www.lajss.org
Latin American Journal of Solids and Structures 4 (2007) 229–246

A combined continuous-discontinuous finite element method for convection-diffusion problems

Philippe R. B. Devloo[a,*], Tiago Forti[a] and Sonia M. Gomes[b]

[a]Faculdade de Engenharia Civil, Arquitetura e Urbanismo,
Universidade Estadual de Campinas, SP – Brazil
[b]Instituto de Matemática, Estatística e Computação Científica,
Universidade Estadual de Campinas, SP – Brazil

Abstract

Discontinuous Galerkin (DGM) method combines the advantages of stability of finite volume method and the accuracy of continuous finite element method (FEM). Applications of the DGM are particularly valuable where the solution presents high-gradients or discontinuities, such as boundary layers and shock problems. A disadvantage of the DGM is the higher computational cost when comparing to classic finite element method, due to the increased number of degrees of freedom. With this motivation, in this paper we explore the idea of combining continuous and discontinuous Galerkin formulations for the simulation of convection-diffusion problems. The computational domain is decomposed into two parts. In one region the solution is supposed to be smooth, and the traditional continuous finite element method is applied. On the other hand, where steep gradients are expected, we use a discontinuous Galerkin formulation. This paper presents numerical results for the combined FEM/DGM method applied to convection-diffusion problems.

Keywords: finite element method, discontinuous Galerkin method, convection-diffusion, boundary layers

1 Introduction

Problems of practical interest, in which convection plays an important role, arise in several applications like meteorology, gas dynamics, turbulent flows, modeling of shallow water, transport of contaminant in porous media, viscoelastic flows and electro-magnetism among many others. The solution of these problems, however, is far from trivial. The exact solution of (non linear) purely convection problems may develop discontinuities such as shock waves. Also in the presence of diffusive terms, but when convection is dominant, the formation of boundary layers may also happen. Thus, when constructing numerical methods for these problems, it is necessary to ensure not only that the discontinuities of the solution are the physical ones, but also that the

*Corresp. author email: phil@fec.unicamp.br Received 18 Sep 2006; In revised form 07 Jul 2007

appearance of such discontinuities does not induce spurious oscillations that spoil the quality of the approximation.

In the approximation of pure diffusion elliptic problems the finite element method (FEM) is the most widely used method. However, for convection dominated hyperbolic problems the FEM presents deficiencies. Originally, the discontinuous Galerkin method (DGM) formulations were intended to solve such kind of hyperbolic conservation laws: to capture the discontinuities that may occur in the solution, the use of discontinuous approximating functions seemed to be appropriate. But recently, discontinuous Galerkin formulations have also been considered to treat problems involving diffusion terms [1, 6, 7, 15].

Like in the FEM, the DGM uses the Galerkin principle for interpolation spaces formed by piecewise polynomials, but no restriction of continuity over the boundaries between neighbor elements is imposed. The DGM can also be interpreted as a generalization of the finite volume method for the particular case where the approximating functions are constant in the interior of each element. In this sense, the expertise already developed for finite volume methods can be applied for the DGM. The increasing interest in these kind of formulations are due to some interesting features: they are locally conservative, the accuracy is obtained by means of high-order polynomials within elements, without any regularity constraint at element interfaces. Furthermore, unstructured meshes and parallelization can be easily handled. The combination of these properties leads to robust solvers with high precision in space and wide stability range.

A disadvantage of DGM is that it produces more degrees of freedom than the FEM. It means that discontinuous Galerkin is more accurate and stable solving problems with discontinuities but more costly than the continuous FEM. With this motivation, this work presents an implementation for an innovating numerical method FEM/DGM for convection-diffusion problems by combining continuous and discontinuous Galerkin methodologies in the same simulation obtaining the advantages of both methods. In regions where the solution is detected to be smooth, continuous elements are employed, and in regions of high gradients or discontinuities, discontinuous elements are employed. This approach was employed by Clint Dawson [8] in 2002 for 1D problems and by the authors [9–11] in 2005 for 2 dimensional problems.

The paper is organized as follows. In Section 2 we describe the different formulations of interest for our applications. For convection-diffusion problems it is necessary to adopt a consistent methodology for both hyperbolic and elliptic operators. For the elliptic operators, we shall consider the weak formulation proposed by Baumann in [2] that can be interpreted as a hybrid formulation where the Lagrange multiplier is replaced by the normal flux across interfaces (section 2.1). For the convection operator, the weak formulation is described in section 2.2. In Section 3 we give some details concerning the computational implementation of the algorithms in the PZ environment (www.labmec.fec.unicamp.br/˜pz). In Section 4, numerical simulation results are presented for two model problems in order to analyze the performance of the methods considered in this work FEM, DGM and FEM/DGM . The first problem has a solution with discontinuity inside the element and the other one refers to a problem with steep boundary layer. Concluding remarks are presented in Section 5.

2 Discontinuous Galerkin formulations

A linear convection-diffusion problem can be stated as follows: find u that satisfies

$$\begin{aligned} -\nabla \cdot (A\nabla u) + div(\beta u) = S \; in \; \Omega \\ u = f \; in \; \Gamma_D \\ (A\nabla u) \cdot n = g \; in \; \Gamma_N \end{aligned} \tag{1}$$

where:
Ω is the domain with Lipschitz boundary $\partial\Omega$;
A is the diffusion tensor
β is the convection vector;
n is the outward normal to $\partial\Omega$;
S, f and g are known smooth functions;
Γ_D is the part of the boundary with Dirichlet conditions;
Γ_N is the part of the boundary with Neumann conditions;
$\Gamma_D \bigcap \Gamma_N = \varnothing$
and $\Gamma_D \bigcup \Gamma_N = \partial\Omega$.

