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Time-dependent behaviour of reacting viscoelastic concrete

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Abstract

Concrete exhibits time-dependent behaviour like basic creep, recovery and stress relaxation. Solidification theory proposed by Bazant and co-workers attempts to simulate the aging creep of hydrating concrete under uniaxial stress. There exists considerable empirical evidence to the effect that even mature concrete exhibits creep-like phenomena on exposure to chemically aggressive environment. The dissolution-precipitation mechanism has been proposed by Benipal for explaining such behaviour of reacting elastic solids. Based on this chemomechanical mechanism, a theory of reacting isotropic linear elastic solids has recently been proposed by the authors. This theory is capable of explaining long-term behaviour, but is incapable of predicting the creep of mature concrete not involved in any reactions. In this paper, a theory of isotropic linear viscoelastic solids applicable to both reacting and inert solids has been proposed. Here, the earlier model dealing with the reacting elastic solids has been named reacting rheological element. The present theory of reacting viscoelastic concrete is based on the above reacting rheological model arranged in parallel with an aging Maxwell rheological element. This theory has been shown to predict satisfactorily the observed time-dependent behaviour of concrete involved in hydration and other reactions simultaneously subjected to multiaxial stress/strain histories. The theoretical significance and the practical relevance of the results obtained have been discussed.

Keywords: chemomechanics, dissolution-precipitation theory, aging Maxwell model, time-dependent behaviour, reacting viscoelastic concrete

1 Introduction

One dimensional classical rheological models such as Kelvin-Voigt model, three-element model, etc., are used in predicting the time-dependent deformation of linear viscoelastic non-aging materials. These models are inadequate to predict the behaviour of materials like concrete exhibiting aging viscoelastic response [4]. After many attempts at proposing empirical models, a proper theory called solidification theory was proposed for predicting aging creep of hydrating

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concrete [2]. As per this theory, creep is caused by presence of viscoelastic hydration products of concrete whereas aging is associated with their volume growth on solidification.

It has been reported [1,5] that even mature concrete, which otherwise does not exhibit much creep, does so on exposure to reacting chemicals. During their service life, concrete structures under the action of service loads are simultaneously exposed to reacting chemicals in their environment. As such, they are expected to exhibit creep-like behavior, apart from the gradual reduction in stiffness and strength. As a matter of fact, the service life may itself be determined by the excessive displacements under service loads as well as chemical deterioration. Thus, there is a need to propose theories for predicting the chemomechanical response of such materials.

Linear aging viscoelasticity theories of homogeneous isotropic materials are the simplest ones of the chemomechanical theories. Under the action of service loads, concrete constituents can justifiably be assumed to be linear in the sense that their mechanical properties like elasticity and viscosity are independent of the magnitude of stress components. Any effect that is stress dependent as plasticity or cracking is not sought to be included in these theories. In contrast, aging is understood here as time-dependent autonomous variation of the above material properties. For example, the hydration of concrete causes gradual increase in stiffness, while the damage caused by softening at elevated temperatures or by chemical deterioration results in gradual stiffness degradation.

Such a linear aging chemomechanical theory called dissolution- precipitation theory (DPT) was proposed [3] in which creep in reacting elastic solids is attributed to be caused by the gradual transformation of the uniaxially stressed reacting elastic constituents into stress-free solute on dissolution. In contrast to the classical viscoelasticity theories as well as the solidification theory (ST), none of the constituent phases has been assumed to be viscoelastic in nature in the above theory. Recently, based on this dissolution precipitation mechanism of chemomechanical behaviour, a dissolution-precipitation theory has been proposed to model the time-dependent mechanical behaviour of reacting homogeneous isotropic linear elastic solids under multiaxial stress/strain histories [7,8]. The above micromechanics-based dissolution precipitation theory has also been recast in a phenomenological form wherein the empirical parameters of the newly proposed 'reacting rheological element' can be quantified by using the available experimental data [6]. It can be observed that the solidification theory as well as the dissolution precipitation theory belongs to the class of linear aging theories.

The dissolution-precipitation theory, at its current stage of development, is incapable of predicting observed time-dependent deformations such as creep, recovery and relaxation exhibited by mature inert concrete not involved in any reactions. This fact implies that concrete is not an elastic solid as assumed in the present theory but a viscoelastic solid capable of exhibiting time-dependent deformations even in the absence of any reactions. In view of these facts, in this paper, the above phenomenological dissolution-precipitation theory has been extended to incorporate the effect of viscoelastic nature of the material. In the rheological reacting viscoelastic model studied here, the above reacting rheological element is arranged in parallel to an aging Maxwell rheological element. The creep, recovery, superposition and relaxation behavior of concrete has been investigated has been delineated and compared with the available empirical data. The temporal variation of the forces in two elements has also been traced. The material modeled is assumed to be isotropic linear aging viscoelastic solid undergoing infinitesimally small deformations.

2 Proposed rheological model

In this paper, the mechanical behaviour of a rheological model composed of above reacting rheological element and aging Maxwell element arranged in-parallel (Fig. 1) has been studied.

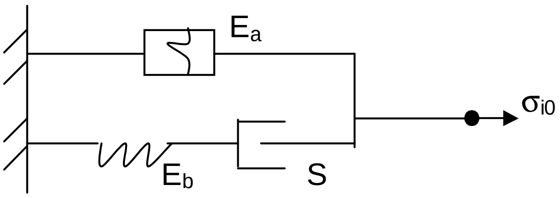


Figure 1: The proposed rheological model.

Let the subscripts a and b pertain to the reacting element and the Maxwell element respectively. The following relations for stress σ and strain ε hold for these rheological elements arranged in parallel.

$$\sigma = \sigma_a + \sigma_b \quad \text{and} \quad \varepsilon = \varepsilon_a = \varepsilon_b$$
$$\dot{\sigma} = \dot{\sigma}_a + \dot{\sigma}_b \quad \text{and} \quad \dot{\varepsilon} = \dot{\varepsilon}_a = \dot{\varepsilon}_b \tag{1}$$

where the superposed 'dot' denotes rate of variation of stresses, strains and other variables with time t.

