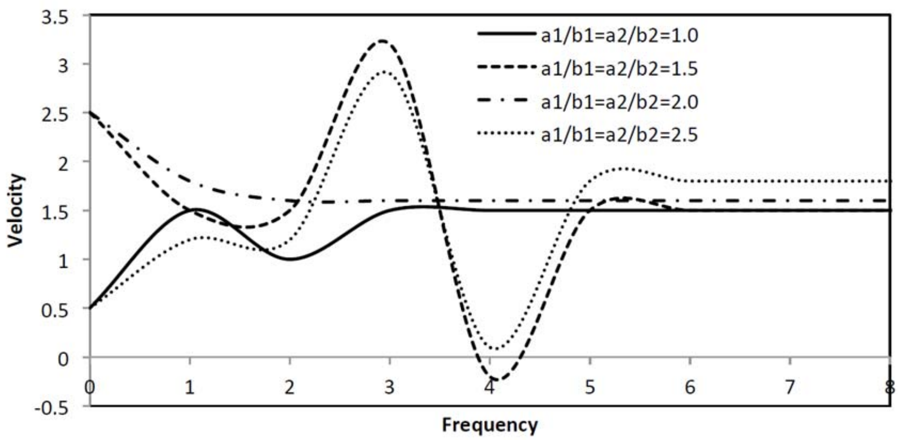

Figure 9: Variation of velocity versus dimensinonless frequency of flexural symmetric modes of thermo-elastic elliptical cross sectional plate.

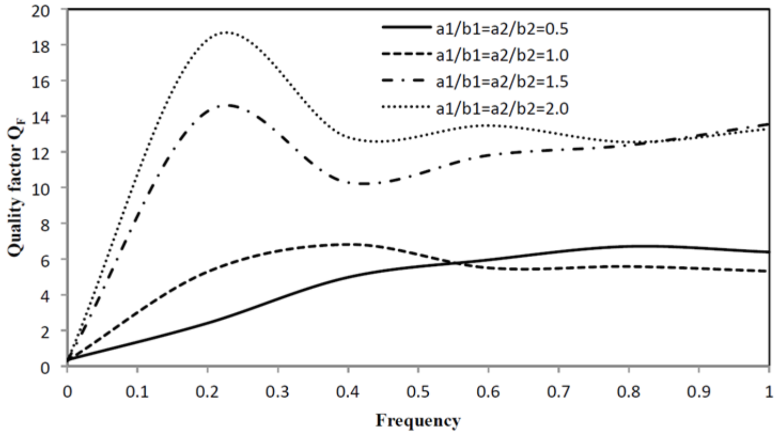

Figure 10: Variation of quality factor versus dimensinonless frequency of flexural symmetric modes of elliptical cross sectional plate.

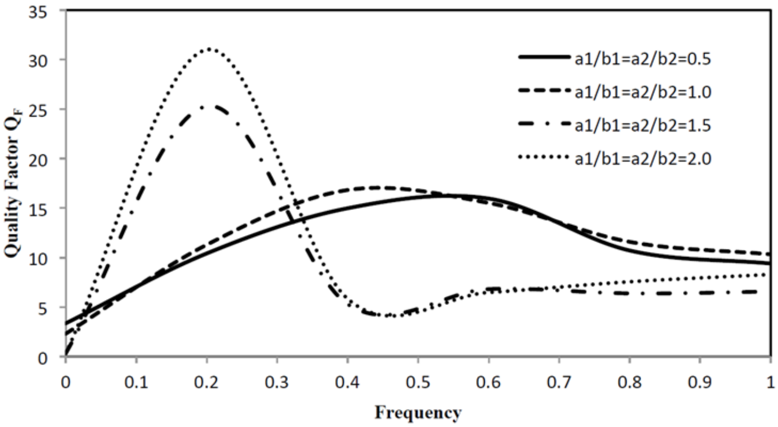

Figure 11: Variation of quality factor versus dimensinonless frequency of flexural symmetric modes of thermo-elastic elliptical cross sectional plate.

6 CONCLUSIONS

In this paper, the wave propagation in a heat conducting thermo elastic plate of elliptical shaped is analyzed by satisfying the boundary conditions on the irregular boundary using the Fourier expansion collocation method and the frequency equation for the longitudinal and flexural (symmetric and antisymmetric) modes of vibrations are obtained. The computed dimensionless frequency, velocity and quality factor are plotted in graphs for longitudinal and flexural (symmetric and antisymmetric) modes with and without thermal signals. From the graphical representation, the effect of thermal energy with thermal relaxation times and the anisotropy of the material on the various considered wave characteristics are more significant and dominant in the flexural modes of elliptical cross sectional plate.