The weak formulation of problem (1) will be constructed in two parts: the weak formulation for the elliptic operator $-\nabla \cdot (A\nabla u)$ followed by the weak formulation for the convection term $div(\beta u)$.

The following functional spaces are defined:

Partition of the domain

Let $P_h = \Omega_e$ be a partition of Ω, $\overline{\Omega} = \bigcup_{e=1}^{nel} \overline{\Omega}_e$, where $\Omega_e \bigcap \Omega_f = \emptyset$ for $e \neq f$ and nel is the number of elements of P_h. In other words, the elements are defined with open boundaries with no intersection between elements. For elements Ω_e and Ω_f with common boundary it is defined $n_f = n_e$ if $e > f$, where n_e is the outward normal of $\partial\Omega_e$.

Approximation Space

Let the space

$$V(P_h) = H^1(P_h) = \left\{ v \,\epsilon\, L^2(\Omega) : v|_{\Omega_e} \in H^1(\Omega_e) \,\forall\, \Omega_e \,\epsilon\, P_h \right\}$$

The notation $\Gamma_{P_h} = \Gamma_D \cup \Gamma_N \cup \Gamma_{int}$ is employed, where Γ_D and Γ_N are the boundaries with Dirichlet and Neumann conditions and Γ_{int} the set of interfaces between elements. The operators [ ] and <> are defined as:

- $[\vartheta] = \vartheta|_{\partial\Omega_e \bigcap \Gamma_{ef}} - \vartheta|_{\partial\Omega_f \bigcap \Gamma_{ef}}\,, \; e > f$

- $< \vartheta >= \frac{1}{2}\left(\vartheta|_{\partial\Omega_e \bigcap \Gamma_{ef}} + \vartheta|_{\partial\Omega_f \bigcap \Gamma_{ef}}\right)$

where Γ_{ef} relates to the common boundary of elements Ω_e and Ω_f.

2.1 Elliptic problem

The elliptic problem can be stated as: find u that satisfies

$$\begin{array}{c} -\nabla \cdot (A\nabla u) = S \ in \ \Omega \\ u = f \ in \ \Gamma_D \\ (A\nabla u) \cdot n = g \ in \ \Gamma_N \end{array} \tag{2}$$

where Ω, A, n, S, f, g, Γ_D are Γ_N the same defined in the problem 1.

The weak formulation proposed by Baumann in [2] can be interpreted as a hybrid formulation where the Lagrange multiplier p is replaced by the normal flux across interfaces $p = (A\nabla u) \cdot n$. The weak formulation is state: find $u \in V(P_h)$ that satisfies

$$B_H(u, v) = F_H(v)\, \forall v \in V(P_h) \tag{3}$$

$$\begin{aligned} B_H(u,v) \quad = \quad & \sum_{e=1}^{Nelem} (\int_{\Omega_e} \nabla v \cdot (A\nabla u) dx) + \int_{\Gamma_D} (u\, (A\nabla v) \cdot n - v\, (A\nabla u) \cdot n) ds + \\ & \int_{\Gamma_{int}} (\langle (A\nabla v) \cdot n \rangle \, [u] - \langle (A\nabla u) \cdot n \rangle \, [v]) ds \end{aligned}$$

$$L_H(v) = \sum_{e=1}^{Nelem} (\int_{\Omega_e} vS dx) + \int_{\Gamma_D} f\, (A\nabla v) \cdot n ds + \int_{\Gamma_N} vg ds.$$

2.2 Linear hyperbolic problem

The liner hyperbolic problem can be stated as follow: find u that satisfies

$$\begin{array}{c} div(\beta u) = S \ in \ \Omega \\ u = f \ in \ \Gamma_- \end{array} \tag{4}$$

where Ω, S and f are defined in problem 1. β is the convection vector, $\Gamma_- = \{x \in \partial\Omega \mid (\beta \cdot n)(x) < 0\}$ and $\Gamma_+ = \partial\Omega \backslash \Gamma_-$.

A weak formulation of the problem 4 may be constructed by multiplying a test function $v \in V(P_h)$, integrating over the domain Ω and integrating by parts:

$$\int_\Omega -(\nabla v \,.\, \beta) u \, d\Omega + \int_{\partial\Omega} (\beta u v) \,.\, n \, ds = \int_\Omega S \, v \, d\Omega,$$

or replacing the integral over Ω by the sum of integrals over the elements $\Omega_e \in P_h$

$$\sum_{e=1}^{nel}\left\{\int_{\Omega_e} -(\nabla v \cdot \beta)u\, d\Omega + \int_{\partial\Omega_e} (\beta u v)\cdot n\, ds\right\} = \sum_{e=1}^{nel}\left\{\int_{\Omega_e} S\, v\, d\Omega\right\}. \tag{5}$$

The convection term over the boundary $\partial\Omega$ is replaced by an upwind numerical flux $F_N \approx \int_{\partial\Omega}(\beta u v)\cdot n\, ds$ (see [14]). The numerical flux is defined as

$$F_N^e = \int_{\Gamma_+^e} v\, u^-(\beta \cdot n_e)\, ds + \int_{\Gamma_-^e} v\, f\,(\beta \cdot n_e)\, ds$$

where $\Gamma_-^e = \{x \in \partial\Omega_e \mid (\beta \cdot n)(x) < 0\}$ and $\Gamma_+^e = \partial\Omega_e \backslash \Gamma_-^e$ and

$$u^- = \lim_{\alpha \to 0^+} u(x - \alpha(\beta.\overrightarrow{n}).\overrightarrow{n}),\ x \in \Gamma^e$$

Figure 1 illustrates the meaning of u^-:

