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A benchmark study on intelligent sampling techniques in Monte Carlo simulation

Abstract

In recent years, new, intelligent and efficient sampling techniques for Monte Carlo simulation have been developed. However, when such new techniques are introduced, they are compared to one or two existing techniques, and their performance is evaluated over two or three problems. A literature survey shows that benchmark studies, comparing the performance of several techniques over several problems, are rarely found. This article presents a benchmark study, comparing Simple or Crude Monte Carlo with four modern sampling techniques: Importance Sampling Monte Carlo, Asymptotic Sampling, Enhanced Sampling and Subset Simulation; which are studied over six problems. Moreover, these techniques are combined with three schemes for generating the underlying samples: Simple Sampling, Latin Hypercube Sampling and Antithetic Variates Sampling. Hence, a total of fifteen sampling strategy combinations are explored herein. Due to space constrains, results are presented for only three of the six problems studied; conclusions, however, cover all problems studied. Results show that Importance Sampling using design points is extremely efficient for evaluating small failure probabilities; however, finding the design point can be an issue for some problems. Subset Simulation presented very good performance for all problems studied herein. Although similar, Enhanced Sampling performed better than Asymptotic Sampling for the problems considered: this is explained by the fact that in Enhanced Sampling the same set of samples is used for all support points; hence a larger number of support points can be employed without increasing the computational cost. Finally, the performance of all the above techniques was improved when combined with Latin Hypercube Sampling, in comparison to Simple or Antithetic Variates sampling.

Keywords

Structural reliability; Monte Carlo simulation; intelligent sampling techniques; benchmark study.

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1 INTRODUCTION

Since the early beginnings in the sixties and seventies, structural reliability analysis has reached a mature stage encompassing solid theoretical developments and increasing practical applications. Structural reliability methods have permeated the engineering profession, finding applications in code calibration, structural optimization, life extension of existing structures, life-cycle management of infrastructure risks and costs, and so on. During the past 30 years, significant advances were obtained in terms of transformation methods (FORM, SORM), as well as in terms of simulation techniques. Transformation methods were found to be efficient in the solution of problems of moderate dimensions and moderate non-linearity. Simulation techniques (Metropolis and Ulam, 1949; Metropolis et al., 1953; Robert and Casella, 2011) have always allowed the solution of highly non-linear high-dimensional problems, although computational cost used to be a serious limitation. This is especially true when failure probabilities are small and limit state functions are given numerically (e.g., finite element models) (Beck and Rosa, 2006). With the recent and exponential advance of computational processing power, Monte Carlo simulation using intelligent sampling techniques is becoming increasingly more viable.

Several intelligent sampling techniques for Monte Carlo simulation have been proposed in recent years (Au and Beck, 2001; Au, 2005; Au et al., 2007; Bucher, 2009; Sichani et al., 2011a; Sichani et al., 2011b; Sichani et al., 2014; Naess et al., 2009; Naess et al., 2012). However, when such techniques are introduced, they are generally compared with one or two existing techniques, and their performance is evaluated over two or three problems. It is difficult to find in the published literature benchmark studies where several sampling techniques are compared for a larger number of problems (Au et al., 2007; Engelund and Rackwitz, 1993; Schuëller and Prandlwarter, 2007). This article presents a benchmark study, comparing Simple or Crude Monte Carlo with four modern sampling techniques: Importance Sampling Monte Carlo, Asymptotic Sampling, Enhanced Sampling and Subset Simulation over six problems. Moreover, these techniques are combined with three schemes for generating the underlying samples: Simple Sampling, Latin Hypercube Sampling and Antithetic Variates Sampling. Hence, a total of fifteen sampling strategy combinations are explored herein. Due to space constrains, results are presented for only three of the six problems studied. The conclusions, however, cover the six problems studied.

The remainder of the article is organized as follows. The structural reliability problem is formulated in Section 1. The basic techniques for generating the underlying samples are presented in Section 2. The intelligent sampling techniques for failure probability evaluation are presented in Section 3. Problems are studied in Section 4, and Concluding Remarks are presented in Section 5.

2 FORMULATION

2.1 Reliability problem

Let $\mathbf{X} = \{X_1, X_2, ..., X_m\}$ be a random variable vector describing uncertainties in loads, material strengths, geometry, and models affecting the behavior of a given structure. A limit state equation $g(\mathbf{X})$ is written such as to divide the failure and survival domains (Madsen et al., 1986; Melchers, 1999):

$$\begin{aligned} \Omega_f &= \left\{ \mathbf{x} \mid g(\mathbf{x}) \le 0 \right\} \quad \text{is the failure domain} \\ \Omega_s &= \left\{ \mathbf{x} \mid g(\mathbf{x}) > 0 \right\} \quad \text{is the survival domain} \end{aligned} \tag{1}$$

The failure probability is given by:

$$P_f = P\left[\mathbf{X} \in \Omega_f\right] = \int_{\Omega_f} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{2}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the probability density function of random vector $\mathbf{X}$. The biggest challenge in solving the simple multi-dimensional integral in equation (2) is that the integration domain is generally not known in closed form, but it is given as the solution of a numerical (e.g., finite element) model. Monte Carlo simulation solves the problem stated in equation (2) by generating samples of random variable vector $\mathbf{X}$, according to distribution function $f_{\mathbf{X}}(\mathbf{x})$, and evaluating weather each sample belongs to the failure or survival domains.

2.2 Crude Monte Carlo Simulation

One straightforward way of performing the integration in equation (2) is by introducing an indicator function $I[\mathbf{x}]$, such that $I[\mathbf{x}] = 1$ if $\mathbf{x} \in \Omega_f$ and $I[\mathbf{x}] = 0$ if $\mathbf{x} \in \Omega_s$. Hence, the integration can be performed over the whole sample space:

$$P_f = \int_{\Omega} I[\mathbf{x}] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \equiv E\left[I[\mathbf{x}]\right] \tag{3}$$

In equation (3), one recognizes that the right-hand term is the expected value ($E[.]$) of the indicator function. This expected value can be estimated from a sample of size n by:

$$E\left[P_f\right] = \overline{P}_f = \overline{I}[\mathbf{x}] = \frac{1}{n}\sum_{j=1}^{n} I\left[\mathbf{x}_j\right] = \frac{n_f}{n} \tag{4}$$

where n_f is the number of samples which belong to the failure domain and n is the total number of samples. The variance of $\overline{P}_f$ is given by:

$$Var\left[\overline{P}_f\right] = \frac{1}{n-1}\sum_{j=1}^{n}\left(I\left[\mathbf{x}_j\right] - \overline{P}_f\right)^2 \tag{5}$$

In Crude Monte Carlo simulation, sample vector $\mathbf{x}_j$ can be generated using the Simple Sampling, Antithetic Variates Sampling or Latin Hypercube Sampling, as detailed in the sequence.

3 BASIC SAMPLING TECHNIQUES

In this paper three basic sampling techniques are investigated: Simple Sampling, Latin Hypercube Sampling and Antithetic Variates Sampling. These basic sampling schemes are not specific for the solution of structural reliability problems: they can be employed in the numerical solution of integrals (like equation (2)), in the spatial distribution of points in a given domain (for surrogate modeling, for instance), and so on. These techniques are further combined with specific techniques for solving structural reliability problem via Monte Carlo simulation, as described in Section 3.

3.1 Simple Sampling

The use of Monte Carlo simulation in solving general problems involving random variables (and/or stochastic processes) requires the generation of samples from random variable vector $\mathbf{X}$. The most straightforward way of generating samples of a vector of random variables is by an inversion of their cumulative distribution function $F_{\mathbf{X}}(\mathbf{x})$:

I. Generate a random vector of components $\mathbf{v}_{(1\times m)} = \{v_j\}$, $j = 1,...,m$, uniformly distributed between 0 and 1;

II. Use the inverse of the cumulative distribution function, such that $\{x_j\} = \{F_{\mathbf{X}}^{-1}(v_j)\}$, $j = 1,...,m$.

When components of vector $\mathbf{X}$ are correlated, the correlation structure can be imposed by pre-multiplication by the Cholesky-decomposition of the correlation matrix. Details are given in Madsen et al. (1986) and Melchers (1999).

