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Modified Bolle – Reissner Theory of Plates Including Transverse Shear Deformations

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Abstract
In this work, new equations for first-order shear deformation plates are deduced taking into account the kinematic assumptions of the Bolle–Reissner theory but considering the equilibrium equations in the deformed configuration for the plate. The system of differential equations deduced is applicable to the calculation of the stresses in isotropic plates and is valid for thin and moderately thick plates. Analytical solutions are also presented in this work which are compared, when possible, with the ones obtained with other refined shear deformation plate theories.

Keywords
Moderately thick plates, analytical solutions, shear deformation.

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1 INTRODUCTION

The current trend in the study of plates can be deduced from the themes of the articles collected by Voyiadjis and Karamanlidis (1990) and Kienzler, Altenbach and Ott (2004). In the first publication it can be seen that out of the seven papers included on theoretical aspects of the analysis, four make a direct reference to moderately thick plates. The second one discusses common roots of different new plate and shell theories and reviews current state-of-the-art developments: higher-order shear deformation theories, zigzag theories, the global–local higher-order deformation theories and the layer-wise laminated plate theories are reviewed in this publication.

A survey of various shear deformation theories on plates can be found in the works of Reddy and Liu (1985), Touratier (1991), Soldatos (1992), Idibi, Karama and Touratier (1997), Karama, Afaq and Mistou (2003), Demasi (2004), Chaudhuri (2005, 2008), Aydogdu (2009), Brischetto, Carrera and Demasi (2009), Mantari, Oktem and Soares (2011) and Meiche et al. (2011). All of these theories are addressed in the context of higher-order shear deformation theories and usually with application to composite structures. In the same way, with respect to its wide dissemination, Zienkiewicz and Taylor's (2000a, 2000b) report can be taken as an example. It can be observed how, from the third edition (Zienkiewicz 1980) to the fifth (Zienkiewicz and Taylor 2000a, 2000b), the treatment of the calculation of plates has changed considerably, both with regard to the methodology and didactics of the flexure of thin plates and in the connection between thin and moderately thick formulations.

All this suggests interest in the subject and the need for synthesis in order to obtain theories which are plausible, are easily comprehensible and which do not involve an excessive degree of complexity to arrive at their solution.

If we focus on classical first-order shear deformation theories, up until now, it is evident that a different methodology was followed to tackle plate problems according to whether thin or moderately thick plates were being studied.

Since Kirchhoff (1850) presented his theory on thin plates summarized in the biharmonic equation, refined plate theories, including shear deformation, have been deduced.

Reissner (1945) introduced the shear deformation effect, proposing a correction of the biharmonic expression, obtaining a new equation valid for moderately thick plates,

$$D \cdot \Delta\Delta w = p - \frac{2 \cdot \mu}{10 \cdot (1 - \mu)} \cdot h^2 \cdot \Delta p \tag{1}$$

where w is the deflection, h is the thickness of the plate, D is the flexural rigidity of the plate and μ is a correction factor.

Simultaneously, Bolle (1947) and Mindlin (1951) presented similar equations for moderately thick plates under different assumptions. Whilst Bolle adopts a parabolic distribution of the transversal stresses through to the thickness, Mindlin assumes it to be constant.

In the first case (Bolle–Reissner theory), a contradiction is assumed to exist between the straightness of the normal element and the distribution presumed for tangential stresses through the thickness.

In 1957, Vlasov exposed the first consistent higher-order plate theory. He established a third-order displacement field that satisfies the stress-free boundary conditions on the top and bottom planes of a plate (Reddy 1990). He also proposed a greater correction, in which it is also assumed that the normal element of the plate bends in such a way that the shear in the plate thickness varies in accordance with parabolic law, obtaining expressions for the displacement u_i such as:

$$u_i = -z \cdot \theta_i - \frac{4 \cdot z^3}{3 \cdot h^2} \cdot \frac{\tau_{iz}^o}{G} \tag{2}$$

where τ_{iz}^o is the transversal stress in the middle surface and the nomenclature is that followed in the following sections. G is the shear modulus and θ_i are the rotations of the normal to the midplane about the x and y axes, θ_y and θ_x, respectively.

Donnell (1976) extends to plates the methodology previously used for beams and approaches the study of moderately thick plates using a series solution for the loading function.

These theories which can be denominated "first-order shear deformation theories", provide accurate solutions for plated structures with applications to problems in different fields of architecture and civil engineering. For really thick plates, higher-order shear terms should be taken into account.

Analytical solutions for Reissner and Mindlin plate equations have been studied in several works. A comparison of these solutions with the higher-order plate theory of Reddy was established in a book by Wang, Reddy and Lee (2000).

The equations proposed in this paper are corrections of the ones proposed by Reissner for the analysis of plates and provide accurate solutions for moderately thick plates and thin plates. Also, duly corrected with the inclusion of nonlinear terms, they permit the study of plates with large deflections (nonlinear calculations).