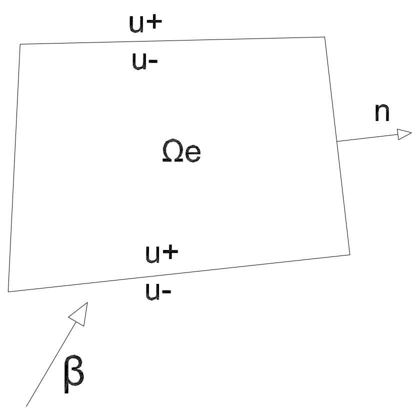

Figure 1: Illustration of u^- and u^+

The weak form is stated as follow: find $u \in V(P_h)$ that satisfies

$$B_C(u,v) = L_C(v)\ \ \forall v \in V(P_h) \tag{6}$$

$$B_C(u,v) = \sum_{e=1}^{nel}\left\{\int_{\Omega_e} -(\nabla v \cdot \beta)u\, d\Omega + \int_{\Gamma_+^e} v\, u^-(\beta \cdot n_e)\, ds\right\}$$

$$L_C(v) = \sum_{e=1}^{nel}\left\{\int_{\Omega_e} S\, v\, d\Omega - \int_{\Gamma_-^e} v\, f\,(\beta \cdot n_e)\, ds\right\}.$$

2.3 Convection diffusion problem

Combining the weak forms for elliptic and convection operators developed in the previous sections, the weak form of the convection-diffusion problem may be stated as: find $u \in V(P_h)$ that satisfies

$$B_{DG}(u, v) = L_{DG}(v) \, \forall v \in H^1(P_h) \tag{7}$$

with

$$B_{DG}(u, v) = B_C(u, v) + B_H(u, v)$$

$$L_{DG}(v) = L_C(v) + L_H(v)$$

B_* and L_* are the bilinear and linear forms defined previously, where:

- B_C is the bilinear form associated with the convection operator
- B_H is the bilinear form associated with the elliptic operator
- and L_* are their respective linear forms.

Artificial diffusion - SUPG

It is well known that FEM method is unstable when applied to convection dominated problems. Several stabilized versions of the FEM have been proposed by means of the introduction of some artificial diffusion or stabilization term [3, 13]. Calle [4, 5] and Santos [17] have also applied an artificial diffusion on the DGM method to reduce the oscillations produced by discontinuities captured in the interior of the element. In the present work a SUPG - *Streamline Upwind Petrov-Galerkin*, also called SD - *Streamline Diffusion* shall be used to stabilize the FEM and to prevent oscillations in the interior of discontinuous elements. It is important to remark that the SUPG is not an artificial dissipation, since the weak formulation with SUPG is still equivalent to the strong problem. The simulation gains stability but remains equivalent to the strong problem.

Considering that the objective of this work is to study convection dominated problems, the SUPG is added only to the convection operator. The absence of SUPG in the elliptic operator modifies the problem and the weak formulation is no longer equivalent to the strong problem. However, since the diffusion contribution in convection dominated problems is small, the error of this simplification might not be significant. In numerical examples it is shown the gain of stability of the FEM with SUPG. But convergence rates are influenced by the non consistency of the actual implementation.

The SUPG terms added to equation (7) are:

$$B_{SUPG}(u,v) = \sum_{e=1}^{nel} \frac{\delta h}{2\,||\beta||} \int_{\Omega_e} (\nabla u.\beta)\,(\nabla v.\beta)\,d\Omega_e, \tag{8}$$

$$L_{SUPG}(v) = \sum_{e=1}^{nel} \frac{\delta h}{2\,||\beta||} \int_{\Omega_e} S\,\nabla v.\beta\,d\Omega_e,$$

where δh is the size of the element Ω_e.

Adding the equation (8) to the original weak formulation (equation 7), the DGM formulation for convection-diffusion problems with SUPG is obtained:

$$B_{DG}(u,v) + B_{SUPG}(u,v) = L_{DG}(v)\,\forall v \in H^1(P_h) \tag{9}$$

3 Computational implementation

The algorithms of the present applications are implemented in the PZ environment [1]. PZ is an environment for developing finite element techniques for *one, two and tridimensional* boundary value problems. It has been designed using the object oriented philosophy and *allows the insertion of new formulations with reduced additional effort. For instance, all hp adaptive functionalities previously implemented in the PZ environment could be re-used for the combined continuous-discontinuous algorithm. In the sequence, we give some important aspects required for the computational implementation of the algorithms.*

3.1 Geometry and interpolation spaces

In the PZ environment, a strong separation is made between the approximation of the geometry and the approximation of the function space. The topology description is made by geometric elements (*TPZGeoEl*) associated to a geometric mesh. The geometric elements are responsible for keeping track of the neighboring information and for the mapping from master element to the real element in the classical FEM. The geometric elements also detect the occurrence of hanging-nodes in the FEM mesh.

The interpolation space is implemented by computational elements. Computational elements are objects of classes derived from the abstract class *TPZCompEl* (Figure 2). The continuous finite element function spaces are implemented by the class *TPZInterpolatedElement* and the discontinuous Galerkin function spaces are implemented by classes *TPZCompElDisc* and *TPZ-InterfaceElement*. The inheritage chain guarantees that the functionalities implemented for the FEM in PZ are available for the DGM.

Figure 3 illustrates a geometric and computational meshes. The approach adopted in this work in combining continuous and discontinuous elements in the same simulation was made

[1](www.labmec.fec.unicamp.br/˜pz)

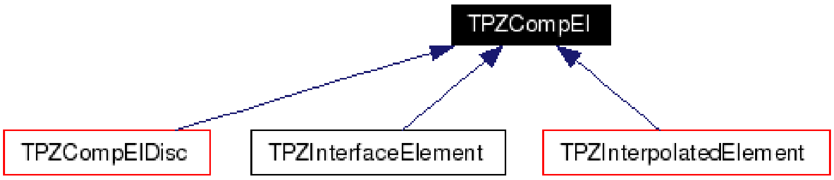

Figure 2: Computational elements

simple by the use of a consistent class interface. In some regions the problem is treated with continuous elements, and other regions with discontinuous elements. In the boundary between these two regions interface elements are applied, which will implement the boundary terms of the discontinuous Galerkin operator. Interface elements are necessary between discontinuous elements and between one discontinuous and one continuous element.