3.2 Antithetic Variate Sampling

In Simple Sampling, a set of random numbers $\mathbf{u}_{(n\times 1)} = \{u_{1j}, u_{2j},..., u_{nj}\}^t$ is employed to obtain n samples of random variable X_j. In Antithetic Variate sampling, the idea is to divide the total number of samples by two, and to obtain two vectors $\mathbf{u} = \{u_1, u_2,..., u_{n/2}\}^t$ and $\bar{\mathbf{u}} = \{1-u_1, 1-u_2,...,1-u_{n/2}\}^t$. Now consider that any random quantity P (including the failure probability, P_f) can be obtained by combining two unbiased estimators P_f^a and P_f^b, such that:

$$P_f^c = \frac{P_f^a + P_f^b}{2} \tag{6}$$

The variance of this estimator is:

$$Var\left[P_f^c\right] = \frac{1}{4}\left(Var\left[P_f^a\right] + Var\left[P_f^b\right] + 2\cdot Cov\left[P_f^a, P_f^b\right]\right) \tag{7}$$

Thus, by making $P_f^a = f(\mathbf{u})$ and $P_f^b = f(\bar{\mathbf{u}})$, a negative correlation is imposed, $Cov\left[P_f^a, P_f^b\right]$ becomes negative, hence the variance of P_f^c is reduced in comparison to the variances of P_f^a or P_f^b. The Antithetic Variates technique, when applied by itself, may not lead to significant improvement; when applied in combination with other techniques, more significant improvements can be achieved.

3.3 Latin Hypercube Sampling

The Latin Hypercube Sampling (LHS) was introduced by McKay et al. (1979). The idea of Latin Hypercube Sampling is to divide the random variable domain in stripes, where each stripe is sampled only once (McKay et al., 1979; Olsson et al., 2003), such as in Figure 1. This procedure guarantees a sparse but homogeneous cover of the sampling space.

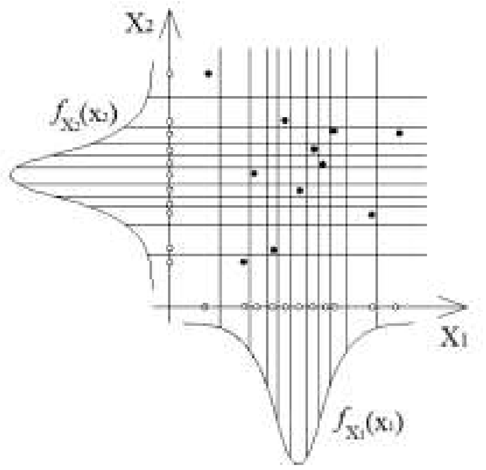

Figure 1: Latin Hypercube Sampling.

To obtain the Latin Hypercube, let m be the number of random variables and n the number of samples. A matrix $\mathbf{P}_{(n\times m)}$ is created, where each column is a random permutation of $1,...,n$. A matrix $\mathbf{R}_{(n\times m)}$ is created, whose elements are uniform random numbers between 0 and 1. Then, matrix $\mathbf{S}$ is obtained as (Olsson et al., 2003):

$$\mathbf{S} = \frac{1}{n}\left(\mathbf{P} - \mathbf{R}\right) \tag{8}$$

The samples are obtained from $\mathbf{S}$ such that:

$$x_{ij} = F_{X_j}^{-1}\left(s_{ij}\right) \tag{9}$$

where $F_{X_j}^{-1}$ is the inverse cumulative distribution function of random variable X_j.

In order to reduce memory consumption, equation (8) can be "solved" is scalar fashion. The following algorithm is adopted:

1. Start the loop for random variable j:
2. Generate the first column of matrix $\mathbf{P}$, as a random permutation n of $1,...,n$;
3. Start the loop for the number of simulations i;
4. Generate a single uniform random number between 0 and 1; thus, it is not necessary to create matrix $\mathbf{R}$;
5. Compute a number s, and use equation (9) to compute the element x_{ij}. Thus, matrix $\mathbf{S}$ also does not need to be computed;
6. Repeat step 4 until $i = n$;
7. Repeat step 2 until $j = m$.

For illustration purposes, Figure 2 shows histograms obtained by Simple Sampling (Figure 2a), Antithetic Variates Sampling (Figure 2b) and Latin Hypercube Sampling (Figure 2c), for a random variable X with normal distribution, with mean equal to 0.0 and standard deviation equal to 0.15. Three thousand samples were used to compute these histograms. One observes the smoother distributions obtained by means of Latin Hypercube Sampling.

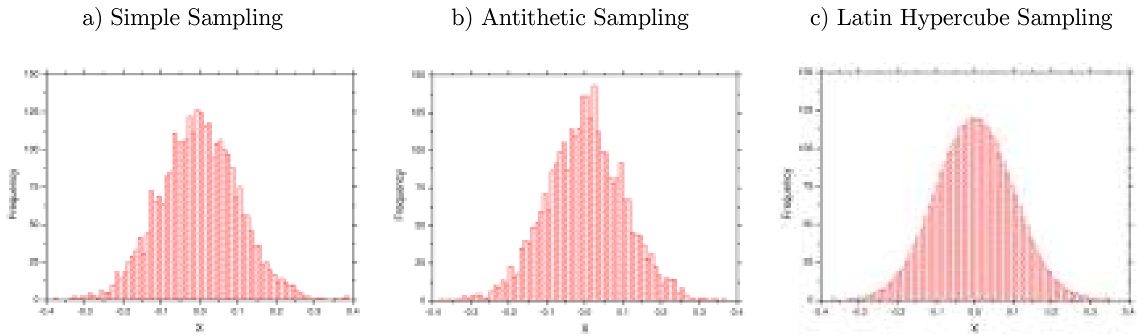

Figure 2: Histograms obtained for 3000 samples of a single random variable $X\tilde{}N(0,\ 0.15)$.

4 INTELLIGENT SAMPLING TECHNIQUES

4.1 Importance Sampling Monte Carlo

Importance Sampling Monte Carlo centered on design points is a powerful technique to reduce the variance in problems involving small and very small failure probabilities. The drawback is that it needs prior location the design points. Design points can be located using well-known techniques of the First Order Reliability Method, or FORM (Madsen et al., 1986; Melchers, 1999). However, finding the design point can be a challenge for highly non-linear problems.

Recall the fundamental Monte Carlo simulation equation (equation 3). If numerator and denominator of this expression are multiplied by a conveniently chosen sampling function $h_{\mathbf{X}}(\mathbf{x})$, the result is unaltered:

$$P_f = \int_{\Omega} I[\mathbf{x}] \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{X}}(\mathbf{x})} h_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \equiv E\left[I[\mathbf{x}] \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{X}}(\mathbf{x})} \right] \tag{10}$$

The expected value, in the right-hand side of equation (10), can be estimated by sampling using:

$$\overline{P}_f = \overline{I}[\mathbf{x}] = \frac{1}{n} \sum_{i=1}^{n} I[\mathbf{x}_i] \frac{f_{\mathbf{X}}(\mathbf{x}_i)}{h_{\mathbf{X}}(\mathbf{x}_i)} \tag{11}$$

By properly choosing the sampling function $h_{\mathbf{X}}(\mathbf{x})$, one can increase the number of "successes", or the number of sampled points falling in the failure domain. This is normally accomplished by centering the sampling function $h_{\mathbf{X}}(\mathbf{x})$ in the design point. In other words, the sampling function is usually the joint probability density function $f_{\mathbf{X}}(\mathbf{x})$, but with the mean replaced by the design point coordinates $\left(\mathbf{x}^*\right)$. However, observe that in comparison to equation (4), each sampled point falling in the failure domain $\left(I\left[\mathbf{x}_i\right]=1\right)$ is associated to a sampling weight $f_{\mathbf{X}}\left(\mathbf{x}_i\right)/h_{\mathbf{X}}\left(\mathbf{x}_i\right) \ll 1$.

For structures or components with multiple failure modes associated as a series system, the sampling function can be constructed by a weighted sum of functions $h_{i\mathbf{X}}\left(\mathbf{x}\right)$, centered at the i^{th} design point, such that:

$$h_{\mathbf{X}}\left(\mathbf{x}\right)=\sum_{i=1}^{nls} p_i h_{i\mathbf{X}}\left(\mathbf{x}\right) \tag{12}$$

where nls is the number of limit states and p_i is the weight related to the limit state i. For a series system, where failure in any mode characterizes system failure, p_i is obtained as:

$$p_i=\frac{\Phi\left(-\beta_i\right)}{\sum_{i=1}^{nls}\Phi\left(-\beta_i\right)} \tag{13}$$

For parallel system, similar expressions are given in (Melchers, 1999).