We also present these new equations disconnected in terms of displacements and moment sum (Marcus moment sum) which permit us to obtain analytical solutions for simply supported plates in static and dynamic analysis. These equations have the same structure formally as the ones proposed by Reismann (1980). Numerical treatment of these equations with the finite difference method is easy and straightforward and avoids complex techniques involving finite elements such as the ones proposed by Batoz and Lardeur (1989), Miehe (1998) or Parisch (1995) which correctly solve this problem.

2 INITIAL ASSUMPTIONS OF THE THEORY INCLUDING SHEAR DEFORMATION, STRAIN DISPLACEMENT EQUATIONS AND CONSTITUTIVE EQUATIONS

Firstly, we propose the hypothesis of the work as follows:

1) The loads, which are distributed, act on the mid-surface of the plate and will be perpendicular to the middle surface. The displacements of the points located in the middle surface are also sensibly perpendicular to the mentioned mid-surface (the middle surface being practically inelastic), although initially we suppose that $u_o(x,y)$ and $v_o(x,y)$, the displacements according to the x and y axes of the points located in the middle surface, are not zero (in order to obtain general equations, even for nonlinear calculations; see Figure 1).

Therefore, the displacements from one point of the plate are given by:

$$\begin{aligned} u &= u_o(x,y) + \theta_y \cdot z \\ v &= v_o(x,y) - \theta_x \cdot z \\ w &= w_o(x,y) \end{aligned} \tag{3}$$

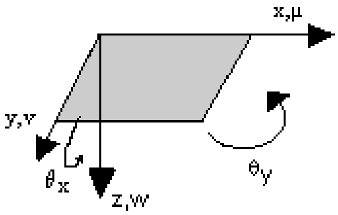

Figure 1: Sign convention. Rotations and displacements.

2) For the tangential stresses, a parabolic distribution throughout the thickness is assumed,

$$\tau_{xz} = \alpha \cdot G \cdot (1 - \frac{4 \cdot z^2}{h^2}) \cdot \gamma_{xz}, \qquad \tau_{yz} = \alpha \cdot G \cdot (1 - \frac{4 \cdot z^2}{h^2}) \cdot \gamma_{yz} \ ; \tag{4}$$

where $\alpha = \frac{5}{4}$.

3) The rotation ω_{xy} of a differential element around the z axis is null for all points of the plate.

$$\omega_{xy} = -\frac{1}{2} \cdot \left(\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x} \right) \cdot z = 0 \tag{5}$$

This important condition is introduced in this section and will be demonstrated in the next one. Based on these initial hypotheses we can deduce the expressions of the strains, rotations and stresses which are necessary for future developments.

The normal strains ε_z, ε_x, ε_y and shear strains γ_{xy}, γ_{yz}, γ_{xz} can be deduced from the displacements taking into account Eq. (3),

$$\varepsilon_z = 0 \quad , \quad \varepsilon_x = \varepsilon_{x_o} + z \cdot \frac{\partial \theta_y}{\partial x} \quad , \quad \varepsilon_y = \varepsilon_{y_o} - z \cdot \frac{\partial \theta_x}{\partial y} \tag{6}$$

$$\gamma_{xy} = \gamma_{xy_o} + z \cdot (\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}) \quad , \quad \gamma_{yz} = -\theta_x + \frac{\partial w}{\partial y} \quad , \quad \gamma_{xz} = -\theta_y + \frac{\partial w}{\partial x} \ . \tag{7}$$

ε_{xo}, ε_{yo}, γ_{xyo} are strains corresponding to points on the middle plane of the plate.

Similarly, rotations around the axis are,

$$\omega_{xz} = \frac{1}{2} \cdot \left(-\theta_y + \frac{\partial w}{\partial x} \right), \omega_{yz} = \frac{1}{2} \cdot \left(\theta_x + \frac{\partial w}{\partial y} \right), \omega_{xy} = -\frac{1}{2} \cdot \left(\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x} \right) \cdot z \tag{8}$$

Supposing elastic behaviour of the material, the stresses are given, according to Hooke's law, by

$$\sigma_x = \frac{E}{1-\mu^2} (\varepsilon_x + \mu \cdot \varepsilon_y) = \frac{E}{1-\mu^2} \left[\varepsilon_{xo} + \mu \cdot \varepsilon_{yo} + z \cdot (\frac{\partial \theta_y}{\partial x} - \mu \frac{\partial \theta_x}{\partial y}) \right] , \tag{9}$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\varepsilon_y + \mu\cdot\varepsilon_x\right] = \frac{E}{1-\mu^2}\left[\varepsilon_{yo} + \mu\cdot\varepsilon_{xo} + z\cdot\left(-\frac{\partial\theta_x}{\partial y} - \mu\frac{\partial\theta_y}{\partial x}\right)\right]$$

$$\tau_{xy} = G\cdot\left[\gamma_{xyo} + z\cdot\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right)\right], \tag{10}$$

$$\tau_{zx} = \alpha\cdot G\cdot\left(1-\frac{4z^2}{h^2}\right)\cdot\left(\theta_y + \frac{\partial w}{\partial x}\right)\ \tau_{yz} = \alpha\cdot G\cdot\left(1-\frac{4z^2}{h^2}\right)\cdot\left(-\theta_x + \frac{\partial w}{\partial y}\right).$$

where E is Young's modulus.