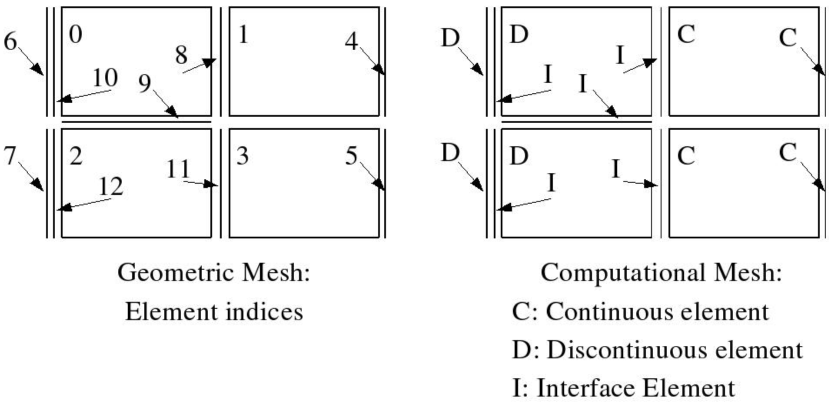

Figure 3: Geometric and computational meshes

The computational elements are responsible for computing the integral terms of the weak formulations. For instance, consider the elliptic DGM formulation of this work (eq. 3). The class *TPZCompElDisc* is responsible for the integration over the discontinuous element Ω_e and the class *TPZInterfaceElement* is responsible for computing the integral over interfaces Γ_{int}, Γ_D e Γ_N. In the FEM the element *TPZInterpolatedElement* is the only responsible for computing the terms of the classical weak form.

The computational elements compute the integral terms using a numerical integration rule. Consider a general integral term defined as the integral of an operator A applied to shape functions φ_i e φ_j over the element domain γ_e:

$$\int_{\gamma_e} A\left(\varphi_i, \varphi_j\right) = \sum_{p=0}^{np-1} A\left(\varphi_i(x_p), \varphi_j(x_p)\right) w_p$$

where x_p denotes the integration point of index p and w_p is its correspondent weight. The computational element constructs the integration rule and performs the loop over the np points computing the shape functions φ_i e φ_j and their derivatives. However the computational element does not compute the operation A. That operator is implemented by a class derived from the abstract class *TPZMaterial* (section 3.2). That separation allows the same computational element to perform different formulations. Examples of formulations implemented in the PZ are scalar convection-diffusion problems (treated in this work), elasticity (beams, plates and solids), Euler's equation, etc, and is illustrated in figure 4.

This work contributes to the PZ enviromnet by allowing the use of interface elements between continuous and discontinuous elements. The discontinuous element constructs its shape functions in the real element and the continuous element constructs the shape functions in master element. A modification has been made in the class *TPZInterfaceElement* to allow the inclusion of interface between continuous and discontinuous elements. An interface integration point is then mapped to the coordinate in the master element of the continuous neighbor element. Then the continuous element compute its shape functions and derivatives and the interface may perform the integral contribution.

3.2 Material classes

In the PZ environment a weak formulation is implemented by a class derived from the abstract class *TPZMaterial*. The class *TPZMaterial* defines the interfaces that a derived class must implement. Important functions are *Contribute* and *ContributeBC*. The function *Contribute* computes the contribution to the stiffness matrix and load vector of the element in a given integration point. The function *ContributeBC* computes the contribution to the stiffness matrix and load vector of an element of type boundary condition at an integration point.

The class *TPZDiscontinuousGalerkin* is derived from the class *TPZMaterial*. The purpose of the class *TPZDiscontinuousGalerkin* is to define the necessary interfaces for the DGM. The function *ContributeInterface* and *ContributeBCInterface* with the already defined (in base class *TPZMaterial*) *Contribute* are the most important functions. The function *ContributeInterface* and *ContributeBCInterface* are responsible for computing the contribution to the stiffness matrix and load vector of an interface element (and interface in boundary condition) at an integration point.

Figure 4 illustrates some material classes implemented in PZ. The class *TPZMatPoisson3d* which implements a scalar convection-diffusion formulation is the class used in this work.

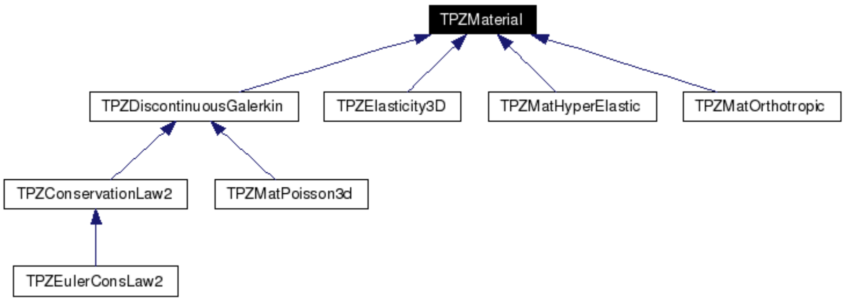

Figure 4: Some material classes in PZ

4 Numerical examples

For an efficiency comparison of the different formulations FEM, DGM and DGM/FEM, we shall consider two model problems.

4.1 Convection diffusion problem

Consider the problem:

$$-\nabla \cdot (A\nabla u) + div(\beta u) = 0 \ in \ (-1,1)^2$$
$$\beta = \{1,1\}^T, \ A = 1.e-6$$

augmented with Dirichlet boundary condition with values showed in figure 5. The solution of this problem has a discontinuities in the interior of the domain.