The application of Importance Sampling Monte Carlo in combination with Simple Sampling or Antitethic Variates Sampling is straightforward. When applying Importance Sampling in combination with Latin Hypercube Sampling, there is a possibility that most samples be generated on one side of the limit state. This can be avoided by adopting a transformation proposed by Olsson et al. (2003). This transformation rotates the Latin Hypercube, on the standard normal space, according the orientation of the design point. This procedure is presented in Figure 3. Further details are given in Olsson et al. (2003).

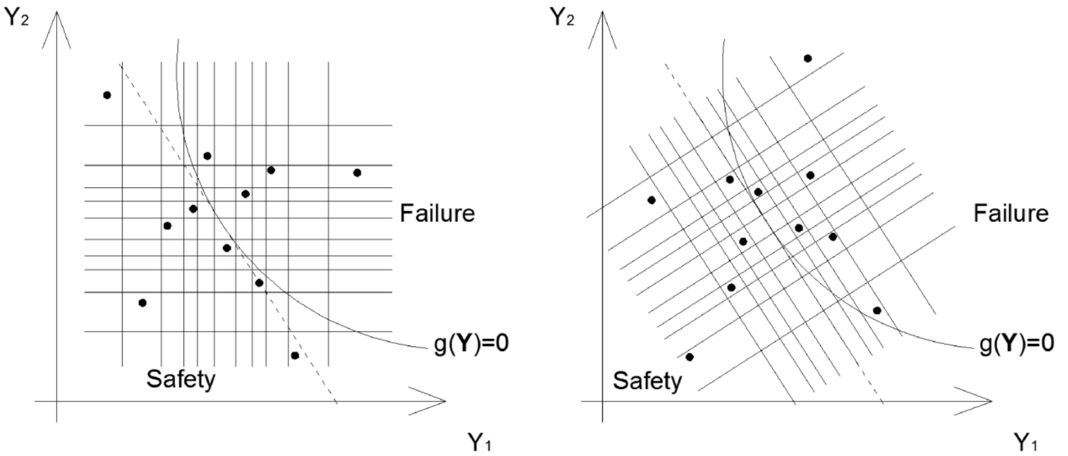

Figure 3: Original and rotated Latin Hypercube, adapted from Olsson et al. (2003).

4.2 Asymptotic Sampling

The Asymptotic Sampling technique (Bucher, 2009; Sichani et al., 2011a; Sichani et al., 2011b; Sichani et al., 2014) was developed based on the asymptotic behavior of failure probabilities as the standard deviation of the random variables tends to zero. One advantage over Importance Sampling is that it does not require previous knowledge of the design point.

Asymptotic Sampling is based on choosing factors $f < 1$, related to the standard deviations of the random variables as:

$$\sigma_i^f = \frac{\sigma_i}{f} \tag{14}$$

where σ_i is the standard deviation of random variable i and σ_i^f is the modified standard deviation for the same random variable. For small values of $f < 1$, larger standard deviations, hence also larger failure probabilities are obtained. For each pre-selected value of f a Monte Carlo Simulation is performed, in order to obtain the reliability index $\beta(f)$. Following Bucher (2009), the asymptotic behavior of β with respect to f can be described by a curve:

$$\frac{\beta}{f} = A + B \cdot f^C \tag{15}$$

Where A, B and C are constants to be determined by nonlinear regression (e.g. least squares method), using $\left(f, \beta/f\right)$ as support points. After finding the regression coefficients, the reliability index for the original problem is estimated by making $f = 1$ in equation (13), such that $\beta = A + B$. Finally, the probability of failure is obtained as $P_f = \Phi(-\beta)$, where $\Phi(.)$ is the standard normal cumulative distribution function. The Monte Carlo simulations for different values of β (support points) can be obtained by Simple Sampling, Latin Hypercube Sampling or Antithetic Variates Sampling.

The parameters influencing the performance of Asymptotic Sampling are the number of support points and the range of f values considered. In this paper, these parameters are not studied: they are fixed at values providing satisfactory responses: 5 support points are employed with f varying from 0.4 to 0.7 or from 0.5 to 0.8, depending on the problem.

4.3 Enhanced Sampling

The Enhanced Sampling technique was proposed by Naess et al. (2009, 2012) and is based on exploring the regularity of the tails of the PDF's. It aims at estimating small or very small failure probabilities for systems. The original limit state function $M = g\left(x_1, x_2, ..., x_m\right)$ is used to construct a set of parametric functions $M(\lambda)$, with $0 \leq \lambda \leq 1$, such that:

$$M(\lambda) = M - (1 - \lambda) \cdot \mu_M \tag{16}$$

where μ_M is the mean value of M. Thus, it is assumed that the behavior of failure probabilities with respect to λ can be represented as:

$$P_f(\lambda) \approx q \cdot \exp\left[-a(\lambda - b)^c\right] \quad \text{with } \lambda \to 1 \tag{17}$$

Using a set of support points $\left[\lambda, P_f\left(\lambda\right)\right]$, the parameters in equation 15 can be found by nonlinear regression. The probability of failure for the original problem is estimated for $\lambda = 1$. One large advantage over Asymptotic Sampling is that, from one single Monte Carlo simulation run, the whole range of parametric functions $M\left(\lambda\right)$ can be evaluated. Hence, a large number of support points can be used, with no penalty in terms of computational cost.

For systems, each limit state or component $M_j = g_j\left(x_1, x_2, ..., x_m\right)$, with $j = 1, ..., nls$, must be evaluated. Thus, a parametric set of equations is obtained, such as:

$$M_j\left(\lambda\right) = M_j - \left(1 - \lambda\right) \cdot \mu_{M_j} \tag{18}$$

Therefore, for a series system the probability of failure is obtained as:

$$P_f = P\left[\bigcup_{j=1}^{nls} \left\{M_j\left(\lambda\right) \leq 0\right\}\right] \tag{19}$$

For a parallel system, the probability of failure is obtained as:

$$P_f = P\left[\bigcap_{j=1}^{nls} \left\{M_j\left(\lambda\right) \leq 0\right\}\right] \tag{20}$$

and for a series system with parallel subsystems, the probability of failure is given by:

$$P_f = P\left[\bigcup_{j=1}^{l} \bigcap_{i \in C_j} \left\{M_i\left(\lambda\right) \leq 0\right\}\right] \tag{21}$$

where C_j is a subset of $1, ..., nls$, for $j = 1, ..., l$.

The parameters of Enhanced Sampling are the number of support points $\left[\lambda, P_f\left(\lambda\right)\right]$ and the values of λ for which failure probabilities are evaluated. Within this paper, these values are kept fixed: 100 support points are used, with λ varying from 0.4 to 0.9.

4.4 Subset Simulation

The Subset Simulation technique was proposed by Au and Beck (2001) aiming to estimate small and very small probabilities of failure in structural reliability. The basic idea of this technique is to decompose the failure event, with very small probability, into a sequence of conditionals events with larger probabilities of occurrence. For the later, small sample Monte Carlo simulation should be sufficient. Since Simple Sampling is not a good option to generate conditional samples, the Markov Chain Monte Carlo and the modified Metropolis-Hastings algorithms are used.

The estimation of conditional probabilities in Subset Simulation depends on the choice of the intermediates events. Consider the failure event E. The probability of failure associated to it is given by:

$$P[E] = P[\mathbf{X} \in \Omega_f] = \int_{\Omega_f} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} = \int_{\Omega} I[\mathbf{x}] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{22}$$

Considering a decreasing sequence of events E_i, such as $E_1 \supset E_2 \supset ... \supset E_m = E$, thus:

$$E_k = \bigcap_{i=1}^{k} E_i, \quad k = 1,...,m \tag{23}$$

Hence, the probability of failure is evaluated as:

$$P_f = P[E] = P\left[\bigcap_{i=1}^{m} E_i\right] = P[E_1] \cdot \prod_{i=2}^{m} P[E_i \mid E_{i-1}] \tag{24}$$

In equation (22), $P[E_1]$ can be evaluated by means of Crude Monte Carlo, using Simple Sampling, Latin Hypercube Sampling or Antithetic Variates Sampling. On the other hand, the conditional probabilities $P[E_i \mid E_{i-1}]$ are estimated by means of Markov Chains using the Modified Metropolis-Hastings algorithm (Au and Beck, 2001; Au, 2005; Au et al., 2007).

In Subset Simulation the intermediate events E_i are chosen in an adaptative way. In structural reliability the probability of failure is estimated by:

$$P_f = P[E] = P[\{g(\mathbf{X}) \leq 0\}] \tag{25}$$

As the failure event is defined by:

$$E = \{g(\mathbf{X}) \leq 0\} \tag{26}$$

The intermediate failure events are defined by:

$$E_i = \{g(\mathbf{X}) \leq b_i\} \tag{27}$$

with $i = 1,...,m$. Hence, the sequence of intermediate events E_i is defined by the set of intermediate limit states.