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Appendix A

The expressions $e_n^i \sim \bar{k}_n^i$ used in Eqs. (20) and (22) are given as follows:

$$\begin{aligned} e_n^i &= 2\left\{n(n-1)J_n(\alpha_i ax) + (\alpha_i ax)J_{n+1}(\alpha_i ax)\right\}\cos 2(\theta-\gamma_i)\cos n\theta \\ &- x^2\left\{(\alpha_i a)^2 + \left[\bar{\lambda} + 2\cos^2(\theta-\gamma_i)\right] + \bar{\lambda} d_i\left(1+i\Omega\varepsilon_3\right)\right\}J_n\left(\alpha_i ax\right)\cos n\theta \\ &+ 2n\left\{(n-1)J_n(\alpha_i ax) - (\alpha_i ax)J_{n+1}(\alpha_i ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i), \quad i=1,2 \end{aligned} \tag{A.1}$$

$$\begin{aligned} e_n^3 &= 2\left\{n(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)\right\}\cos n\theta \quad \cos 2(\theta-\gamma_i) \\ &+2\left\{[n(n-1) - (\alpha_3 ax)^2]J_n(\alpha_3 ax) + (\alpha_3 ax)J_{n+1}(\alpha_3 ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i) \end{aligned} \tag{A.2}$$

$$\begin{aligned} e_n^4 &= 2\left\{n(n-1)Y_n(\alpha_3 ax) - (\alpha_3 ax)Y_{n+1}(\alpha_3 ax)\right\}\cos n\theta \quad \cos 2(\theta-\gamma_i) \\ &+2\left\{[n(n-1) - (\alpha_3 ax)^2]Y_n(\alpha_3 ax) + (\alpha_3 ax)Y_{n+1}(\alpha_3 ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i) \end{aligned} \tag{A.3}$$

$$\begin{aligned} e_n^i &= 2\left\{n(n-1)Y_n(\alpha_i ax) + (\alpha_i ax)Y_{n+1}(\alpha_i ax)\right\}\cos 2(\theta-\gamma_i)\cos n\theta \\ &- x^2\left\{(\alpha_i a)^2 + \left[\bar{\lambda} + 2\cos^2(\theta-\gamma_i)\right] + \bar{\lambda} d_i\left(1+i\Omega\varepsilon_3\right)\right\}Y_n\left(\alpha_i ax\right)\cos n\theta \\ &+ 2n\left\{(n-1)Y_n(\alpha_i ax) - (\alpha_i ax)Y_{n+1}(\alpha_i ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i), \quad i=5,6 \end{aligned} \tag{A.4}$$

$$\begin{aligned} f_n^i &= 2\left\{[n(n-1) - (\alpha_i ax)^2]J_n(\alpha_i ax) + (\alpha_i ax)J_{n+1}(\alpha_i ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i) \\ &+ \quad 2n\left\{(\alpha_i ax)J_{n+1}(\alpha_i ax) - (n-1)J_n(\alpha_i ax)\right\}\sin n\theta \cos 2(\theta-\gamma_i), i=1,2 \end{aligned} \tag{A.5}$$

$$\begin{aligned} f_n^3 &= 2n\left\{(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i) \\ &-\left\{2(\alpha_3 ax)J_{n+1}(\alpha_3 ax) - [(\alpha_3 ax)^2 - 2n(n-1)]J_n(\alpha_3 ax)\right\}\sin n\theta \cos 2(\theta-\gamma_i) \end{aligned} \tag{A.6}$$

$$\begin{aligned} f_n^4 &= 2n\left\{(n-1)Y_n(\alpha_3 ax) - (\alpha_3 ax)Y_{n+1}(\alpha_3 ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i) \\ &-\left\{2(\alpha_3 ax)Y_{n+1}(\alpha_3 ax) - [(\alpha_3 ax)^2 - 2n(n-1)]Y_n(\alpha_3 ax)\right\}\sin n\theta \cos 2(\theta-\gamma_i) \end{aligned} \tag{A.7}$$

$$\begin{aligned} f_n^i &= 2\left\{[n(n-1) - (\alpha_i ax)^2]Y_n(\alpha_i ax) + (\alpha_i ax)Y_{n+1}(\alpha_i ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i) \\ &+ \quad 2n\left\{(\alpha_i ax)Y_{n+1}(\alpha_i ax) - (n-1)Y_n(\alpha_i ax)\right\}\sin n\theta \cos 2(\theta-\gamma_i), i=5,6 \end{aligned} \tag{A.8}$$

$$k_n^i = d_i\left\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)J_n\left(\alpha_i ax\right) - \left(\alpha_i ax\right)J_{n+1}\left(\alpha_i ax\right)\cos(\theta-\gamma_i)\cos n\theta\right\}, i=1,2 \tag{A.9}$$

$$k_n^3 = 0.0\,,\ k_n^4 = 0.0 \tag{A.10}$$

$$k_n^i = d_i\left\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)Y_n\left(\alpha_i ax\right) - \left(\alpha_i ax\right)Y_{n+1}\left(\alpha_i ax\right)\cos(\theta-\gamma_i)\cos n\theta\right\}, i=5,6 \tag{A.11}$$