After integrating through the plate thickness, we obtain the axial and bending stresses,

$$N_x = \frac{E\cdot h}{1-\mu^2}(\varepsilon_{xo} + \mu\cdot\varepsilon_{yo})\ ;$$

$$N_y = \frac{E\cdot h}{1-\mu^2}(\varepsilon_{yo} + \mu\cdot\varepsilon_{xo})\ ; \tag{11}$$

$$N_{xy} = G\cdot h\cdot\gamma_{xyo}$$

$$M_x = D\cdot\left(\frac{\partial\theta_y}{\partial x} - \mu\frac{\partial\theta_x}{\partial y}\right)\ ;$$

$$M_y = D\cdot\left(-\frac{\partial\theta_x}{\partial y} + \mu\frac{\partial\theta_y}{\partial x}\right); \tag{12}$$

$$M_{xy} = -\frac{1-\mu}{2}D\cdot\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right).$$

Analogously, shear stresses are,

$$Q_{xz} = \frac{5\cdot E\cdot h}{12\cdot\ 1+\mu}\cdot\left(\frac{\partial w}{\partial x} + \theta_y\right), Q_{yz} = \frac{5\cdot E\cdot h}{12\cdot\ 1+\mu}\cdot\left(\frac{\partial w}{\partial y} - \theta_x\right) \tag{13}$$

Finally, in the classical theory of plates, it is usual to define equivalent transverse shear forces V_x, V_y given by,

$$V_x = Q_x - \frac{\partial M_{xy}}{\partial y}, V_y = Q_y - \frac{\partial M_{xy}}{\partial x} \tag{14}$$

which are useful in defining the boundary conditions of the problem and the effective shear forces acting on the elastic plate.

Up until now, only the definition of the strains and stresses has been taken into account for a first-order shear deformation plate theory (FSDT) without using the third assumption of the work. Establishing the classical equilibrium equations in the undeformed configuration of the plate, we would obtain the classical equations of a FSDT. Notwithstanding this, incorporating this new assumption (hypothesis 3), important conclusions will be derived and new consistent plate equations will be obtained.

3 MODIFIED BOLLE–REISSNER EQUATIONS: EQUILIBRIUM EQUATIONS OF THE PLATE ELEMENT

It is well known that the equations of equilibrium for a plate element can be obtained, in accordance with the theory of elasticity, by simply integrating, throughout the thickness of the plate, the internal equilibrium equations of elasticity (Reissman 1980). In this manner an identical system is derived to that obtained by directly establishing the equilibrium of the plate element in the non-deformed geometry of the latter.

We can recall these equations,

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz} = 0\,; \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_{yz} = 0 \tag{15}$$

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + q = 0 \tag{16}$$

and the mechanical boundary conditions,

$$N_{x\nu} = \overline{N}_{x\nu}; N_{y\nu} = \overline{N}_{y\nu}; V_z + \frac{\partial M_{\nu s}}{\partial s} = \overline{V}_z + \frac{\partial \overline{M}_{\nu s}}{\partial s}; M_\nu = \overline{M}_\nu \tag{17}$$

with

$$\begin{aligned} M_\nu &= M_x l^2 + 2M_{xy} l\ m + M_y m^2; V_z = Q_x l + Q_y m \\ M_{x\nu} &= M_x l + M_{xy} m; M_{y\nu} = M_{xy} l + M_y m \\ M_{\nu s} &= -M_{x\nu} m + M_{y\nu} l \end{aligned} \tag{18}$$

where the symbol $(\,\bar{}\,)$ indicates the prescribed stresses along the boundary. $M_{\nu s}$ and $\overline{M}_{\nu s}$ are the twisting moments, M_ν and $\overline{M}_\nu$ are the bending moments and V_z is the shearing force acting in the direction of the z axis. (l, m) are the direction cosines of the normal ν drawn outwardly on the boundary; $l = \cos(x,\nu)$ and $m = \cos(y,\nu)$. The coordinate s is taken along the boundary C, such that ν, s and z form a right-handed system.

In order to obtain the relationship which permits us to simplify and generalize the Bolle–Reissner equations, we need to establish the equilibrium equations for a deformed geometry (bent plate).

This methodology is used by Rekach (1978) and has been proposed by Bazant (2003) in several works, in order to obtain accurate solutions to structural problems. This procedure is emphasized in the study of plates and shells.