The constitutive equation of the rate type for the reacting element derived earlier for sustained state of stress [7,8] has been modified here for the general case of variable stresses in the following form:

$$\dot{\varepsilon} = \frac{\varepsilon\,\dot{\alpha}}{(p + q\alpha)} + \frac{\dot{\sigma}_a}{r(p + q\alpha)} \tag{2}$$

where α represents the fractional extent of the reaction occurred and increases monotonically with time t. The $\alpha - t$ relation quantifying the reaction kinetics has been presented later. From the authors' earlier work on micromechanical aspects [7, 8], the parameters p and q are identified to represent the relative stiffness of the inert phase and the reacting products deposited respectively. The parameter r denotes the absolute stiffness of the reacting phase before the start of the reaction. Its value can be estimated from the following expression knowing the values of the volume fraction v_{10} of the reacting phase before the start of the reaction and its elastic modulus E_1.

$$r = v_{10} E_1 \tag{3}$$

Let E_a, representing the instantaneous stiffness of the reacting element, be given by

$$E_a = r(p + q\alpha) \tag{4}$$

From the above equations, one obtains the constitutive equation of the reacting element in the following simpler form

$$\dot{\varepsilon}_a = \frac{r\varepsilon\,\dot{\alpha}}{E_a} + \frac{\dot{\sigma}_a}{E_a} \tag{5}$$

The corresponding constitutive equation for the Maxwell element with elastic constant E_b and instantaneous viscosity S is stated as

$$\dot{\varepsilon}_b = \frac{\dot{\sigma}_b}{E_b} + \frac{\sigma_b}{S} \tag{6}$$

Using equation (1) and eliminating σ_a from the above equations (5) and (6), one obtains the following governing equation for the proposed rheological model

$$\frac{\sigma_b}{S} + \left(\frac{1}{E_a} + \frac{1}{E_b}\right)\dot{\sigma}_b = \frac{\dot{\sigma} + r\varepsilon\,\dot{\alpha}}{E_a} \tag{7}$$

It has been established by the authors [7,8] that the relevant equations given above for the one dimensional case applied to all the principal stress directions separately.

In the next section, the creep, recovery, superposition and relaxation behaviour of reacting viscoelastic concrete has been analyzed qualitatively. The empirical calibration of the model and the quantitative prediction of the material response have been undertaken in the subsequent section.

3 Qualitative analysis of material behaviour

3.1 Creep

Consider a material point under the action of triaxial normal stresses σ_i and exhibiting time-dependent normal strains ε_i. Let a state of average stress σ_{i0} be applied at any time t=t_0, i.e., $\alpha = \alpha_0$, and kept sustained thereafter. As $\dot{\sigma}_i = 0$, the material exhibits creep even though the element stresses σ_{ia} and σ_{ib} keep on varying with time. Thus, the differential equations (1) and (7) reduce to

$$\begin{array}{l} \frac{\sigma_{ib}}{S} + \left(\frac{1}{E_a} + \frac{1}{E_b}\right)\dot{\sigma}_{ib} = \frac{r\varepsilon_i\,\dot{\alpha}}{E_a} \\ \dot{\sigma}_{ia} = -\,\dot{\sigma}_{ib}\,and\;\dot{\varepsilon}_i = \dot{\varepsilon}_{ia} = \dot{\varepsilon}_{ib} \end{array} \tag{8}$$

Let σ_{ai0} and σ_{bi0} be the initial stresses in reacting element and Maxwell element respectively and ε_{ai0} and ε_{bi0} are the corresponding instantaneous strains (at t=t$_0$) just after the application of stresses. In view of equation (1),

$$\sigma_{i0} = \sigma_{ai0} + \sigma_{bi0} \quad \varepsilon_{ai0} = \varepsilon_{bi0} = \varepsilon_{i0} \tag{9}$$

Just before loading, the elements are stress-free and strain-free. The instantaneously applied stress components σ_{i0} introduce stress increments σ_{ai0} and σ_{bi0} in the reacting and Maxwell elements respectively. The corresponding instantaneous incremental elastic strain components are given by the Hooke's Law as follows:

$$\varepsilon_{i0} = \frac{\sigma_{aj0}\left[\delta_{ij} - (1-\delta_{ij})\nu\right]}{E_{a0}} = \frac{\sigma_{bj0}\left[\delta_{ij} - (1-\delta_{ij})\nu\right]}{E_b} \tag{10}$$

Here, $E_{a0} = r(p+q\alpha_0)$ is the instantaneous modulus of elasticity of reacting element at t=t$_0$ i.e., at $\alpha = \alpha_0$, Knowing that the parallel elements share the applied stresses σ_{i0}in proportion to their relative stiffness, the element normal stress increments σ_{ai0} and σ_{bi0} are obtained as

$$\begin{aligned} \sigma_{a10} &= \frac{E_{a0}\sigma_{10}}{E_{a0}+E_b} \quad \sigma_{a20} = \frac{E_{a0}\sigma_{20}}{E_{a0}+E_b} \quad \sigma_{30} = \frac{E_{a0}\sigma_{30}}{E_{a0}+E_b} \\ \sigma_{b10} &= \frac{E_b\sigma_{10}}{E_{a0}+E_b} \quad \sigma_{b20} = \frac{E_b\sigma_{20}}{E_{a0}+E_b} \quad \sigma_{b30} = \frac{E_b\sigma_{30}}{E_{a0}+E_b} \end{aligned} \tag{11}$$

Assuming the stress in the reacting element to remain constant during the incremental time interval Δt, or equivalently $\Delta\alpha$, both the elements exhibit creep strains. In view of the relevant equations derived earlier [7, 8], the total strain in reacting element at time t$_1$= t$_0$ + Δt or at $\alpha_1 = \alpha_0 + \Delta\alpha$ is obtained as

$$\varepsilon_{ai} = \varepsilon_{ai0}\left[\frac{p+q\alpha_1)}{p+q\alpha_0}\right]^{\frac{1}{q}} \tag{12}$$