For convenience, the conditional probabilities are established previously, such that $P[E_i \mid E_{i-1}] = P_0$. Also, the number of samples n_{SS} (e.g. $n_{SS} = 500$) at each subset is previously established. Hence, sets of samples $X_{0,k}$, with $k = 1,...,n_{SS}$ are obtained. The limit state functions are evaluated for $X_{0,k}$, resulting in vector $Y_{0,k} = g(X_{0,k})$. Components of vector $Y_{0,k}$ are arranged in increasing order, resulting in vector $Y^+_{0,k}$. The intermediate limit of failure b_1 is established as the sample $Y^+_{0,k}$ for which $k = P_0 n_{SS}$, such that $P[\{Y^+_{0,P_0 n_{SS}} \leq b_1\}] = P_0$. Thus, there are $P_0 n_{SS}$ samples on the "failure" domain defined by intermediate limit b_1. From each of these samples, by means of Markov Chain simulation, $(1 - P_0) n_{SS}$ conditional samples $X_{(1|0),k}$ are generated, with distribution $P(. \mid E_1)$. The limit state function is evaluated for these samples resulting in vector $Y_{(1|0),k} = g(X_{(1|0),k})$ and in ordered vector $Y^+_{(1|0),k}$, both related to intermediate limit of failure b_2, where $k = P_0 n_{SS}$. Therefore, $P[\{Y^+_{(1|0),P_0 n_{SS}} \leq b_2\}] = P_0$. Thus, the next intermediate event $E_2 =$

$\{Y \leq b_2\}$ is defined. One can notice that $P[E_2 \mid E_1] = P[Y \leq b_2 \mid Y \leq b_1] = P_0$. The $P_0 n_{SS}$ conditional samples will be the seeds of the conditional samples for the following level. Repeating the process, one generates conditional samples until the final limit b_m is reached, such that $b_m = 0$. This process is illustrated in Figure 4.

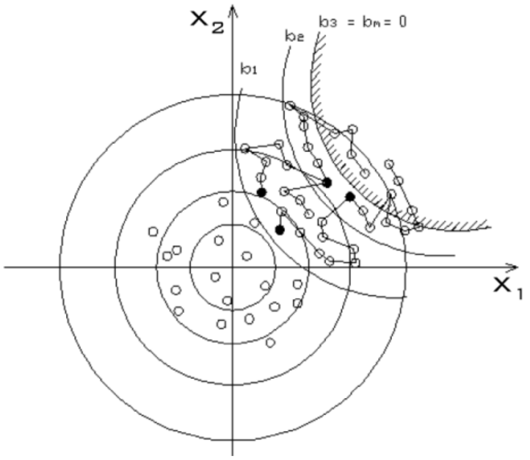

Figure 4: Subset sample generation using Markov Chains.

The random walk is defined by its probability distribution (e.g. uniform) and by the standard deviation σ_i^{RW}, which is considered as the product of a value α by the standard deviation of the problem, such that $\sigma_i^{RW} = \alpha \cdot \sigma_i$. Parameters of this algorithm are α, the number of samples for each subset (n_{SS}) and the conditional probability P_0. Different values are used for these parameters for different problems, as detailed in the next Section. In all cases, the random walk is modeled by a uniform probability distribution function.

5 COMPARATIVE PERFORMANCE OF SIMULATION STRATEGIES

In this study, three basic sampling techniques are combined with Crude Monte Carlo and with four modern sampling techniques: Importance Sampling Monte Carlo, Asymptotic Sampling, Enhanced Sampling and Subset Simulation. Hence, fifteen sampling schemes are investigated with respect to their performance in solving six structural reliability problems. Due to space limitations, only results for the three most relevant problems are presented herein. Conclusions reflect the six problems studied.

The following analysis procedure consists in two steps. The first step is a study of the convergence of the probability of failure and its coefficient of variation for increasing numbers of samples. Since the required number of samples for Subset Simulation is much lower than for the other techniques, the convergence study is not performed for Subset Simulation. The second step is a comparison of the results for all techniques, including Subset Simulation, considering a small number of samples.

For Examples 1 and 2, the limit state functions are analytic; hence processing time is not a relevant issue. Example 3 involves a Finite Element model with physical and geometrical nonlinearities; hence processing time is more relevant.

5.1 Example 1: non-linear limit state function

The first example has a nonlinear limit state function, where the random variables are modeled by non-Gaussian probability distribution functions. The limit state function is (Melchers and Ahamed, 2004):

$$g\left(X_1, X_2, X_3, X_4, X_5, X_6\right) = 2 \cdot X_1 \cdot X_2 \cdot X_3 \cdot X_4 - \frac{X_5 \cdot X_6^2}{8} \quad (28)$$

where X_i, with $i = 1,\ldots,6$, are the random variables. The parameters of the probability distributions are given in Table 1.

Random Variable	Distribution	Mean	St. Dev.
X_1	Weibull (minima)	4.0	0.1
X_2	Log-normal	25.0	2.0
X_3	Gumbel	0.875	0.1
X_4	Uniform	20.0	1.0
X_5	Exponencial	100.0	100.0
X_6	Normal	150.0	10.0

Table 1: Random variables of Example 1 (Melchers and Ahamed, 2004).

The reference probability of failure is obtained using Crude Monte Carlo simulation with Simple Sampling and 2×10^9 samples. The reference probability of failure is 2.777×10^{-5} ($\beta = 4.031$). This example aims to evaluate the performance of the intelligent sampling techniques in a problem with nonlinear limit state function involving non-Gaussian distribution functions.

Figures 5, 6 and 7 show convergence plots for the mean and coefficient of variation (c.o.v) of the P_f, for Simple Sampling, LHS and Asymptotic Sampling, respectively. Convergence results were evaluated for number of samples varying from 1×10^3 to 1×10^6. Results are shown for Crude, Importance, Asymptotic and Enhanced Sampling. For Asymptotic Sampling, the parameter f varies from 0.4 to 0.7, with 5 support points in this range. Subset simulation is not included, because computations are truly expensive for large numbers of samples. On the other hand, only a few samples are required to achieve similar results with Subset simulation, as observed in Table 2.

Two striking results can be observed in Figures 5 to 7. For all basic sampling techniques, the c.o.v. for Importance Sampling converges very fast to near zero. This is a very positive result. On the other hand, it is observed that the failure probability also converges very fast, but with a bias w.r.t. the reference result. This bias, although small and acceptable, is of some concern, and is introduced by the sampling function. It is also observed, in Figures 5 to 7, that for Asymptotic Sampling the c.o.v. convergence is quite unstable, with results oscillating significantly, and convergence for P_f also shows some bias. Both results are especially true for Simple and Anthytetic Variates Sampling, and less so for LHS. In fact, is observed that LHS improves the results for all sampling techniques illustrated in Figures 5 to 7.

a) Mean b) Coefficient of variation.

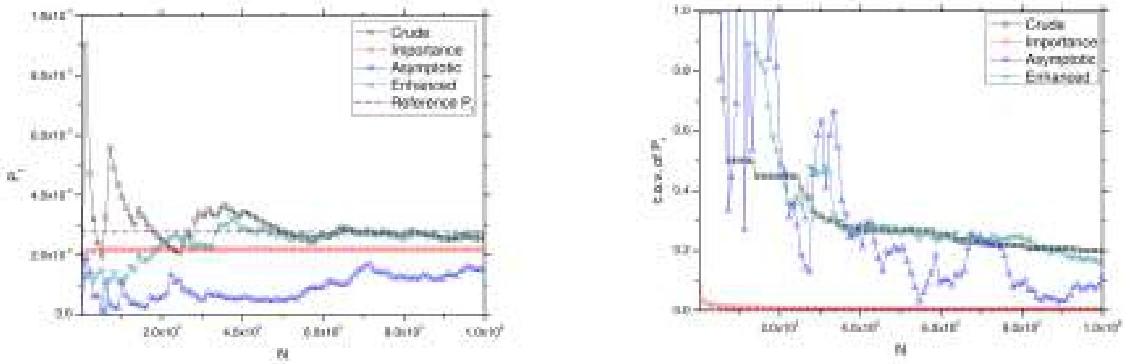

Figure 5: Convergence of P_f (a) and its c.o.v. (b) using Simple Sampling in Example 1.

a) Mean b) Coefficient of variation.