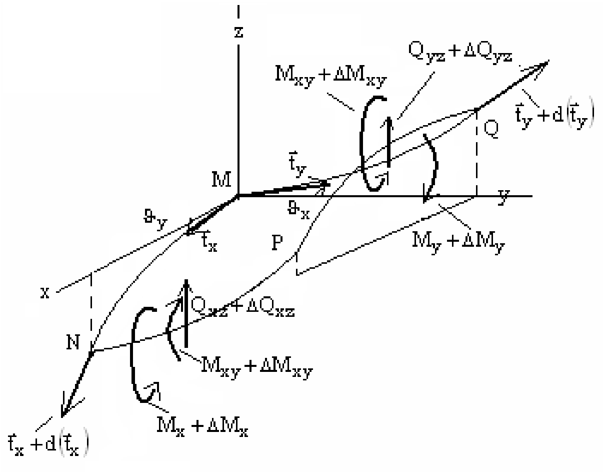

Figure 2: Equilibrium of a plate element in a deformed configuration: bending moments and shear forces.

The Frenet formulae permit to us to write (Nayak 2012)

$$\frac{\partial \vec{t}_x}{\partial x} = -\frac{\partial \theta_y}{\partial x}\cdot \vec{k}\,;\frac{\partial \vec{t}_x}{\partial x} = \frac{\partial \theta_x}{\partial x}\cdot \vec{k}\,;\frac{\partial \vec{t}_x}{\partial y} = -\frac{\partial \theta_y}{\partial y}\cdot \vec{k}\,;\frac{\partial \vec{t}_y}{\partial y} = -\frac{\partial \theta_x}{\partial y}\cdot \vec{k} \tag{19}$$

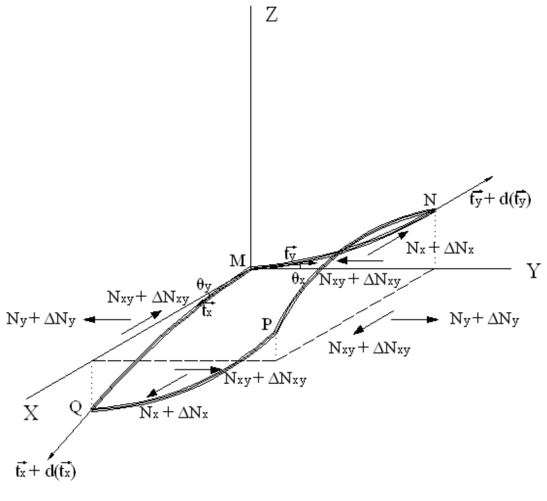

Figure 3: Equilibrium of a plate element in a deformed configuration: membrane forces.

Let Figures 2 and 3 represent a plate element subjected to stresses in which we have omitted the operators on the unseen sides for greater clarity of the figure, and on the other side the real geometric configuration which is produced by the equilibrium that the element adopts once the plate is bent.

To obtain the necessary equations it is only necessary to consider the equations of static equilibrium in the element

$$\sum \vec{F} = \vec{0} = \sum \vec{F}_{\text{ext}} + \sum \vec{F}_{QM} + \sum \vec{F}_{MN} + \sum \vec{F}_{PN} + \sum \vec{F}_{PQ} \tag{20}$$

$$\sum \vec{F}_{QM} = (-N_x \cdot \vec{t}_x - N_{xy} \cdot \vec{t}_y - Q_{xz} \cdot \vec{k}) \tag{21}$$

$$\sum \vec{F}_{MN} = (-N_y \cdot \vec{t}_y - N_{xy} \cdot \vec{t}_x - Q_{yz} \cdot \vec{k}) \tag{22}$$

$$\sum \vec{F}_{PM} = \left[N_x \cdot \vec{t}_x + d(N_x \cdot \vec{t}_x) + N_{xy} \cdot \vec{t}_y + d(N_{xy} \cdot \vec{t}_y) + Q_{xz} \cdot \vec{k} + d(Q_{xz} \cdot \vec{k}) \right] \cdot ds_1 \tag{23}$$

$$\sum \vec{F}_{PQ} = \left[N_y \cdot \vec{t}_y + d(N_y \cdot \vec{t}_y) + N_{xy} \cdot \vec{t}_x + d(N_{xy} \cdot \vec{t}_x) + Q_{yz} \cdot \vec{k} + d(Q_{yz} \cdot \vec{k}) \right] \cdot ds_2 \tag{24}$$

bearing in mind that we proceed from the QM edge to the PN edge, varying according to the x axis (with $y =$ constant) so that, for example

$$d(N_x \cdot \vec{t}_x) = \frac{\partial N_x}{\partial x} \cdot dx \cdot \vec{t}_x + \frac{\partial \vec{t}_x}{\partial x} \cdot dx \cdot N_x \tag{25}$$

and from the MN edge to the QP with $x = constant$, we obtain,

$$d(N_{xy} \cdot \vec{t}_x) = \frac{\partial N_{xy}}{\partial y} \cdot dy \cdot \vec{t}_x + N_{xy} \cdot \frac{\partial \vec{t}_x}{\partial y} \cdot dy \tag{26}$$

where $\mathrm{ds}_1 \approx \mathrm{dx}, \mathrm{ds}_2 \approx \mathrm{dy}$, and for the values of those derived from unit vectors $\vec{t}_x$ and $\vec{t}_y$ we find that the vector equation yields