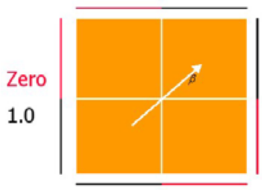

Figure 5: Problem boundary conditions

Analyzing figures 6, 7, 8 and 9, some aspects are remarkable. First of all, the discontinuous Galerkin method is stable for this problem. The use of SUPG improves the stability and the quality of the solution. It is also important to observe that the oscillations present in the interior of the element do not propagate to the neighbor elements, which is an advantage of the DGM when comparing to FEM. The quality of the FEM solution is increased with the use of SUPG.

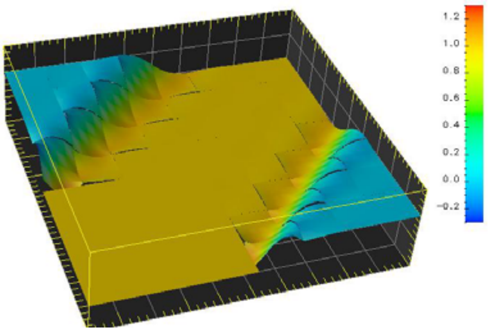

Figure 6: Solutions with h uniform refinement - $p = 2$ (Discontinuous Galerkin method)

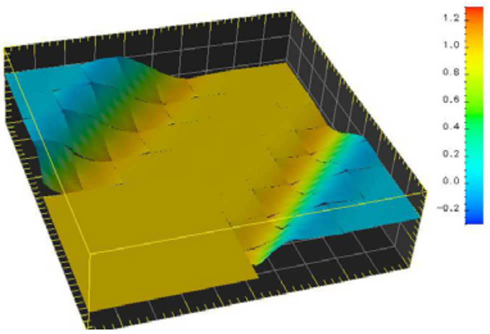

Figure 7: Solutions with h uniform refinement - $p = 2$ (Discontinuous Galerkin method with SUPG)

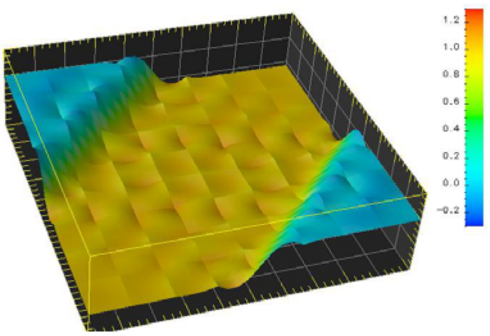

Figure 8: Solutions with h uniform refinement - $p = 2$ (Continuous finite element method)

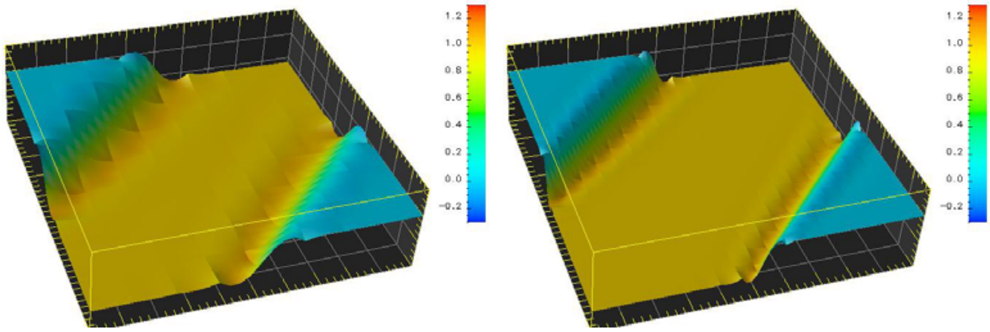

Figure 9: Solutions with h uniform refinement - $p = 2$ (Continuous finite element method with SUPG)

4.2 Boundary layer problem

Consider the boundary layer problem treated in Houston, Schwab and Sli [12]. The differential problem is

$$-\nabla \cdot (A\nabla u) + div(\beta u) = S \ em \ \Omega = (0,1)^2$$

$$S = 2 + \frac{(1-x)^2 - (1-x) - (1-y) + (1-y)^2}{A\,(1-e^{-1/A})\,e^{(1-x)\,(1-y)/A}} - x - y$$

$$A = 0.01,\ \beta = \{1,1\}^T$$

with Dirichlet boundary condition

$$u|_{\Gamma_D} = \begin{cases} 0 & for\, x = 1 \\ 0 & for\, y = 1 \\ -\left(-1 + e^{y/A} + y - e^{1/A} y\right)\left(-1 + e^{1/A}\right)^{-1} & for\, x = 0 \\ -\left(-1 + e^{x/A}) + x - e^{1/A} x\right)\left(-1 + e^{1/A}\right)^{-1} & for\, y = 0 \end{cases}$$

The analytical solution is given by

$$u(x,y) = x + y\,(1-x) + (e^{-1/A} - e^{-(1-x)\,(1-y)/A})\,(1 - e^{-1/A})^{-1}$$

which is shown in figure 10. For this example the solutions of DGM and FEM are compared. It is also proposed to use both elements in the same simulation. In the vicinity of the boundary layer discontinuous elements are adopted in order to have a stable solution. And in the region the solution is smooth continuous elements are employed. The mesh illustrated in figure 10 is

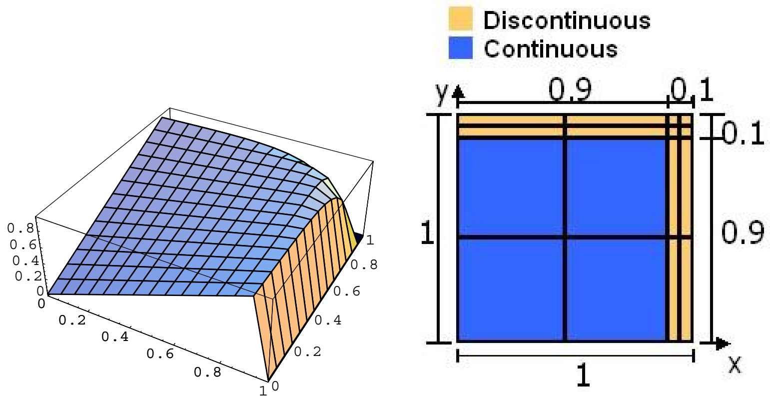

Figure 10: Exact solution of the boundary layer problem (left) and reference mesh (right)