Similarly, total strain in Maxwell element due to creep at time t$_1$ is given by the well known expression

$$\varepsilon_{bi} = \varepsilon_{bi0} + \frac{\left[\delta_{ij} - (1-\delta_{ij})\nu\right]\sigma_{bj0}}{S} \cdot \Delta t \tag{13}$$

As the Maxwell element and the reacting element are in-parallel, their strains at any time are expected to be equal, even though their stresses can vary continuously with the passage of time. Yet, as can be observed, their strains at the instant t$_1$ turn out to be unequal and the stresses are assumed to be constant in the derivation of above equations. The required strain compatibility condition is sought to be satisfied by imposing instantaneously the incremental stresses $\Delta\sigma_{ai}$ and $\Delta\sigma_{bi}$ on the two elements at the end of the time interval. As no additional external stresses are applied during this time interval, the incremental stresses are related as

$$\Delta\sigma_{ai} + \Delta\sigma_{bi} = 0 \tag{14}$$

The instantaneous strain increments $\Delta\varepsilon_{ai}$ and $\Delta\varepsilon_{bi}$ due to these stress increments $\Delta\sigma_{ai}$ and $\Delta\sigma_{bi}$ respectively are

$$\Delta\varepsilon_{ai} = \frac{[\delta_{ij} - (1-\delta_{ij})\nu]\,\Delta\sigma_{aj}}{E_{a1}} \quad \Delta\varepsilon_{bi} = \frac{[\delta_{ij} - (1-\delta_{ij})\nu]\,\Delta\sigma_{bj}}{E_b}. \tag{15}$$

where $E_{a1} = r\,[p + q\alpha_1]$. Using the equations (1), (8), (9), (11) and (13), the total strains ε_i in both the elements at t_1 are obtained as

$$\varepsilon_{ai} + \Delta\varepsilon_{ai} = \varepsilon_{bi} + \Delta\varepsilon_{bi} = \varepsilon_i \tag{16}$$

$$\varepsilon_{ai} + \frac{[\delta_{ij} - (1-\delta_{ij})\nu]\,\Delta\sigma_{aj}}{E_{a1}} = \varepsilon_{bi} + \frac{[\delta_{ij} - (1-\delta_{ij})\nu]\,\Delta\sigma_{bj}}{E_b} = \varepsilon_i \tag{17}$$

Using the equations (1), (8), (11) and (15), the values of $\Delta\sigma_{bi}$ and $\Delta\sigma_{ai}$ required for strain compatibility are estimated as

$$\Delta\sigma_{b1} = \frac{\sigma_{10}E_{a1}E_b}{(E_{a0}+E_b)\,(E_{a1}+E_b)}\left[\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right]$$

$$\Delta\sigma_{b2} = \frac{\sigma_{20}E_{a1}E_b}{(E_{a0}+E_b)\,(E_{a1}+E_b)}\left[\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right]$$

$$\Delta\sigma_{b3} = \frac{\sigma_{30}E_{a1}E_b}{(E_{a0}+E_b)\,(E_{a1}+E_b)}\left[\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right]$$

In view of the equations (8), (11), (12) and (3.1), the final state of stresses after the first time interval in reacting and Maxwell elements are $\sigma_{bi} = \sigma_{bi0} + \Delta\sigma_{bi}$ and $\sigma_{ai} = \sigma_{ai0} + \Delta\sigma_{ai}$. The relevant expressions obtained as

$$\sigma_{b1} = \frac{E_b\sigma_{10}}{E_{a0}+E_b}\left[1 + \frac{E_{a1}}{(E_{a1}+E_b)}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]$$

$$\sigma_{b2} = \frac{E_b\sigma_{20}}{E_{a0}+E_b}\left[1 + \frac{E_{a1}}{(E_{a1}+E_b)}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]$$

$$\sigma_{b3} = \frac{E_b\sigma_{30}}{E_{a0}+E_b}\left[1 + \frac{E_{a1}}{(E_{a1}+E_b)}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]$$

$$\sigma_{a1} = \frac{E_{a0}\sigma_{10}}{E_{a0}+E_b}\left[1 + \frac{E_bE_{a1}}{E_{a0}\,(E_{a1}+E_b)}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]$$

$$\sigma_{a2} = \frac{E_{a0}\sigma_{20}}{E_{a0}+E_b}\left[1 + \frac{E_bE_{a1}}{E_{a0}\,(E_{a1}+E_b)}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]$$

$$\sigma_{a3} = \frac{E_{a0}\sigma_{30}}{E_{a0}+E_b}\left[1 + \frac{E_bE_{a1}}{E_{a0}\,(E_{a1}+E_b)}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right] \tag{18}$$

Though all the normal strains are equal in the reacting and Maxwell element, using the equations (8), (12) and (17), different expressions for final strains in these elements just after the end of first time interval are obtained as

$$
\begin{aligned}
\varepsilon_{a1} &= \varepsilon_{10}\left[\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} + \frac{E_b}{E_{a1}+E_b}\left\{1-\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} + \frac{E_b}{S}\Delta t\right\}\right]\\
\varepsilon_{a2} &= \varepsilon_{20}\left[\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} + \frac{E_b}{E_{a1}+E_b}\left\{1-\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} + \frac{E_b}{S}\Delta t\right\}\right]\\
\varepsilon_{a3} &= \varepsilon_{30}\left[\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} + \frac{E_b}{E_{a1}+E_b}\left\{1-\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} + \frac{E_b}{S}\Delta t\right\}\right]\\
\varepsilon_{b1} &= \varepsilon_{10}\left[1+\frac{E_b}{S}\Delta t + \frac{E_{a1}}{E_{a1}+E_b}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]\\
\varepsilon_{b2} &= \varepsilon_{20}\left[1+\frac{E_b}{S}\Delta t + \frac{E_{a1}}{E_{a1}+E_b}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]\\
\varepsilon_{b3} &= \varepsilon_{30}\left[1+\frac{E_b}{S}\Delta t + \frac{E_{a1}}{E_{a1}+E_b}\left\{\left(\frac{p+q\alpha_1}{p+q\alpha_0}\right)^{\frac{1}{q}} - \frac{E_b}{S}\Delta t - 1\right\}\right]
\end{aligned}
\tag{19}
$$

Following the above procedure for the succeeding time intervals, the time-dependent response of the system under sustained stresses σ_{i0} as well as the temporal variation of the stresses in the two individual elements can be calculated. The states of element stress so obtained are considered to be constant during the next time interval and the above state of strain plays the role of initial state of strain for the same time interval.