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0; \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{27}$$

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + q - N_x \cdot \frac{\partial \theta_y}{\partial x} + N_y \cdot \frac{\partial \theta_x}{\partial y} + N_{xy} \cdot \left(\frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y}\right) = 0 \tag{28}$$

The mechanical boundary conditions associated with these equilibrium equations are,

$$\begin{aligned} &N_{x\nu} = \overline{N}_{x\nu}; N_{y\nu} = \overline{N}_{y\nu}; \\ &Q_x l + Q_y m + N_{x\nu} \cdot \theta_y + N_{y\nu} \cdot \theta_x + \frac{\partial M_{\nu s}}{\partial s} = \overline{V}_z + \frac{\partial \overline{M}_{\nu s}}{\partial s}; \\ &M_\nu = \overline{M}_\nu, \end{aligned} \tag{29}$$

which are different from Eqs. (17) and (18) due to the in-plane stress resultants N_x, N_y, N_{xy} which have contributions to the equation of equilibrium in the direction of the z axis due to the inclination of the middle surface.

Proceeding in the same manner, the vector equation of moment equilibrium is,

$$\sum \overrightarrow{M} = \sum \overrightarrow{M}_{QM} + \sum \overrightarrow{M}_{MN} + \sum \overrightarrow{M}_{PN} + \sum \overrightarrow{M}_{PQ} - Q_{xz} \cdot dx \cdot dy \cdot \vec{t}_y + Q_{yz} \cdot dx \cdot dy \cdot \vec{t}_x + +N_{xy} \cdot dx \cdot dy \cdot \vec{k} - \\ -N_{xy} \cdot dx \cdot dy \cdot \vec{k} = \vec{0} \tag{30}$$

and after substitution

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz} = 0\,; \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_{yz} = 0 \tag{31}$$

$$M_x \cdot \frac{\partial \theta_x}{\partial x} + M_y \cdot \frac{\partial \theta_y}{\partial y} + M_{xy} \cdot (\frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y}) = 0 \tag{32}$$

However, if in the sixth equation we substitute M_x, M_y and M_{xy} for their values as shown in the previous section, it is transformed into

$$\frac{D \cdot (1+\mu)}{2} \cdot (\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x}) \cdot (\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}) = 0 \tag{33}$$

but if we note that

$$\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} = \frac{1}{D \cdot (1+\mu)} \cdot (M_x + M_y) \tag{34}$$

cannot be zero for any point on the plate, we conclude that

$$\frac{\partial \theta_y}{\partial y} = -\frac{\partial \theta_x}{\partial x} \tag{35}$$

which implies that the rotation ω_{xy} of a differential element around the z axis is null for all points of the plate,

$$\omega_{xy} = -\frac{1}{2} \cdot \left(\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x} \right) \cdot z = 0, \tag{36}$$

and demonstrates the inapplicability of the second Bolle equation.

It must be emphasized that this important condition is verified identically for thin plates. If we recall the expressions of the rotations around the axes for Kirchhoff plate theory,

$$\theta_x = \frac{\partial w}{\partial y}, \theta_y = -\frac{\partial w}{\partial x} \tag{37}$$

and since,

$$\frac{\partial \theta_x}{\partial x} = \frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial \theta_y}{\partial x}, \tag{38}$$

the magnitude $\dfrac{M_x + M_y}{1+\mu}$ in Eq. (34) , is called the moment sum or Marcus moment.

After substituting the values of shear stresses and Eq. (36) into the equilibrium equations, and omitting the subscript indicative of the displacements corresponding to points on the middle surface, we find the following system of differential equations,

$$u''_{x^2} + \frac{1+\mu}{2} \cdot v''_{xy} + \frac{1-\mu}{2} \cdot u''_{y^2} = -(w'_x \cdot w''_{x^2} + \mu \cdot w'_y \cdot w''_{xy}) - \frac{1-\mu}{2} \cdot (w'_y \cdot w''_{xy} + w'_x \cdot w''_{y^2}), \tag{39}$$

$$\frac{1-\mu}{2} \cdot v''_{x^2} + \frac{1+\mu}{2} u''_{xy} + v''_{y^2} = -(w'_y \cdot w''_{y^2} + \mu \cdot w'_x \cdot w''_{xy}) - \frac{1-\mu}{2} \cdot (w'_y w''_{x^2} + w'_x w''_{xy}), \tag{40}$$

$$\Delta w - \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot \left[-q + N_x \cdot \frac{\partial \theta_y}{\partial x} - N_y \cdot \frac{\partial \theta_x}{\partial y} + N_{xy} \cdot (\frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y}) \right], \tag{41}$$

$$\Delta \theta_y = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_y + \frac{\partial w}{\partial x}) ; \quad \Delta \theta_x = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_x - \frac{\partial w}{\partial y}) . \tag{42}$$

These equations are quite different from the ones deduced considering the equilibrium in the non-deformed configuration and without considering Eq. (36).