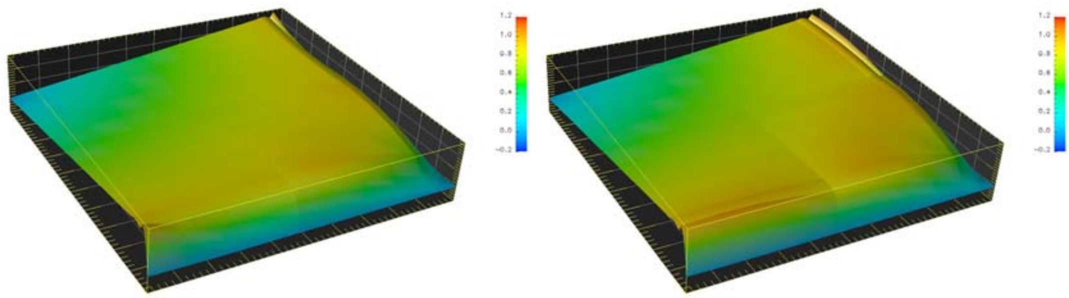

Figure 11: Continuous finite element solution with SUPG (left) and without SUPG (right) - $p = 2$, 16 elements - 81 equations

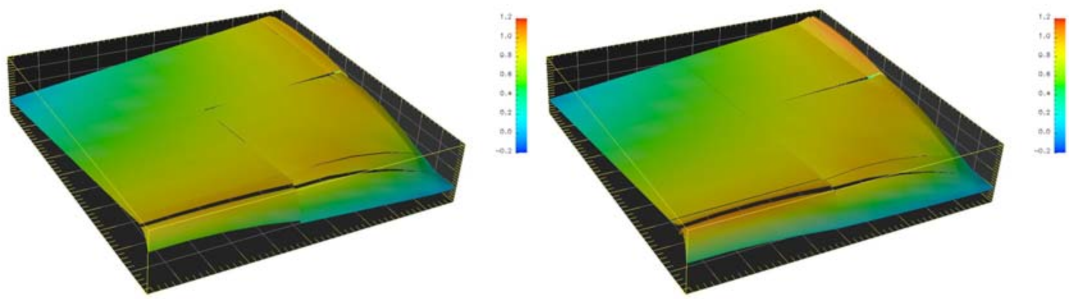

Figure 12: Discontinuous Galerkin solution with SUPG (left) and without SUPG (right) - $p = 2$, 16 elements - 144 equations

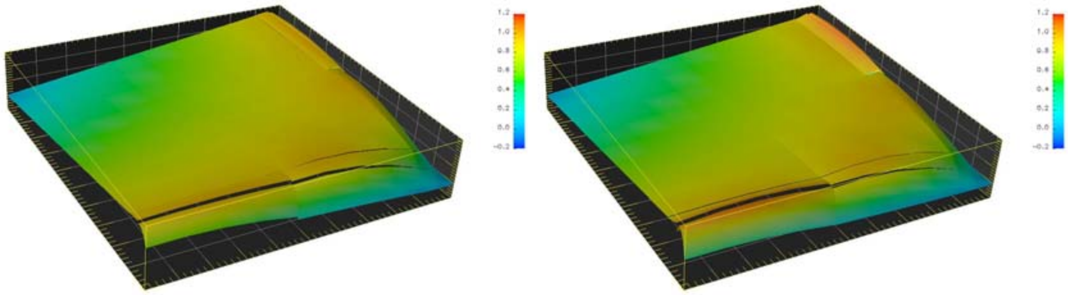

Figure 13: Combined continuous-discontinuous solution with SUPG (left) and without SUPG (right)- $p = 2$, 16 elements - 133 equations

used. From this first mesh, uniform refinements are performed to construct the other simulation meshes. Results are shown in figures 11, 12 and 13.

In the simulations where the SUPG term is employed the convergence rates are close to 1.0 for the three cases (FEM, DG, FEM/DGM). That convergence rate is small comparing to the rates obtained without SUPG. It may be occurred because the SUPG term was applied only in the convection operator. The SUPG stabilizes the solution but the weak formulation is not equivalent to the strong problem reducing the convergence rate of the simulation.

Figures 14, 15, 16, 17, 18 and 19 present the convergence rates for h refinement without the use of SUPG. The convergence rates lead to the expected values.