As and when $\dot{\alpha} = 0$, the reacting element expected to behave like an elastic spring. Such a situation can arise on the completion of the ongoing reaction or during the absence of one of the reactants. Hence, this model becomes indistinguishable from the three-element model of general linear substance with the governing differential equation as

$$\frac{\sigma_b}{S} + \left(\frac{1}{E_a} + \frac{1}{E_b}\right)\dot{\sigma}_b = \frac{\dot{\sigma}}{E_a} \tag{20}$$

The following well known expression for the creep response of such a model is deduced from the above formulation.

$$\varepsilon_i = \frac{\left[\delta_{ij} - (1-\delta_{ij})\nu\right]\sigma_{j0}}{E_a}\left\{1 - \frac{E_b}{E_a+E_b}e^{-\frac{tE_aE_b}{S(E_a+E_b)}}\right\} \tag{21}$$

3.2 Recovery

To recapitulate, the material is subjected to a state of stress σ_{i0} imposed at $\alpha = \alpha_0$ and sustained thereafter. The future material response in the form of strains ε_i and the element stresses σ_{ai}

and σ_{bi} has been obtained above. Let the above material be unloaded at $\alpha = \alpha_1$. Equivalently, a state of stress equal and opposite to σ_{i0} is considered to be applied at $\alpha = \alpha_1$. Thereafter, the material is not under the action of any applied stresses even though the internal stresses in the form of varying element stresses do exist. In this section, a numerical algorithm is presented for obtaining material response after unloading, i.e., for $\alpha > \alpha_1$.

Following the numerical algorithm adopted for obtaining the creep response, the total strains as well as the elemental stresses are obtained at the instant just before unloading. As discussed above, the process of unloading involves an application of a state of stress ‘ -σ_{i0}’ at α_1. The element stresses introduced instantaneously are obtained by using the equation (3) and replacing E_{a0} by E_{a1}. The corresponding changes in the state of strain upon unloading are obtained by using the Hooke’s Law and the relevant elastic moduli. The final state of strain and the element stresses at an instant just after unloading are then obtained by superposing the above computed values. These strain values constitute the initial values of the strains for computing the future material response by using the procedure adopted for obtaining creep strains as well as element stresses.

3.3 Principle of superposition

The additive principle of superposition derived by the authors [7, 8] holds good for this model. Because of the complexity of the problem, the above contention has not been proved, but it has been verified to be true in this paper with the help of two states of stress applied at different times. The values of the relevant material parameters estimated later using the available experimental data have been used in this numerical study. Let a state of stress ‘A’ (10 MPa, 8 MPa and 5 MPa) be applied at any time t_0=8 days and sustained thereafter. Another state of stress ‘B’ (8 MPa, 3 MPa and -3 MPa) is imposed at t_1= 50 days in addition to the previous stresses. The states of strain at 100 days due to the above states of stress applied separately as well as applied simultaneously are calculated with the help of the equations (27). The strains at t_2=100 days due to state of stress ‘A’ are $\varepsilon_{1A} = 406.7 \times 10^{-6}$, $\varepsilon_{2A} = 290.5 \times 10^{-6}$ and $\varepsilon_{3A} = 154.1 \times 10^{-6}$. Similarly, strains at t_2=100 due stress ‘B’ are $\varepsilon_{1B} = 404.1 \times 10^{-6}$, $\varepsilon_{2B} = 113.7 \times 10^{-6}$ and $\varepsilon_{3B} = -151.6 \times 10^{-6}$. The strains due to combined state of stress (A+B) are determined as $\varepsilon_{1(A+B)} = 810.8 \times 10^{-6}$, $\varepsilon_{2(A+B)} = 404.2 \times 10^{-6}$ and $\varepsilon_{3(A+B)} = 2.5 \times 10^{-6}$ respectively.

The sum of the strains due to states of stress ‘A’ and ‘B’ are obtained as

$$\varepsilon_{1A} + \varepsilon_{1B} = 406.7 \times 10^{-6} + 404.1 \times 10^{-6} = 810.8 \times 10^{-6} = \varepsilon_{1(A+B)}$$

$$\varepsilon_{2A} + \varepsilon_{2B} = 290.5 \times 10^{-6} + 113.7 \times 10^{-6} = 404.2 \times 10^{-6} = \varepsilon_{2(A+B)}$$

$$\varepsilon_{3A} + \varepsilon_{3B} = 154.1 \times 10^{-6} - 151.6 \times 10^{-6} = 2.5 \times 10^{-6} = \varepsilon_{3(A+B)}$$

Thus, as can be seen, the validity of the principle of superposition is verified.

3.4 Stress relaxation

Consider a concrete material point exhibiting creep under the action of a constant state of average stress σ_{i0} applied at t_0. The total strains as well as the states of stress gradually change with time in both of the elements. Let, at any instant of time t_1, the state of strain is denoted by $\overline{\varepsilon}_i$ and the states of stress in the reacting and Maxwell elements are denoted by $\overline{\sigma}_{ai}$ and $\overline{\sigma}_{bi}$ respectively. These states of stress and strain are determined by using the algorithm developed above. If this state of strain is kept constant thereafter, i.e., $\dot{\varepsilon}_i = \dot{\varepsilon}_{ai} = \dot{\varepsilon}_{bi} = 0$, the equations (5) and (6) assume the following rate form.

$$r\overline{\varepsilon}_i\,\dot{\alpha} + \dot{\sigma}_{ai} = 0 \quad \text{for reacting element} \tag{22}$$

$$\frac{\sigma_{bi}}{S} + \frac{\dot{\sigma}_{bi}}{E_b} = 0 \quad \text{for Maxwell element} \tag{23}$$

The equation (22) for the reacting element can be rewritten in the following rate and differential forms.