For a given function $w(x,y)$, Eqs. (39) and (40) are linear with respect to the unknown functions $u(x,y)$ and $v(x,y)$ and can be solved exactly or approximately while taking into account the boundary conditions.

For linear plates, these expressions are,

$$(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x}) - \Delta w = \frac{6}{5} \cdot \frac{p}{G \cdot h} , \tag{43}$$

$$\Delta \theta_y - \frac{1+\mu}{2} \cdot \frac{\partial}{\partial y} (\frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y}) = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_y + \frac{\partial w}{\partial x}) , \tag{44}$$

$$\Delta \theta_x - \frac{1+\mu}{2} \cdot \frac{\partial}{\partial x} (\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x}) = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_x + \frac{\partial w}{\partial y}) , \tag{45}$$

which are the modified Bolle–Reissner equations.

4 MODIFIED BOLLE–REISSNER EQUATIONS: DECOUPLED SYSTEM DISPLACEMENTS ROTATIONS

For linear analysis, Eqs. (39) and (40) are automatically satisfied with $u = v = 0$ and the other three leave us with:

$$\Delta w - \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = -\frac{12 \cdot (1+\mu) \cdot p}{5 \cdot E \cdot h}, \tag{46}$$

$$\Delta \theta_y = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_y + \frac{\partial w}{\partial x}), \tag{47}$$

$$\Delta \theta_x = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_x - \frac{\partial w}{\partial y}). \tag{48}$$

In order to obtain a consistent system of differential equations for plate analysis, we derive the first equation from Eqs. (43)–(45) and bearing in mind Eq. (36), we obtain

$$\frac{\partial^2}{\partial x^2} \Delta w + \frac{\partial}{\partial x} \Delta \theta_y = -12 \cdot \frac{(1+\mu)}{5 \cdot E \cdot h} \cdot \frac{\partial^2 p}{\partial x^2}, \tag{49}$$

$$\frac{\partial^2}{\partial y^2} \Delta w + \frac{\partial}{\partial y} \Delta \theta x = -12 \cdot \frac{(1+\mu)}{5 \cdot E \cdot h} \cdot \frac{\partial^2 p}{\partial y^2}. \tag{50}$$

which after addition results in

$$\Delta\Delta w + \frac{\partial}{\partial x} \Delta \theta_y - \frac{\partial}{\partial y} \Delta \theta_x = -12 \cdot \frac{(1+\mu)}{5 \cdot E \cdot h} \cdot \Delta p \tag{51}$$

Whilst from Eqs. (43) and (44) we find

$$\frac{\partial}{\partial x} \Delta \theta_y - \frac{\partial}{\partial y} \Delta \theta_x = \frac{5 \cdot (1-\mu)}{h^2} \cdot (\frac{\partial \theta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta_x}{\partial y} + \frac{\partial^2 w}{\partial y^2}) = \frac{5 \cdot (1-\mu)}{h^2} \cdot \frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot (-p) \tag{52}$$

and substituting into the previous equation

$$\Delta\Delta w = \frac{p}{D} - \frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot \Delta p \tag{53}$$

so that the equation system for the linear case turns out to be

$$\Delta\Delta w = \frac{p}{D} - \frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot \Delta p \tag{54}$$

$$\Delta \theta_x - \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_x - \frac{\partial w}{\partial y}) = 0 \tag{55}$$

$$\Delta \theta_y - \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_y + \frac{\partial w}{\partial x}) = 0 \tag{56}$$

for the case of geometrical non-linearity (large displacements) we obtain

$$u''_{x^2} + \frac{1+\mu}{2} \cdot v''_{xy} + \frac{1-\mu}{2} \cdot u''_{y^2} = -(w'_x \cdot w''_{x^2} + \mu \cdot w'_y \cdot w''_{xy}) - \frac{1-\mu}{2} \cdot (w'_y \cdot w''_{xy} + w'_x \cdot w''_{y^2}) \tag{57}$$