$$\begin{aligned} \bar{e}_n^j &= 2\left\{n(n-1)J_n(\alpha_i ax) + (\alpha_i ax)J_{n+1}(\alpha_i ax)\right\}\cos 2(\theta-\gamma_i)\sin n\theta \\ &- x^2\left\{(\alpha_i a)^2 + \left[\bar{\lambda} + 2\cos^2(\theta-\gamma_i)\right] + d_i\left(1+i\Omega\varepsilon_3\right)\right\}J_n\left(\alpha_i ax\right)\cos n\theta \\ &- 2n\left\{(n-1)J_n(\alpha_i ax) - (\alpha_i ax)J_{n+1}(\alpha_i ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i), i=1,2 \end{aligned} \tag{A.12}$$

$$\begin{aligned}\bar{e}_n^3 &= 2\left\{n(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)\right\}\sin n\theta \quad \cos 2(\theta-\gamma_i)\\ &-2\left\{[n(n-1)-(\alpha_i ax)^2]J_n(\alpha_3 ax) + (\alpha_3 ax)J_{n+1}(\alpha_3 ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i)\end{aligned} \tag{A.13}$$

$$\begin{aligned}\bar{e}_n^4 &= 2\left\{n(n-1)Y_n(\alpha_3 ax) - (\alpha_3 ax)Y_{n+1}(\alpha_3 ax)\right\}\sin n\theta \quad \cos 2(\theta-\gamma_i)\\ &-2\left\{[n(n-1)-(\alpha_i ax)^2]Y_n(\alpha_3 ax) + (\alpha_3 ax)Y_{n+1}(\alpha_3 ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i)\end{aligned} \tag{A.14}$$

$$\begin{aligned}\bar{e}_n^j &= 2\left\{n(n-1)Y_n(\alpha_i ax) + (\alpha_i ax)Y_{n+1}(\alpha_i ax)\right\}\cos 2(\theta-\gamma_i)\sin n\theta\\ &- \quad x^2\left\{(\alpha_i a)^2 + \left[\bar{\lambda} + 2\cos^2(\theta-\gamma_i)\right] + d_i\left(1+i\Omega\varepsilon_3\right)\right\}Y_n\left(\alpha_i ax\right)\cos n\theta\\ &- 2n\left\{(n-1)Y_n(\alpha_i ax) - (\alpha_i ax)Y_{n+1}(\alpha_i ax)\right\}\cos n\theta \sin 2(\theta-\gamma_i), i=5,6\end{aligned} \tag{A.15}$$

$$\begin{aligned}\bar{f}_n^i &= 2\left\{[n(n-1)-(\alpha_i ax)^2]J_n(\alpha_i ax) + (\alpha_i ax)J_{n+1}(\alpha_i ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i)\\ &- \quad 2n\left\{(\alpha_i ax)J_{n+1}(\alpha_i ax) - (n-1)J_n(\alpha_i ax)\right\}\cos n\theta \cos 2(\theta-\gamma_i), i=1,2\end{aligned} \tag{A.16}$$

$$\begin{aligned}\bar{f}_n^3 &= 2n\left\{(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i)\\ &+\left\{2(\alpha_3 ax)J_{n+1}(\alpha_3 ax) - [(\alpha_3 ax)^2 - 2n(n-1)]J_n(\alpha_3 ax)\right\}\cos n\theta \cos 2(\theta-\gamma_i)\end{aligned} \tag{A.17}$$

$$\begin{aligned}\bar{f}_n^4 &= 2n\left\{(n-1)Y_n(\alpha_3 ax) - (\alpha_3 ax)Y_{n+1}(\alpha_3 ax)\right\}\sin n\theta \sin 2(\theta-\gamma_i)\\ &+\left\{2(\alpha_3 ax)Y_{n+1}(\alpha_3 ax) - [(\alpha_3 ax)^2 - 2n(n-1)]Y_n(\alpha_3 ax)\right\}\cos n\theta \cos 2(\theta-\gamma_i)\end{aligned} \tag{A.18}$$

$$\bar{k}_n^i = d_i\left\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)J_n\left(\alpha_i ax\right) + \left(\alpha_i ax\right)J_{n+1}\left(\alpha_i ax\right)\cos(\theta-\gamma_i)\sin n\theta\right\}, i=1,2 \tag{A.19}$$

$$\bar{k}_n^3 = 0.0\,,\ \bar{k}_n^4 = 0.0 \tag{A.20}$$

$$\bar{k}_n^i = d_i\left\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)Y_n\left(\alpha_i ax\right) + \left(\alpha_i ax\right)Y_{n+1}\left(\alpha_i ax\right)\cos(\theta-\gamma_i)\sin n\theta\right\}, i=5,6 \tag{A.21}$$