$$\begin{aligned} \dot{\sigma}_{ai} &= -r\overline{\varepsilon}_i\,\dot{\alpha} \\ d\sigma_{ai} &= -r\overline{\varepsilon}_i d\alpha \end{aligned} \tag{24}$$

The symbol $\dot{\sigma}_{ai}$ can be interpreted as the rate of decrease of the stress component σ_{ai} with the progress of reaction and is proportional to the instantaneous rate of reaction. This implies that there is no stress relaxation in the absence of reactions. The equivalent differential relation gives the incremental decrease in the same stress component occurred during time interval dt. To obtain the value of the residual stress component σ_{ai} at some future instant $t \geq t_1$, this differential equation has to be integrated with respect to α as below:

$$\begin{aligned} \sigma_{ai} &= -r\overline{\varepsilon}_i \int_{\alpha_1}^{\alpha} d\alpha + C \\ \sigma_{ai} &= -r\overline{\varepsilon}_i(\alpha - \alpha_1) + C \end{aligned} \tag{25}$$

The constant of integration C is obtained from the condition that, at $\alpha = \alpha_1$, $\sigma_{ai} = \overline{\sigma}_{ai}$. This condition gives $C = \overline{\sigma}_{ai}$. Thus, the values of the residual stress components in the reacting element are given by the following expression:

$$\sigma_{ai} = \overline{\sigma}_{ai} - r\overline{\varepsilon}_i(\alpha - \alpha_1) \tag{26}$$

With the progress of reaction, the reacting element experiences gradual stress relaxation. The values of the stress components σ_{ai} decrease from $\overline{\sigma}_{ai}$ to the following minimum value as the reaction proceeds to completion ($\alpha = 1$).

$$(\sigma_{ai})_{\min} = \overline{\sigma}_{ai} - r\overline{\varepsilon}_i(1 - \alpha_1) \tag{27}$$

Thus, the reacting element generally has some residual state of stress. In view of the unique $\alpha - t$ relation, the value of the residual stresses can be obtained at any future instant of time $t \geq t_1$.

Similarly, the expression for the residual stress components in the Maxwell element is obtained, in terms of empirical constant C defined later, by integrating the equation (23) with respect to time t as follows:

$$\sigma_{bi} = \overline{\sigma}_{bi} \left[\frac{1 + Ct_1}{1 + Ct} \right]^{\frac{E_b}{CS_0}} \tag{28}$$

As is well known, the Maxwell element experiences complete asymptotic stress relaxation. If the Maxwell element is considered separately, the residual state of stress vanishes asymptotically. The value of the residual average stresses σ_i at any time $t \geq t_1$ is obtained by adding the instantaneous values of the stresses in both the elements as follows

$$\sigma_i = \overline{\sigma}_{ai} - r\overline{\varepsilon}_i(\alpha - \alpha_1) + \overline{\sigma}_{bi} \left[\frac{1 + Ct_1}{1 + Ct} \right]^{\frac{E_b}{CS_0}} \tag{29}$$

It can be observed that, apart from the material properties, the residual average stresses also dependent upon t_0 and t_1 as well as up on the reaction kinetics.

4 Computational study

4.1 Empirical calibration of the model

To predict the material response to any stress history at any stage of reaction, it is necessary to estimate the material parameters like a, p, q, r and S. Here, the material constants have been estimated using creep-time plot (Fig. 9.18 of reference [4]) of a particular uniaxial creep test conducted on a concrete/mortar specimen undergoing hydration reaction in a particular ambient chemical environment. The creep curve gives the variation of axial creep strain with time t. The mix proportion of concrete was taken as 1:3.5:3.5 with 0.72 water-cement ratio.

Following the approach adopted earlier [3, 8], the effect of the ambient environment as well as of the chemical nature of the reactants on the reaction kinetics is represented through an α-t relation. As such, the parameter α is a measure of intrinsic time. In the present case, this relation is calibrated from the data on hydration kinetics at room temperature. As earlier, the form of the α-t relation has been assumed to be

$$\alpha = \frac{t}{a + t} \tag{30}$$

This gives the instantaneous rate of reaction as

$$\overset{\bullet}{\alpha} = \frac{a}{(a + t)^2} \tag{31}$$

Thus, the empirical constant a turns out to be equal to the reciprocal of the initial rate of reaction at t=0 as follows

$$\overset{\bullet}{\alpha} = \frac{1}{a} \tag{32}$$

For better predictions, more complex α-t relation can be assumed. The reaction kinetics represented by the relation depends upon several factors such as, the nature of concrete constituents, nature and concentration of chemical reactants, temperature, amount of solvent like pore water, etc.

The value of the dashpot coefficient S for the conventional Maxwell element is expected to vary with the progress of reaction. Its instantaneous value has been assumed to vary with time as

$$S = S_0(1 + Ct) \tag{33}$$

where S_0 is initial viscosity constant at the beginning of hydration and C is an empirical constant.

Once the values to the empirical material parameters are determined, the above computational algorithm can be used for predicting creep. After many attempts at predicting the observed creep behaviour under uniaxial compressive stress [4] by assigning different combinations of values to the empirical material parameters, the following values have been found to give the best fit within acceptable error: $a = 10$, p=1.25, $q = -0.50$, C=90 and S_0=150. In the present case involving hydration, the reacting phase is identified to be cement. From the mix proportions, its volume fraction before the start of the reaction is estimated to be about 0.1. Based on the arguments presented elsewhere [8] concerning the disruptive effect of dissolution of cement grains on the integrity of concrete, the effective volume fraction of the reacting phase is estimated to be 0.25. So far as their elastic modulus E_1 is concerned, unhydrated cement grains are indistinguishable from hard rock particles. Assuming the value of modulus of elasticity of the reacting phase as $4.8\text{X}10^4$ MPa, the value of the parameter r is estimated to be $1.2\text{X}10^4$ MPa. The value of E_b has been assed to be 25000 MPa. Using the above value of parameter a, the resulting hydration kinetics is depicted in Fig. 2 in the form of $\alpha - t$ plot. It should be remembered that the reaction kinetics of concrete depends upon the type of cement, chemical admixtures if any, ambient temperature, etc. The above $\alpha - t$ relation has been deduced from the empirical data on creep of the given concrete under uniaxial compression and in a particular ambient environment. This observation applies to other empirical constants as well. It should be kept in mind that the above empirical parameters of the model should be used to predict behaviour of same concrete in the same ambient environment.