$$\frac{1-\mu}{2} \cdot v''_{x^2} + \frac{1+\mu}{2} \cdot u''_{xy} + v''_{y^2} = -(w'_y \cdot w''_{y^2} + \mu \cdot w'_x \cdot w''_{xy}) - \frac{1-\mu}{2} \cdot (w'_y \cdot w''_{x^2} + w'_x \cdot w''_{xy}) \tag{58}$$

$$\Delta\Delta w = -\frac{1}{D} \cdot \left[-p + N_x \cdot \frac{\partial \theta_y}{\partial x} - N_y \cdot \frac{\partial \theta_x}{\partial y} + 2 \cdot N_{xy} \cdot \frac{\partial \theta_x}{\partial x} \right] - \frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot \Delta p \tag{59}$$

$$\Delta\theta_x - \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_x - \frac{\partial w}{\partial y}) = 0 \tag{60}$$

$$\Delta\theta_y - \frac{5 \cdot (1-\mu)}{h^2} \cdot (\theta_y + \frac{\partial w}{\partial x}) = 0 \tag{61}$$

In order to obtain the governing equations disconnected in displacements and moment sum and taking into account Eq. (36), we can write Eqs. (59)– (61) as

$$\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} + \Delta w = \frac{M}{D} + \Delta w = -\frac{12 \cdot 1+\mu}{5 \cdot E \cdot h} \cdot P \tag{62}$$

$$\frac{\partial}{\partial x} \Delta\vartheta_y - \frac{\partial}{\partial y} \Delta\vartheta_x = \Delta\left(\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} \right) = \frac{\Delta M}{D} = -\frac{P}{D} \tag{63}$$

and it yields,

$$\begin{aligned} \Delta M &= -P \\ \Delta w &= -\frac{M}{D} - \frac{6}{5 \cdot G \cdot h} \cdot P \end{aligned} \tag{64}$$

These equations have the same structure as the ones proposed by Reissman (1980) and offer a wide range of possibilities of obtaining analytical solutions for simply supported plates as will be seen in the following examples.

Otherwise, a Fourier series solution is always possible, taking into account the appropriate boundary conditions.

5 ANALYTICAL ELASTIC SOLUTIONS BASED ON FOURIER SERIES

In order to obtain analytical solutions to the system of differential equations (49)–(51), Fourier series solutions are proposed. In order to compare the solutions obtained with other methods and theories, we have chosen simply supported plates and completely clamped plates subjected to different loads.

Example 1: a simply supported isotropic rectangular plate of dimensions a by b subjected to the transverse load $p = p(x,y)$ on surface z = −h/2 acting in the upward z direction as given below

$$p = p_o \cdot \text{sen}\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b} \tag{65}$$

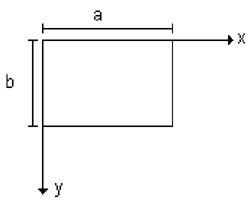

Figure 4: Coordinate system. Plate dimensions.

The boundary conditions are satisfied if the displacements, w, are expressed by

$$w = c \cdot \text{sen}\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b} \tag{66}$$

where a and b are defined according to Figure 4. After derivation and substitution in Eq. (49), we find

$$c = \frac{p_o}{D \cdot \pi^4 \cdot (\frac{1}{a^2} + \frac{1}{b^2})} \cdot \left[\frac{1}{(1/a^2 + 1/b^2)} + \frac{\pi^2 h^2}{5 \cdot (1-\mu)} \right] \tag{67}$$

where the second summation is of little relative importance compared to the first when we deal with a thin plate. Hence, it constitutes a correction to the solution obtained for this type of plate and it is clear that it acquires some relevance as thickness h increases.

Substituting this coefficient in the displacement field of Eq. (53), we obtain,

$$w = \frac{p_o a^4}{D \cdot \pi^4 \cdot (\frac{1}{a^2} + \frac{1}{b^2})^2} \cdot \left[1 + \frac{\pi^2 h^2 (\frac{1}{a^2} + \frac{1}{b^2})}{5 \cdot (1-\mu) a^2} \right] \cdot \text{sen}\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b} \tag{68}$$

In the same way, if seeking to satisfy for

$$M_y\Big]_{x=0} = M_y\Big]_{x=a} = (M_x)_{y=0} = (M_x)_{y=b} = 0 \tag{69}$$

for θ_x and θ_y we adopt

$$\theta_x = c_1 \cdot \text{sen}\frac{\pi x}{a} \cdot \cos\frac{\pi y}{b}, \qquad \theta_y = c_2 \cdot \cos\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b} \tag{70}$$

and substituting in Eqs. (50) and (51) we determine c1 and c2 and thus, after substitution in the general expressions, we obtain:

$$M_x = -\frac{p_o a^2 (1 + \upsilon \frac{a^2}{b^2})}{\pi^2 \cdot (1+\frac{a^2}{b^2})^2} \cdot \text{sen}\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b};$$

$$M_y = -\frac{p_o b^2 (1 + \upsilon \frac{a^2}{b^2})}{\pi^2 \cdot (1+\frac{a^2}{b^2})^2} \cdot \text{sen}\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b} \quad (71)$$

Shear stresses are obtained in a similar way,

$$Q_x = \frac{p_o a}{\pi \cdot (1+\frac{a^2}{b^2})} \cdot \cos\frac{\pi x}{a} \cdot \text{sen}\frac{\pi y}{b};$$

$$Q_y = \frac{p_o b}{\pi \cdot (1+\frac{a^2}{b^2})} \cdot \text{sen}\frac{\pi x}{a} \cdot \cos\frac{\pi y}{b} \quad (72)$$

which coincides exactly with Reissner equations.