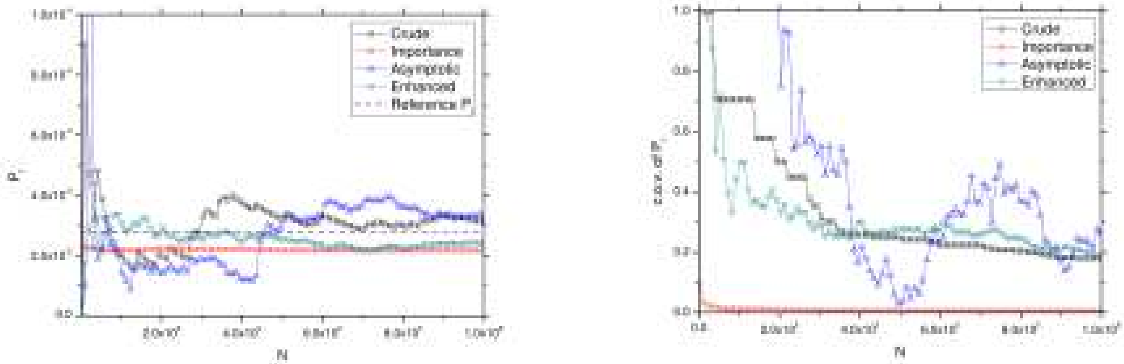

Figure 6: Convergence of P_f (a) and its c.o.v. (b) using Latin Hypercube Sampling in Example 1.

a) Mean b) Coefficient of variation.

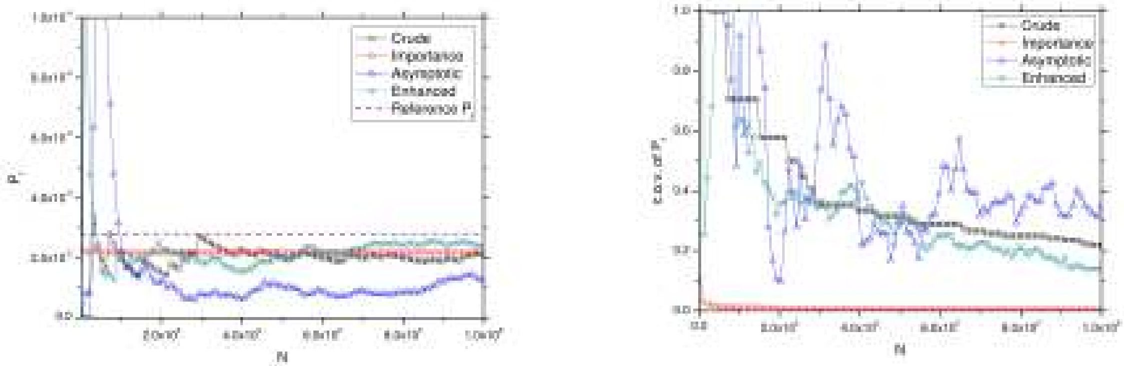

Figure 7: Convergence of P_f (a) and its c.o.v. (b) using Antithetic Sampling in Example 1.

A quantitative comparison of the performance of all sampling techniques, in solution of Problem 1, is presented in Table 2. These results are computed for a much smaller number of samples: 2,300. In Subset Simulation the following parameters are adopted: $n_{SS} = 500$, $\alpha = 1.5$, and $P_0 = 0.1$.

One observes in Table 2 that Crude Monte Carlo and Asymptotic Sampling do not lead to any results for such a small number of samples. Importance Sampling leads to the smallest c.o.v.s, but with deviations of around 18 to 30% from the reference result. Enhanced Sampling works, but deviations and c.o.v.s are rather large. For such a small number of samples, Subset Simulation provides the best results, with acceptable c.o.v.s and deviations varying from around 5 to 60%. These deviations could be further reduced by a marginal increase in the number of samples.

Estimation Technique		P_f	c.o.v.	Deviation (%)
Simple Sampling	Crude Monte Carlo	--	--	--
	Importance Sampling	1.973×10^{-5}	0.0551	28.9521
	Asymptotic Sampling	--	--	--
	Enhanced Sampling	1.152×10^{-4}	0.5941	314.8362
	Subset Simulation	2.640×10^{-5}	0.4741	4.9334
Latin Hypercube Sampling	Crude Monte Carlo	--	--	--
	Importance Sampling	1.947×10^{-5}	0.0587	29.8884
	Asymptotic Sampling	--	--	--
	Enhanced Sampling	5.129×10^{-7}	144.2610	98.153
	Subset Simulation	1.670×10^{-5}	0.5101	39.8632
Antithetic Variates Sampling	Crude Monte Carlo	--	--	--
	Importance Sampling	2.258×10^{-5}	0.0661	18.6892
	Asymptotic Sampling	--	--	--
	Enhanced Sampling	1.369×10^{-5}	4.1940	50.7022
	Subset Simulation	1.078×10^{-5}	0.5559	61.1811

Table 2: Comparison of results for 2,300 samples in Example 1.

5.2 Example 2: Hiper-estatic structural system (truss)

In order to investigate the performance of the intelligent sampling techniques in a structural system problem, an hiper-estatic truss is studied. Service failure is characterized by failure of any component (bar) of the truss, which can be due to buckling (compressed bars) or to yielding (tensile bars). System failure, characterized by failure of a second bar (any), given failure of the hiper-estatic bar (any), is not considered in this study (Verzenhassi, 2008). The truss and its dimensions are shown in Figure 8. The geometrical properties of truss bars are shown in Table 3. Table 4 shows the random variables considered.

Six limit state functions are employed to solve the problem: four functions related to elastic buckling of bars 1, 2, 3 and 6:

$$g_i\left(E_i, L_i, V, H\right) = \frac{\pi^2 E_i I_i}{L_i^2} + \left(a_i V + b_i H\right), \quad \text{with } i = 1,2,3,6 \tag{29}$$

and two functions related to the yielding of bars 4 and 5:

$$g_i\left(A_i, f_{y_i}, V, H\right) = A_i \cdot f_{y_i} - \left(a_i V + b_i H\right), \quad \text{with } i = 4,5 \tag{30}$$

where E_i is the Young's modulus, I_i is the moment of inertia of the U-shaped steel section, L_i is bar length, A_i is the transversal section area, f_{y_i} is the steel yielding stress, V is the vertical load, H is the horizontal load, a_i is the fraction of the vertical load acting at each bar, b_i is the fraction of the horizontal load acting at each bar. The random variables are described in Table 4. A correlation of 0.1 between V and H is considered.

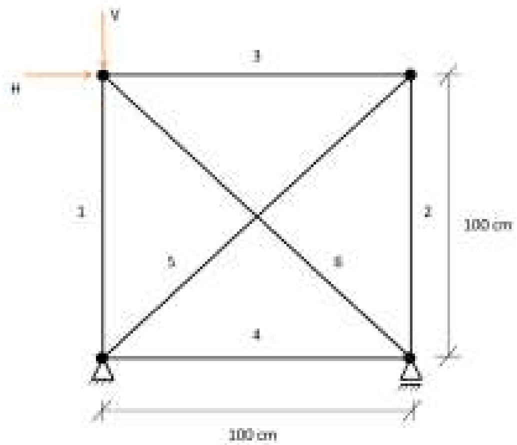

Figure 8: Hiper-static truss studied in Example 2.

Bar	U Shape	A (cm^2)	I (cm^4)
1	U 50x25x2.00	1.87	1.13
2	U 50x25x2.00	1.87	1.13
3	U 50x25x2.00	1.87	1.13
4	U 50x25x2.00	1.87	1.13
5	U 50x25x2.00	1.87	1.13
6	U 75x40x1.20	1.81	2.97

Table 3: Geometrical properties of the truss bars of Example 2 (Verzenhassi, 2008).

Random Variable	Distribution	mean	c.o.v
E_i (MN/cm^2)	Lognormal	20.5	0.05
f_{y_i} (kN/cm^2)	Lognormal	25	0.05
V (kN)	Lognormal	10	0.2
H (kN)	Lognormal	10	0.3

Table 4: Random variables of Example 2.

The reference probability of failure is obtained from a Crude Monte Carlo simulation with 2×10^9 samples using Simple Sampling: $P_f = 1.139\times10^{-4}$ ($\beta = 3.686$).