4.2 Quantitative prediction of material response

Using the relevant equations derived above, the creep behaviour of reacting viscoelastic concrete subjected to diverse biaxial and triaxial stress histories has been predicted. The empirical data on creep and recovery for the chosen stress histories is available in the literature [4]. Such data have also been plotted in the same figures for direct comparison with the quantitative theoretical

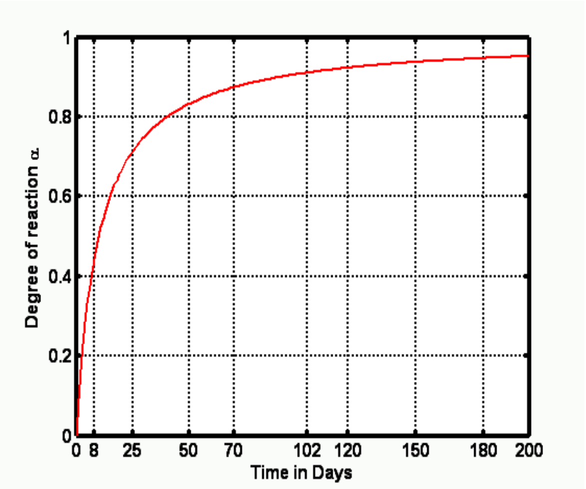


Figure 2: Theoretical prediction of degree of reaction vs time.

predictions. The time variation of the element stresses has also been presented. For the chosen biaxial and triaxial stress histories, the effect of age at loading has been investigated. The stress relaxation behaviour has also been predicted. Using a numerical illustrative example, the principle of superposition has also been validated. The computational algorithm developed is based on incremental procedure in the time domain.

5 Discussion

In this paper, a chemico-mechanical theory of reacting isotropic linear aging viscoelastic solids has been constructed. The time-dependent deformations are caused by both the dissolution-precipitation mechanism and the viscoelastic nature of the material. The proposed theory (DPT) and the well-known solidification theory (ST) developed by Bazant and co-workers seek to simulate the observed behaviour of concrete at all ages. Both of these theories, being phenomenological engineering theories, lack the basis in rational continuum thermo-mechanics. As these theories share the objective as well as broad phenomenological approach, the DPT and ST are competing theories.

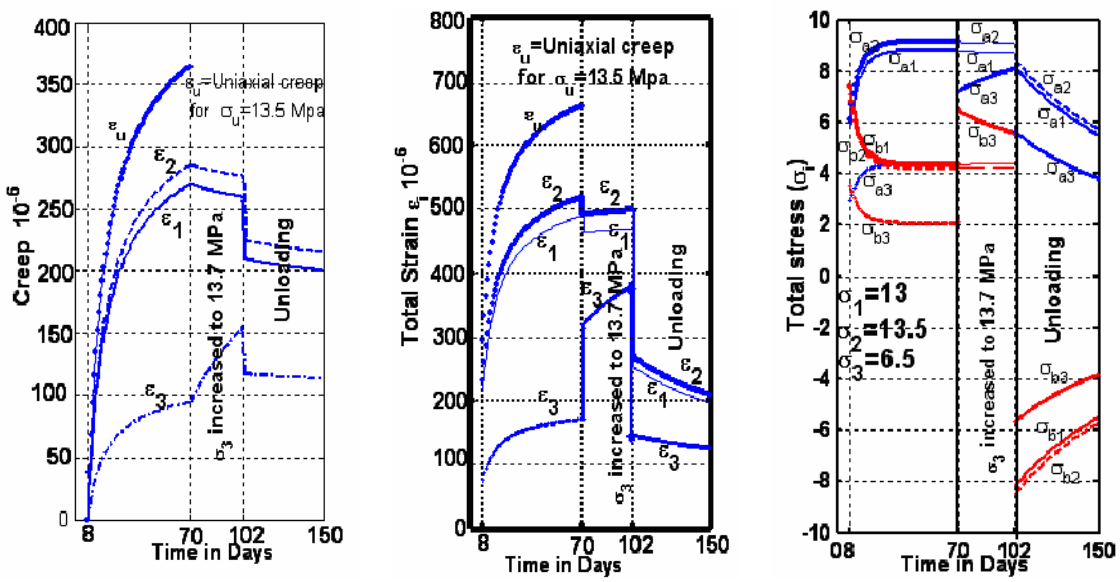
Where the DPT of the reacting elastic solids was first proposed in 1995 [3], it was motivated, albeit to a partial extent, by the then available ST. Partly, the motivation was provided by the need to explain the observed time-dependent deformations in reacting mature concrete. DPT shares with ST the hypothesis that the volume of solid reaction products grows with their precipitation/solidification out of the solution phase and, at the instant of precipitation /solidification, these reaction products are stress-free. Beyond this, their dissimilarities define the distinguishing characteristics of these theories. First of all, DPT identifies a new mechanism

– DP Mechanism – of time-dependent deformations even in elastic solids when they are involved in some reactions. To be specific, ST is incapable of predicting the time-dependent deformation exhibited by mature (elastic) concrete on exposure to reacting environment. Also, ST presumes that the unreacted phase to be stress-free. Such an assumption is not justified in mature concrete with all of its constituent phases being in solid state. Hence, the scope of ST is limited to only hydrating concrete but only at early age. Because of the fact that ST can not simulate the mechanical behaviour of reacting mature concrete, durability of concrete constituents in service subjected to simultaneous action of loads and chemical environment is outside its scope of validity. In contrast, the DPT constitutes a major theoretical contribution to the durability mechanics of concrete. The DP theory and the earlier version of ST both lack the capability to explain the observed phenomenon of aging creep of mature concrete. The solidification theory has been modified later by incorporating the effect of 'micro-prestress' to simulate such phenomenon (Baweja et al 1997).