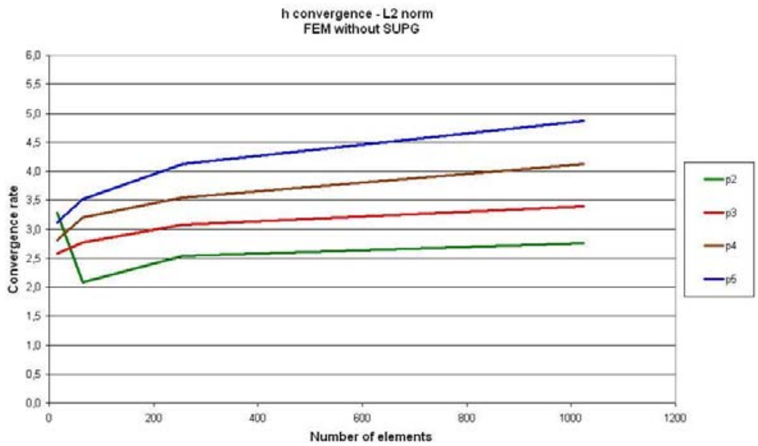

Figure 14: h convergence rates - $L2$ norm - FEM without SUPG

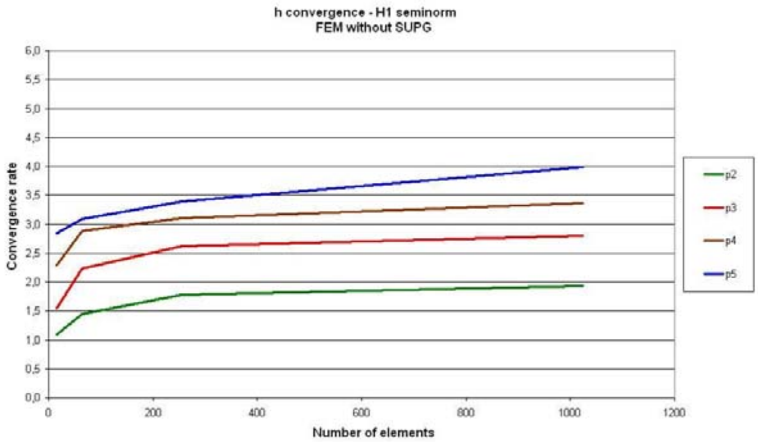

Figure 15: h convergence rates - $H1$ semi-norm - FEM without SUPG

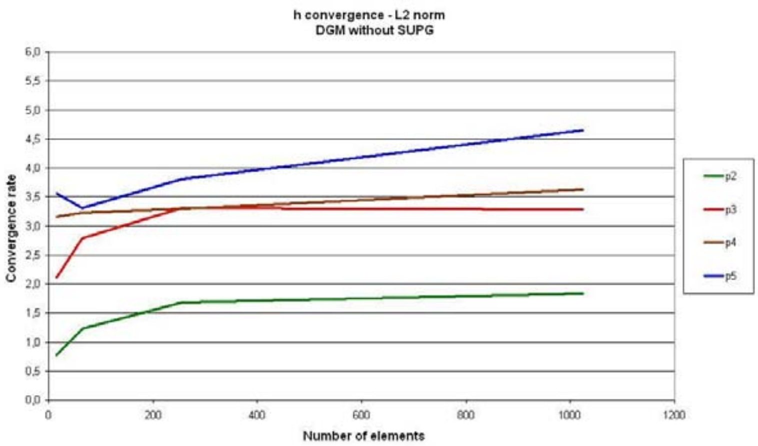

Figure 16: h convergence rates - $L2$ norm - DGM without SUPG

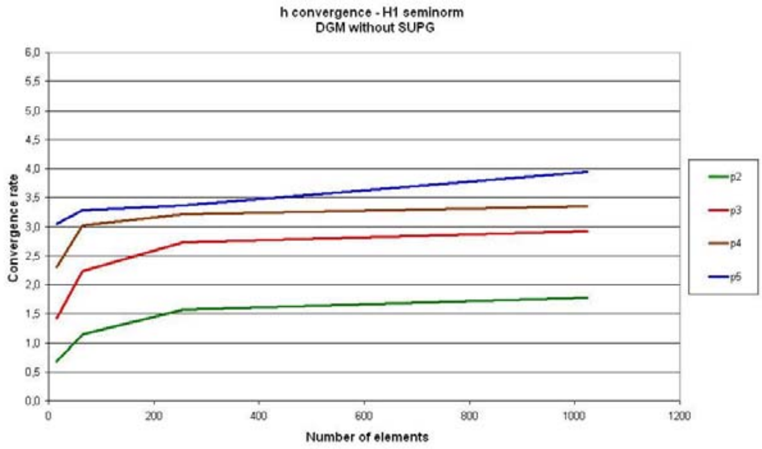

Figure 17: h convergence rates - $H1$ semi-norm - DGM without SUPG

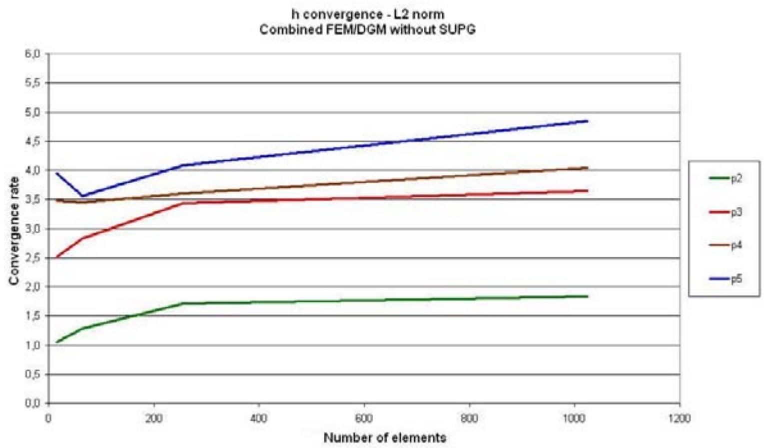

Figure 18: h convergence rates - $L2$ norm - Combined FEM/DGM without SUPG

The results indicate that the convergence rates are going to the analytical rates given in [2] and [16]. The continuous finite element method must achieve a $p + 1$ convergence rate in $L2$ norm and a p rate in $H1$ semi-norm.

The discontinuous Galerkin method (it is shown results with the Baumann formulation) must achieve a p convergence rate for even p (sub-optimal rate) and $p+1$ for odd p in $L2$ norm. In $H1$ semi-norm it must converge in p order.

The combined FEM/DGM presents the same convergence rates of the DGM. However the rates of the combined simulation are slightly better than the rates of the purely DGM simulation.