Figures 9, 10 and 11 show the convergence plots for the mean and coefficient of variation (c.o.v) of the P_f, for Simple Sampling, LHS and Asymptotic Sampling, respectively. Convergence results were evaluated for number of samples varying from 1×10^3 to 1×10^5. Results are shown for Crude, Asymptotic and Enhanced Sampling. For Asymptotic Sampling, the parameter f varies from 0.5 to 0.8, with 5 support points. Importance Sampling is not included because the sampling function is composed following equation (12). Hence, the convergence plot could not be obtained. Results for Importance Sampling and for Subset simulation are shown in Table 5, for a fixed (and smaller) number of samples.

One can notice in Figures 9, 10 and 11 that Enhanced Sampling performs very well, comparing to Crude Monte Carlo and to Asymptotic Sampling, w.r.t. convergence of the P_f and of its coefficient of variation. The use of LHS (Figure 10) is advantageous for the studied techniques, since faster and smoother convergence is observed.

a) Mean b) Coefficient of variation.

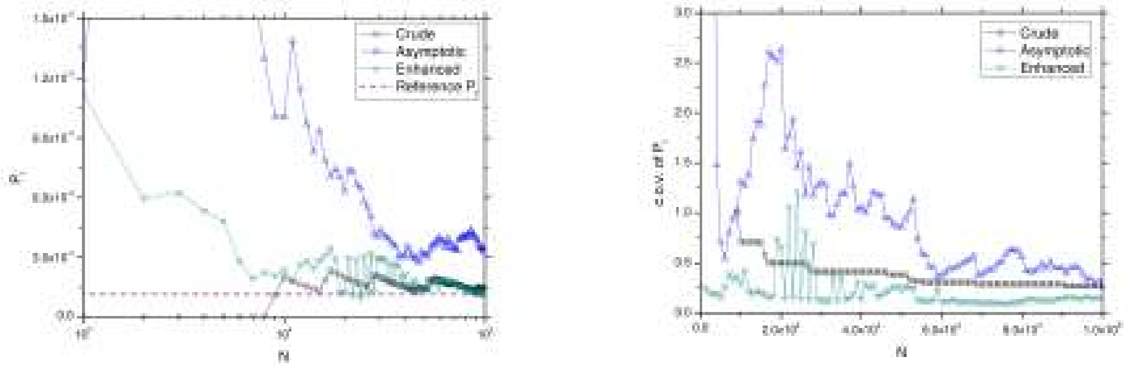

Figure 9: Convergence of P_f (a) and its c.o.v. (b) using Simple Sampling in Example 2.

a) Mean b) Coefficient of variation.

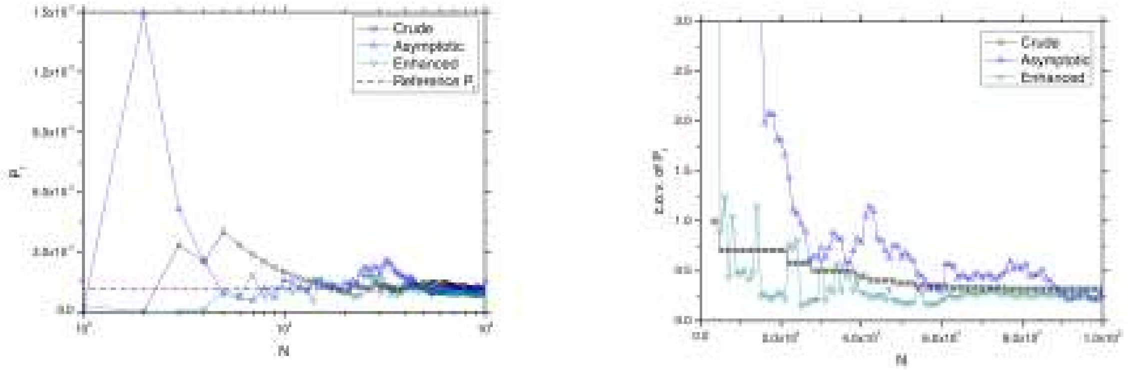

Figure 10: Convergence of P_f (a) and its c.o.v. (b) using Latin Hypercube Sampling in Example 2.

a) Mean b) Coefficient of variation.

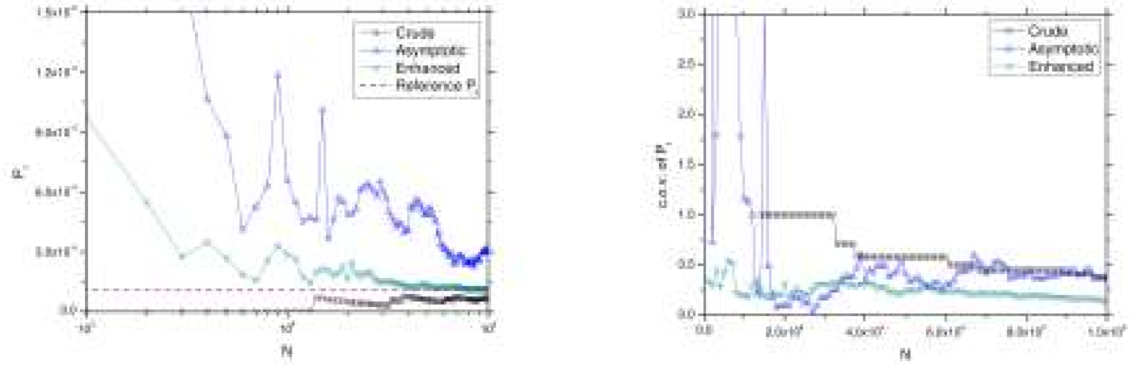

Figure 11: Convergence of P_f (a) and its c.o.v. (b) using Antithetic Sampling in Example 2.

The comparison of all intelligent sampling techniques is presented in Table 5, computed for a smaller number of samples: 3,700. For Subset Simulation, the following parameters are adopted: $n_{SS} = 1{,}000$; $\alpha = 1.5$; and $P_0 = 0.1$.

Estimation Technique		P_f	c.o.v.	Deviation (%)
Simple Sampling	Crude Monte Carlo	--	--	--
	Importance Sampling	1.201×10^{-4}	0.0287	5.4434
	Asymptotic Sampling	2.382×10^{-3}	1.2903	1991.3082
	Enhanced Sampling	6.209×10^{-4}	0.1657	445.1273
	Subset Simulation	1.590×10^{-4}	0.3221	39.5961
Latin Hypercube Sampling	Crude Monte Carlo	--	--	--
	Importance Sampling	1.111×10^{-4}	0.0260	2.4583
	Asymptotic Sampling	3.879×10^{-4}	10.4964	240.5619
	Enhanced Sampling	6.830×10^{-5}	1.6205	40.0351
	Subset Simulation	1.188×10^{-4}	0.3406	4.3020
Antithetic Variates Sampling	Crude Monte Carlo	--	--	--
	Importance Sampling	1.171×10^{-4}	0.0292	2.8095
	Asymptotic Sampling	1.229×10^{-3}	3.1715	979.0167
	Enhanced Sampling	3.846×10^{-4}	0.2525	237.6646
	Subset Simulation	1.280×10^{-4}	0.3205	12.3793

Table 5. Comparison of results for 3,700 samples in Example 2.

In Table 5, one observes that Crude Monte Carlo does not lead to any results for such a small number of samples. Importance Sampling presents a very good performance in comparison with the other techniques, with very small c.o.v.s and small and acceptable deviations from the reference P_f. Subset Simulation also presents an acceptable performance, with small c.o.v.s and larger, but still acceptable deviations from the reference result. The use of Latin Hypercube Sampling is beneficial for all techniques, for this problem.

5.3 Example 3: non-linear steel frame tower

An optimized plane steel frame transmission line tower (Figure 12) is analyzed by finite elements. The problem is based on Gomes and Beck (2013), where structural optimization considering expected consequences of failure was addressed. The mechanical problem is modeled by beam elements with three nodes, with three degrees of freedom per node. The frame is composed by L-shaped steel beams. The Finite Element Method with positional formulation (Coda and Greco, 2004; Greco et al., 2006) is adopted to solve the geometrical non-linear problem. Moreover, the material is assumed elastic perfectly plastic.

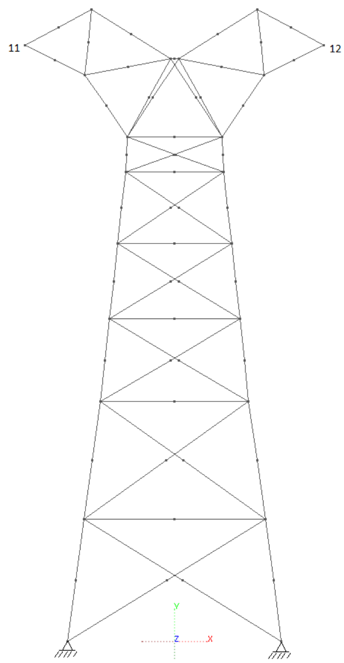

Figure 12: FE model of the power line tower addressed in Example 3.