Available experimental data on the creep of concrete under triaxial and biaxial compressive stress histories [4] has been used for empirical calibration and validation of the proposed model. The triaxial stress history used involves application of triaxial compressive sate of stress [$\sigma_1 =$ 13 MPa, $\sigma_2 = 13.5 MPa, \sigma_3 = 6.5 MPa$] at 8 days, increasing σ_3 to 13.7 MPa at 70 days and followed by unloading at 102 days. The empirical model parameters determined above has been used to obtain the material response up to 150 days for this load history. Theoretical prediction of material response under such stress history presupposes the validity of principle of superposition. The required principle of superposition has been proved earlier for the reacting element [7] and for the linear Maxwell element this principle is known to be valid. For the proposed rheological model of the reacting viscoelastic concrete, the superposition principle has been numerically verified above.

Comparison of predicted creep and total strains with the experimental data shown in Fig. 3a and 3b respectively reveals the basic soundness of the proposed model. It can be observed that the model predicts instantaneous state of strain introduced upon loading. Under sustained state of stress the creep occurs and the reacting concrete has been shown to exhibit gradual increase of strains. The instantaneous application of additional stress σ_3 results in an instantaneous increase in the corresponding strain ε_3 but an instantaneous decrease of strain in the lateral directions due to Poisson's effect. Upon unloading, the instantaneous elastic recovery is followed by creep recovery. However, in contrast to the conventional three element model, the residual strains do not vanish asymptotically. Apart from these expected qualitative theoretical predictions, the quantitative predictions of the model have also been found to be satisfactory. In particular, the error in the predicted strain does not exceed ten percent.

It is well known that, as the reacting element and the Maxwell element are arranged in parallel, the sum of the individual element stresses is always equal to the applied stresses. The variation of these element stresses with time separately for each normal stress has been plotted in Fig. 3c. The aim is to compare, from a different angle, the proposed model with the corresponding classical three element model. Let it be recapitulated that, in the conventional

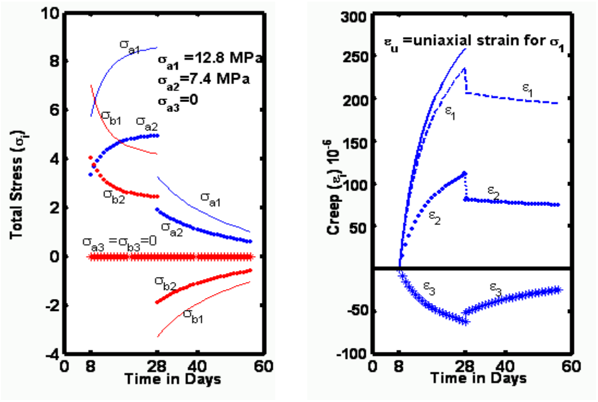


(a) creep (b) total strains (c) element stresses

Figure 3: Theoretical predictions in concrete under triaxial stress history.

three element model under sustained uniaxial stress, the stress in the Maxwell element decreases with the progress of time and vanishes asymptotically. This is because of the fact that, with progressive decrease in the rate of creep, the stress in the dashpot and so in the Maxwell element decreases. After unloading, the stresses in the Maxwell and spring elements in the three-element rheological model balance each other and both vanish asymptotically. Similarly, it has been predicted for the proposed rheological model, as shown in the Fig. 3c, that the states of stress applied instantaneously at 8 days and 70 days are shared by the two elements in the ratio of their relative instantaneous stiffness. Under sustained state of stress, relative contribution of Maxwell element has been predicted to decrease with time. Upon unloading, the element normal stresses in reacting element and Maxwell element balance each other at all times. These element stresses decrease with time and vanish asymptotically as the strain rate vanishes. As pointed out in the preceding paragraph, a certain state of strain is reached asymptotically. Both the elements attain the same state of strain. However, the spring of the Maxwell element is strain-free, the entire strain being suffered by the dashpot.

Similarly, temporal variation of element stresses and creep strains of concrete subjected to biaxial loading and unloading has been shown in Fig. 4. A biaxial state of compressive stress $[\sigma_1 =$ 12.8 MPa, $\sigma_2 = 7.4 MPa, \sigma_3 = 0 MPa]$ is applied at 8 days and removed at 28 days. The expected qualitative behaviour patterns are obtained. It can be observed that the stress σ_3 remains zero in both the elements. The quantitative theoretical predictions have been found to be close to the experimental observations [4]. It can also be observed that, as expected, reacting viscoelastic concrete has been predicted to exhibit creep strains under sustained loading as well as residual strains after unloading to be of the same sense as the instantaneous elastic strains upon loading.

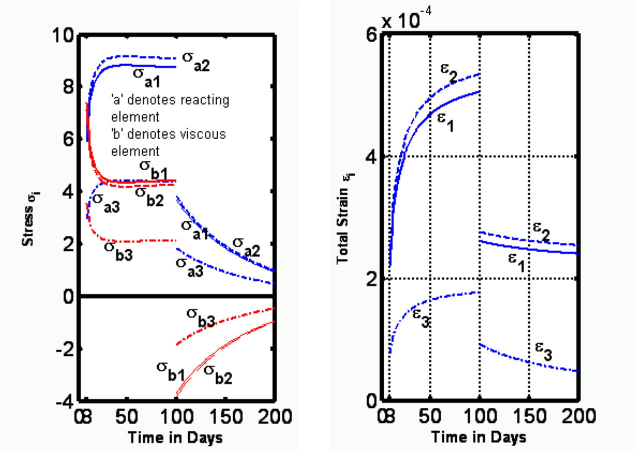


(a) (b)

Figure 4: Theoretical prediction of concrete response to loading-unloading under biaxial stresses: (a) Element stresses (b) Creep.