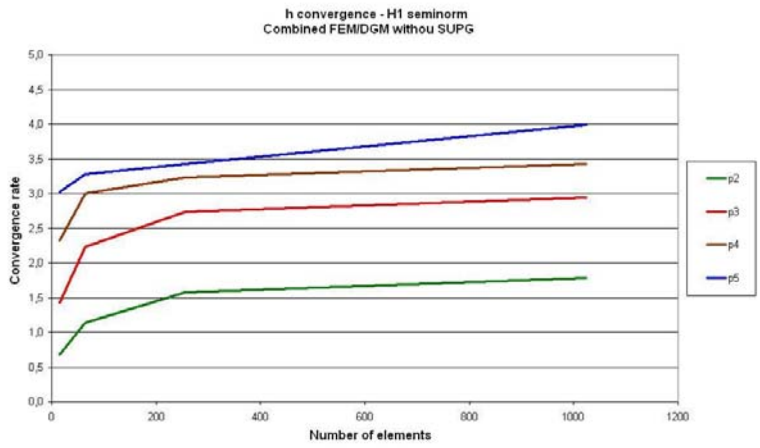

Figure 19: h convergence rates - $H1$ semi-norm - Combined FEM /DGM without SUPG

5 Conclusions

The discontinuous Galerkin method DGM is suitable for convection-diffusion problems. In the presented tests the method did not require artificial diffusion SUPG which is necessary in the continuous finite element method FEM. In the FEM the oscillations propagates in a great region of the domain. In the DGM, the oscillations are present only in the region of the discontinuity and do not propagate to the rest of the domain. When alining the interface of elements with the discontinuity no oscillation is observed with DGM.

However, one disadvantage of DGM is its higher computational cost. To overcome this drawback, it was adopted the FEM/DGM strategy: discontinuous elements in regions of high gradients and continuous elements in smoothness regions of the solution. With such approach, the stability advantage of the DGM is obtained with reduced number of degrees of freedom by means the use of continuous elements where stability is not a concern. The combined method FEM/DGM also showed to be consistent.

The implementation of the FEM/DGM method opens the path to the development of adaptive continuous/discontinous approximations, where strong gradients or shocks can be determined a-posteriori.

Acknowledgments. The authors gratefully acknowledge the financial support from the following Brazilian agencies: CNPq, Capes, Fapesp, Embraer and Cenapad-SP.

References

[1] I. Babuska, C. E. Baumann, and J. T. Oden. A discontinuous hp finite element method for diffusion problems: 1-d analysis. *Computer and Mathematics with Applications*, 37:103–122, 1999.

[2] Carlos Erik Baumann. *An HP-Adaptive Discontinuous Finite Element Method for Computational Fluid Dynamics*. PhD thesis, University of Texas at Austin, 1997.

[3] A. Brooks and T. Hughes. Streamline upwind/petrov-galerkin formulations for convection dominated flows with particular emphasis on the incompressible navier-stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 32, 1982.

[4] Jorge L. D. Calle, Philipe R. B. Devloo, and Snia M. Gomes. Stabilized discontinuous galerkin method for hyperbolic equations. *Computer Methods in Applied Mechanics and Engineering*, 194:1861–1874, 2005.

[5] Jorge Lizardo Daz Calle. *O Mtodo de Galerkin Descontnuo com Difusividade Implcita e H-Adaptabilidade Baseada em Tcnicas Wavelet*. PhD thesis, IMECC - Unicamp, 2002.

[6] E. G. D. Carmo and G. B. Alvarez. A new upwind function in stabilized finite element formulations, using linear and quadratic elements for scalar convection-diffusion problems. *Computer Methods in Applied Mechanics and Engineering*, 193:2383–2402, 2004.

[7] Bernardo Cockburn and Chi-Wang Shu. The local discontinuous galerkin method for time dependent convection-diffusion systems. *SIAM Journal Numerical Analysis*, 35:2440–2463, 1998.

[8] Clint Dawson and Jennifer Proft. Coupling of continuous and discontinuous galerkin methods for transport problems. *Computer Methods in Applied Mechanics and Engineering*, 191:3213–3231, 2002.

[9] P. R. B. Devloo, Tiago Forti, and Snia M. Gomes. Mtodos de elementos finitos contnuo e descontnuo combinados aplicado a problemas de conveco-difuso. In *XXVI CILAMCE*, 2005.

[10] P. R. B. Devloo, Snia M. Gomes, and Tiago Forti. Object oriented implementation of combined hp-adaptive continuous-discontinuous fem - abstract. In *8th USNCCM*, 2005.

[11] Tiago L. D. Forti. O mtodo de galerkin descontnuo aplicado a problemas de conveco difuso. Master's thesis, FEC-UNICAMP, 2005.

[12] Paul Houston, Christoph Schwab, and Endre Sli. Discontinuous hp-finite element methods for advection-diffusion-reaction problems. *SIAM J. Numer. Anal.*, 39:2133–2163, 2002.

[13] T. J. Hughes and A. Brooks. A theorical framework for petrov-galerkin methods with discontiuous weighting functions. applications to the streamline-upwind procedures. *Finite Elements in Fluids*, 4, 1982. ed. Gallagher,Wiley.

[14] Randall J. LeVeque. *Numerical Methods for Conservation Laws*. Birkhuser Verlag, 1990.

[15] J. Tinsley Oden, Ivo Babuska, and Carlos Erik Baumann. A discontinuous hp finite element method for diffusion problems. *Journal of Computational Physics*, 146:491–519, 1998.

[16] S. Prudhomme, F. Pascal, J. T. Oden, and A. Romkes. Review of a priori error estimation for discontinuous galerhin methods. Technical report, Texas Institute for Computational and Applied Mathematics, 2000.

[17] Erick Slis Raggio Santos. Desenvolvimento de mtodo implcito para simulador numrico tri-dimensional de escoamentos compressveis. Master's thesis, FEC - Unicamp, 2004.