The limit state function is defined based on the load-displacement curve (Figure 13) for top nodes 11 and 12. Because several configurations had to be tested in the optimization analysis performed in Gomes and Beck (2013), a robust limit state function was implemented. The same limit state function is employed herein: $g\left(\mathbf{x},\mathbf{d}\right) = 75° - \tan^{-1}\left(\Delta\delta / \Delta L\right)$, where $\Delta\delta$ is the increment in mean displacement in centimeters, for nodes 11 and 12; ΔL is the increment in the non-dimensional load factor; and 75° is the critical angle considered.

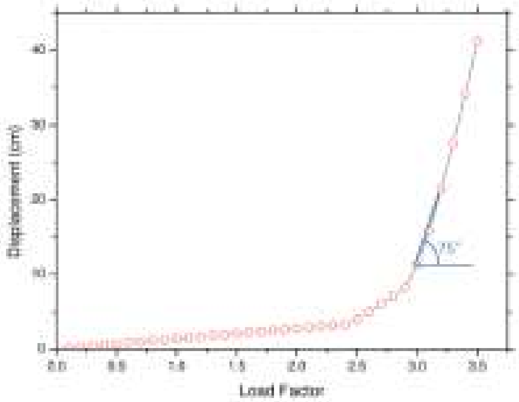

Figure 13: Load factor × mean displacement diagram for top nodes 11 and 12.

5.3.1 Loading

The gravity load from aligned cables is considered. The design wind load on cables and structure is calculated based on Brazilian code ABNT NBR 6123: *Wind loads in buildings*. The design wind is taken as $v_0 = 45$m/s. Thus, the characteristic wind load is calculated as $v_k = S_1 S_2 S_3 v_0$, where S_1, S_2 and S_3 are the topographical ($S_1 = 1$), the rugosity ($S_2 = 1.06$) and statistical ($S_3 = 1$) factors, yielding $v_k = 47.7$ m/s. This characteristic wind is taken at a height of 10 meters. The variation of wind velocity with height follows a parabolic shape, such that $v(z) = \sqrt{0.1 \cdot z \cdot v_k^2}$, where z is height. To model uncertainty in the wind load, an non-dimensional random variable V is introduced, such that $V(z) = v(z) \cdot V$. Thus, the wind pressure is evaluated from wind velocities as $q(z,V) = 0.613 \cdot (v(z) \cdot V)^2$, where pressure is given in N/m^2, for normal atmospheric conditions (1 atm) and temperature (15 °C). The wind load acts at each element on the tower. The drag coefficient is $C_a = 2.1$, which is the maximum value for prismatic beams with L-shaped sections. Cables of 2.52 cm diameter were adopted, with an influence area of 300 m and drag coefficient of $C_a = 1.2$. For these values, one obtains the random force $F_a(V)$ acting in the horizontal direction on nodes 11 and 12, such that $F_a(V) \cong 19{,}279.65 \cdot V^2$.

Random variables considered in this problem are the Young's modulus (E) and the non-dimensional wind variable (V). The random variables are described in Table 6.

Random Variable	Distribution	Mean	c.o.v
V	Gumbel	0.95	0.13
E (GPa)	Lognormal	207	0.03

Table 6: Random variables of Example 3 (Gomes and Beck, 2013).

5.3.2 Results

In Gomes and Beck (2013), probability of failure is evaluated by FORM, and values $P_{f,FORM} = 1.088\times10^{-4}$ ($\beta = 3.6976$) are obtained for the tower configuration in Figure 12. In this paper, several Monte Carlo Simulation techniques are adopted for evaluating this failure probability. FORM is very efficient, and efficiency is fundamental for solving the structural optimization problem. However, FORM provides only approximate results for problems with non-linear limit state functions.

The reference probability of failure is evaluated by Crude and by Importance Sampling Monte Carlo simulation, using Simple Sampling, Latin Hypercube Sampling and Antithetic Variates Sampling. 1×10^5 samples are employed for each solution. Results are given in Table 7, where one observes that Crude Monte Carlo with LHS and Importance Sampling Monte Carlo present probabilities of failure very close to the result obtained by FORM. This shows that, in spite of the non-linearity of the limit state function, FORM provides accurate results. For the remainder of the analysis, the average value of $P_f = 1.096\times10^{-4}$ is used as a reference.

Estimation Technique		P_f	c.o.v.	Processing Time	Deviation from average (%)
Crude Monte Carlo	Simple Sampling	1.400×10^{-4}	0.2672	11 h e 2 min	27.74
	Latin Hypercube Sampling	1.10×10^{-4}	0.3015	10 h e 17 min	0.36
	Antithetic Variates Sampling	1.50×10^{-4}	0.2582	11 h e 1 min	36.86
Importance Sampling Monte Carlo	Simple Sampling	1.099×10^{-4}	0.0038	13 h e 36 min	0.27
	Latin Hypercube Sampling	1.089×10^{-4}	0.0038	13 h e 41 min	0.64
	Antithetic Variates Sampling	1.096×10^{-4}	0.0038	13 h e 43 min	0.00

Table 7: Comparison of Crude Monte Carlo and Importance Sampling, using 10^5 samples.

Figures 14, 15 and 16 show the convergence plots for the mean and coefficient of variation (c.o.v) of the Pf, for Simple Sampling, LHS and Asymptotic Sampling, respectively. Convergence results were evaluated for number of samples varying from 1×10^3 to 1×10^5. Results are shown for Crude, Importance, Asymptotic and Enhanced Sampling. In Asymptotic Sampling, parameter f varies from 0.5 to 0.8, with 5 support points.

a) Mean b) Coefficient of variation.

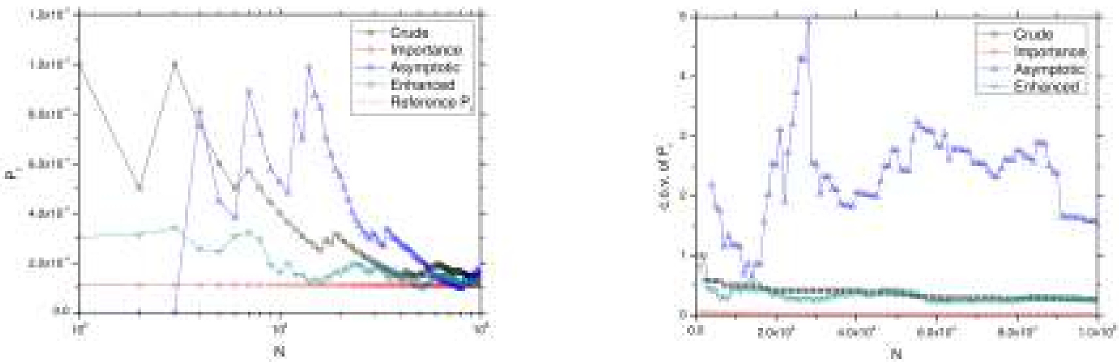

Figure 14: Convergence of P_f (a) and its c.o.v. (b) using Simple Sampling in Example 3.

a) Mean b) Coefficient of variation.

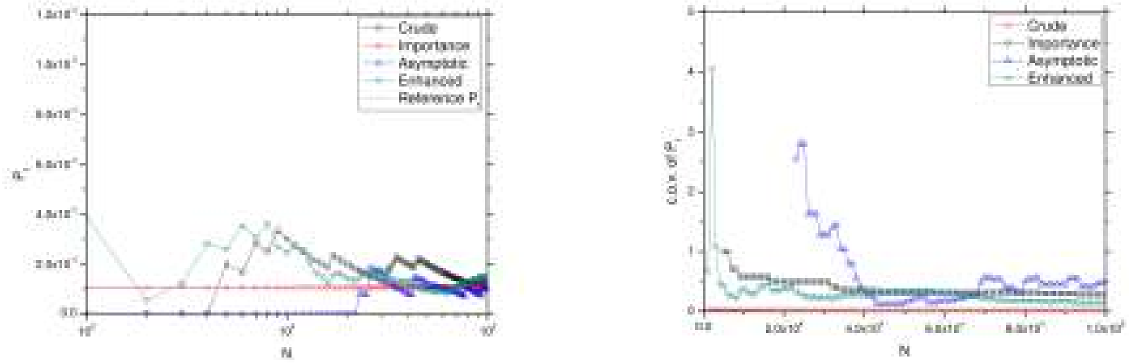

Figure 15: Convergence of P_f (a) and its c.o.v. (b) using Latin Hypercube Sampling in Example 3.

a) Mean b) Coefficient of variation.