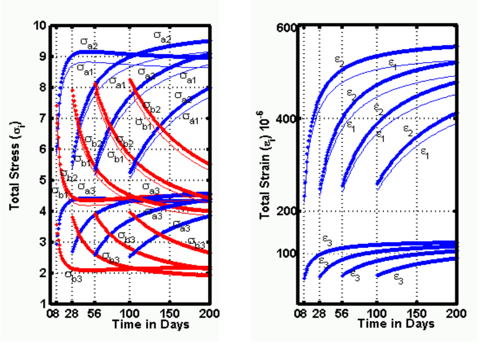
It can be observed from Fig. 3 and Fig. 4 that reacting viscoelastic concrete has been predicted to exhibit positive creep recovery after unloading. This prediction is confirmed by experimental data on concrete [4]. However, the reacting elastic concrete has been shown to exhibit negative creep recovery [8]. Similarly, negative creep recovery has been predicted by solidification theory of hydrating concrete [2]. In the two theories, viz, dissolution precipitation theory [DPT] and solidification theory [ST], this phenomenon has been studied theoretically in terms of the concept of divergence. Here, no such analytical derivation has been given, but it has been shown by numerical examples that, in this respect, the model of reacting viscoelastic concrete proposed in this paper is better than the above two theories.

Additionally, temporal variation of element stresses and total strains in concrete subjected to a loading-unloading cycle of triaxial compressive stresses [$\sigma_1 = 13MPa, \sigma_2 = 13.5MPa$, $\sigma_3 = 6.5MPa$] has been shown in Fig. 5. The effect of age at loading of the above triaxial state of stress on element stresses and total strains have been plotted in Fig. 6. The qualitative predictions of the model in both of these illustrative examples are satisfactory. For example, creep at any age is predicted to be lesser when the concrete is loaded at later ages. The temporal variation of element stresses, being more complex, deserves an explanation. It should be realized that the elastic modulus and viscosity of the Maxwell element increase with the passage of time. Just after loading, and at the asymptotic stage, the applied stresses are shared by elements in the ratio of their instantaneous relative stiffness. However, under the sustained loading the variation of the element stresses in the presence of ongoing reactions is governed by their viscosity parameters.



(a) (b)

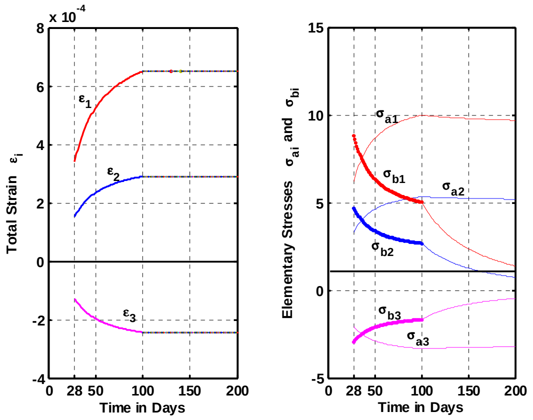
Figure 5: Theoretical prediction of concrete response to loading-unloading under triaxial stresses: (a) Element stresses (b) Total strain.



(a) (b)

Figure 6: Theoretical prediction of effect of age at loading on concrete behaviour under sustained triaxial state of stresses: (a) Element stress (b) Total strain.

The phenomenon of stress relaxation under the chosen triaxial states of stress has also been investigated. A state of triaxial compression $[\sigma_1 = 15MPa, \sigma_2 = 8MPa, \sigma_3 = -5MPa]$ has been applied at 28 days. Under sustained state of stress, the reacting viscoelastic concrete exhibits creep. At 100 days, the total state of strain is kept constant as shown in Fig. 7a. It can be observed from Fig. 7b that the element stresses gradually decrease with the passage of time and can even undergo change of sense. The temporal variation of total stresses has been shown in Fig. 8. It can be observed from these figures that the total stresses do not vanish asymptotically and that the residual total stresses do not experience change of sense.



(a) (b)

Figure 7: Theoretical prediction of creep and relaxation of concrete under triaxial state of stress: a) Total strains b) Element stresses.

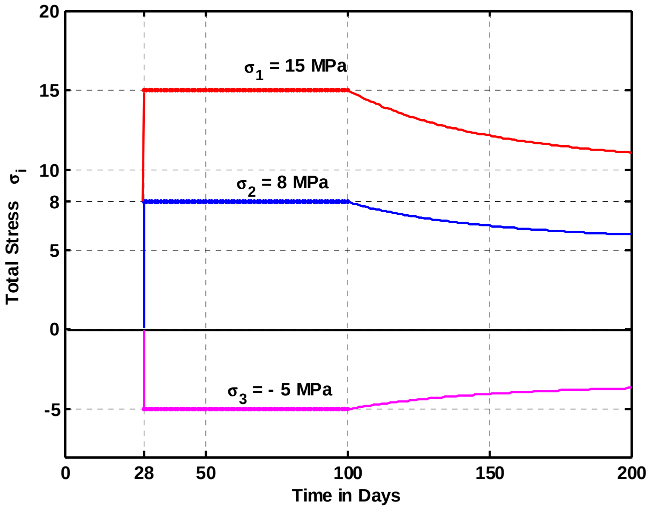


Figure 8: Theoretical prediction of stress-relaxation.

6 Conclusions

The chemico-mechanical model of reacting homogeneous isotropic linear elastic concrete proposed earlier has been extended in this paper to reacting viscoelastic concrete. The proposed rheological model is obtained by arranging in parallel the reacting element with Maxwell element with time-dependent viscosity. The proposed model has been calibrated by using available experimental data.

The creep, recovery and stress relaxation behaviour of same hydrating concrete in the same ambient environment subjected to multiaxial stress has been predicted. Comparison of the theoretical predictions with the experimental data shows the basic soundness of the proposed model. Plots showing temporal variations of element stresses have been shown to provide deeper insight into the phenomenon.

The proposed model is similar to the solidification theory of hydrating concrete, but there exist certain inherent dissimilarities also. Both these theories belong to the class of linear aging viscoelasticity theories. While the scope of the solidification theory is restricted to hydrating early age concrete, the proposed model is capable of predicting time-dependent behaviour of concrete involved in all types of reactions. As such, it has obvious significance for durability mechanics of concrete. Both of the theories are incapable of predicting aging creep in mature concrete not involved in any reactions including hydration. Recently, the solidification theory has been extended in the form of microprestress solidification theory to remove this lacuna. It is proposed to extend the proposed rheological model also along these lines.

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