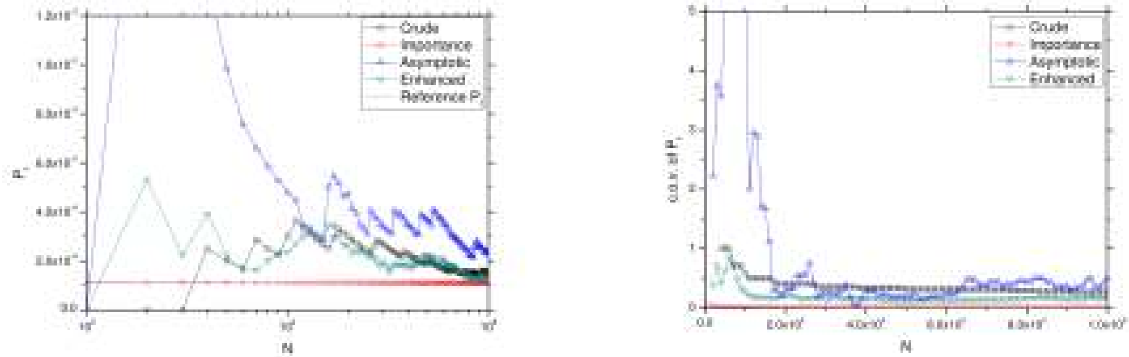

Figure 16: Convergence of P_f (a) and its c.o.v. (b) using Antithetic Sampling in Example 3.

In Figs. 14 to 16, one observes that convergence of Importance Sampling, for this problem, is very fast and accurate, for all basic sampling techniques. Also, it can be clearly seen that LHS improves the results for all sampling techniques, reducing oscillations during convergence. The convergence behavior of Asymptotic Sampling is very unstable, both for the c.o.v. and the P_f; hence this technique does not perform very well for this problem. Enhanced Sampling works much better, providing results similar to Crude sampling for this number of samples.

A quantitative comparison of the performance of all sampling techniques, in solution of Problem 3, is presented in Table 8. These results are computed for 14,800 samples. In Subset Simulation the following parameters are adopted: $n_{SS} = 4000$, $\alpha = 1.5$, $P_0 = 0.1$.

One observes in Table 8 that Importance Sampling and Subset simulation out-perform the other methods in terms of small c.o.v. and small deviation from the reference solution. Asymptotic and Enhanced sampling show a similar and average performance. Latin Hipercube Sampling improves the results for most methods, especially for Subset Simulation.

5.3.3 Processing Time

The processing time to compute the results in Table 8 (14,800 samples) are given in Table 9. One observes that processing times are similar for all techniques, except for Importance Sampling: this technique takes a little longer to compute the weights for each sample, following equation (11). Processing times are very similar for the basic sampling techniques. By comparing results of Table 9 with results of Table 7, one observes the massive gain in processing time that is obtained using the intelligent sampling techniques addressed herein. Computing times are much smaller, but the quality of the solutions (small deviation and c.o.v.) are similar.

Estimation Technique		P_f	c.o.v.	Deviation (%)
Simple Sampling	Crude Monte Carlo	2.703×10^{-4}	0.4999	146.62
	Importance Sampling	1.090×10^{-4}	0.0099	0.55
	Asymptotic Sampling	2.697×10^{-4}	8.0852	146.08
	Enhanced Sampling	1.327×10^{-4}	0.4103	21.08
	Subset Simulation	1.960×10^{-4}	0.1970	78.83
Latin Hypercube Sampling	Crude Monte Carlo	6.757×10^{-5}	--	38.35
	Importance Sampling	1.091×10^{-4}	0.0100	0.46
	Asymptotic Sampling	8.300×10^{-5}	1.5114	24.27
	Enhanced Sampling	1.659×10^{-4}	0.3106	51.37
	Subset Simulation	1.110×10^{-4}	0.2046	1.28
Antithetic Variates Sampling	Crude Monte Carlo	2.703×10^{-4}	0.4999	146.62
	Importance Sampling	1.086×10^{-4}	0.0100	0.91
	Asymptotic Sampling	1.393×10^{-4}	4.7499	27.10
	Enhanced Sampling	1.650×10^{-4}	0.3941	50.55
	Subset Simulation	1.623×10^{-4}	0.1963	48.08

Table 8: Comparison of results for 14,800 samples in Example 3.

	Simple	LHS	Anthitetic
Crude Monte Carlo	1 h and 37 min	1 h and 31 min	1 h and 38 min
Importance Monte Carlo	2 h	2 h and 2 min	2 h and 2 min
Asymptotic Sampling	1 h and 16 min	1 h and 15 min	1 h and 14 min
Enhanced Sampling	1 h and 27 min	1 h and 30 min	1 h and 30 min
Subset Simulation	1 h and 52 min	1 h and 50 min	1 h and 51 min

Table 9: Processing time for 14,800 samples in Example 3.

5.3.4 Processing Time

In order to further evaluate the performance of Subset Simulation using smaller numbers of samples, the following parameters are adopted: $n_{SS} = 2{,}000$; $\alpha = 2$; and $P_0 = 0.1$. Results are presented in Table 10.

Technique	Number of Samples	P_f	c.o.v.	Deviation (%)	Processing time (min)
Simple Sampling	9,200	9.539×10^{-5}	0.2461	12.97	58
LHS	7,400	1.055×10^{-4}	0.2446	3.74	43
Asymptotic Sampling	7,400	1.065×10^{-4}	0.2463	2.83	44

Table 10: Comparison of results using Subset Simulation for Example 3.

Results presented in Table 10 show that Subset Simulation is an effective tool for estimating small probabilities of failure in problems involving numerical evaluation of limit state functions. The original solution via Crude Monte Carlo took approximately 11 hours to compute a probability of failure with coefficient of variation of 0.27 and relative deviations of: 28.68% (Simple Sampling), 1.1% (Latin Hypercube Sampling) and 37.87% (Antithetic Variates Sampling). In comparison, Subset Simulation took approximately 45 minutes to estimate the same probability of failure with coefficient of variation of 0.25 and deviations around 3.7%.

5 CONCLUDING REMARKS

This paper presented a benchmark study on intelligent and efficient sampling techniques in Monte Carlo Simulation. Crude Monte Carlo simulation was compared to Importance Sampling, Asymptotic Sampling, Enhanced Sampling and to Subset Simulation. These five sampling schemes were combined with Simple Sampling, Latin Hypercube Sampling and Antithetic Variates Sampling, resulting in fifteen sampling strategy combinations. The performance of these strategies was investigated for six problems, but results for only three were presented herein. The conclusions below reflect the six problems studied.

It was observed that use of Latin Hypercube Sampling had a significant and positive influence for all sampling techniques with which it was combined. LHS has led to smoother convergence curves and more accurate results for most cases studied. Antithetic Variates also produced better results than Simple Sampling.

Importance Sampling using design points was found to be one of the most efficient techniques for solving problems with small and very small failure probabilities. Convergence is extremely fast, and with a couple of samples one obtains accurate failure probabilities with small sampling error (c.o.v.). However, for one of the problems studied, it was observed that the sampling function introduced some bias in the results, making failure probabilities converge to an inexact yet acceptable value. Also, it is known that use of Importance Sampling can be a problem for highly non-linear problems for which the design point(s) cannot be found.

Results obtained with Asymptotic Sampling were inaccurate for a number of problems, perhaps because a small number of support points were used. The drawback with this scheme is that a new set of samples has to be computed for each additional support point; hence there is a string compromise between the number of support points and the accuracy with which failure probabilities can be estimated for each support point.

Although similar to Asymptotic Sampling, the Enhanced Sampling technique presented consistent results for all problems studied. Enhanced sampling has a large advantage over Asymptotic

Sampling, because the same set of samples is used for all support points. Hence, there is no computational penalty for using many support points, and the accuracy of the regression is improved.

Subset Simulation performed extremely well for all problems studied, resulting in accurate estimates of failure probabilities, with very small sampling errors. This explains why Subset Simulation has become so popular among structural reliability researchers, and is being applied extensively in the solution of both time invariant and time variant reliability problems.

Finally, it can be said that Subset Simulation, Enhanced Sampling and Importance Sampling, aided by Latin Hypercube sampling, are efficient ways of solving reliability problems, with a much smaller number of samples than required in Crude Monte Carlo Simulation.

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