Notwithstanding this, if we analyze maximum horizontal bending stress, for $a = b = 3h$ and $\upsilon = 0,3$, $z = h/2$,

$$\sigma_x\left(x,y,-\frac{h}{2}\right) = -1,99 p_o \cdot sen\frac{\pi x}{a} \cdot sen\frac{\pi y}{b}; \quad (73)$$

which improves the result obtained with the Reissner equations and reduces the error with respect to the solution with the theory of elasticity,

$$\sigma_x\left(x,y,-\frac{h}{2}\right) = -2,12 p_o \cdot sen\frac{\pi x}{a} \cdot sen\frac{\pi y}{b}. \quad (74)$$

To conclude this example, identical results are obtained considering the general solutions of the system of differential equations expressing Fourier series for the deflections and the rotations.

If we consider that the rotations and deflections can be expressed in the form,

$$w = \sum w_n(y) \cdot \text{sen}\frac{m \cdot \pi \cdot x}{a}; \quad \theta_y = \sum T_y(y) \cdot sen\frac{m \cdot \pi \cdot x}{a}; \quad \theta_x = \sum T_x(y) \cdot \cos\frac{m \cdot \pi \cdot x}{a} \quad (75)$$

then we can express the system in Eqs. (54)-(56) as,

$$-(\frac{m\cdot\pi}{a})T_x(y)+T'_y(y)+(\frac{m\cdot\pi}{a})^2 w_n(y)-w''_n(y)=\frac{6}{5\cdot G\cdot h}q_n(y) \quad (76)$$

$$-\left[(\frac{m\cdot\pi}{a})^2+\frac{5\cdot(1-\mu)}{h^2}\right]T_x(y)+\frac{1-\mu}{2}T''_x(y)-\frac{5\cdot(1-\mu)}{h^2}(\frac{m\cdot\pi}{a})w_n(y)=0 \quad (77)$$

$$-\frac{1-\mu}{2}\left[(\frac{m\cdot\pi}{a})^2+\frac{10}{h^2}\right]T_y(y)+T''_x(y)-\frac{5\cdot(1-\mu)}{h^2}w'_n(y)=0 \quad (78)$$

whose complicated analytical solution imposing the appropriate boundary conditions for simply supported plates and considering $q\ (x,y)=q_0$ is,

$$\begin{aligned} w=&\frac{P}{24D}\ x^4-2ax^3+a^3x\ +\frac{Pa^4}{D}\left(\sum_{m=1,3,5..}A_m ch\frac{m\pi y}{a}+\frac{m\pi y}{a}sh\frac{m\pi y}{a}\ sen\frac{m\pi x}{a}\right)-\\ &-\frac{12\ 1+\mu\ Pa^2}{5Et}\left(\frac{x^2-ax}{2}+\sum 2B_m\ m\pi\ ^2 ch\frac{m\pi y}{a}sen\frac{m\pi x}{a}\right)\end{aligned} \quad (79)$$

with

$$A_m=-\frac{2\ \alpha_m th\alpha_m+2}{m^5\pi^5 ch\alpha_m},B_m=\frac{2}{m^5\pi^5 ch\alpha_m},\alpha_m=\frac{m\pi b}{2a} \quad (80)$$

which can be reduced to Eq. (68) rearranging the terms of these last equations.

Following the methodology known for thin plates, the solution to the case of a rectangular plate with clamped edges is obtained by superimposing the previous problem, of the supported plate, to the cases of supported plates only subjected to bending moments at the edges and taking into account that the rotations on the edges of the clamped plate are zero.

Example 2: free vibration of a simple supported isotropic rectangular plate. Applying the principle of d'Alembert to establish dynamic equilibrium equations for the study of transverse oscillations of plates, we need only consider in the equilibrium equation (49) the inertia forces rather than static loads P.

The problem to solve is,

$$\Delta\Delta\overline{w}=-\frac{\gamma\cdot h}{D}\cdot\ddot{w}-\frac{12\cdot(1+\mu)\gamma}{5\cdot E\cdot h}\cdot\Delta\ddot{w}\,; \quad (81)$$

expressing the deflection as,

$$w=\ C_1\cdot\cos\ f\cdot\tau\ +C_2\cdot sen\ f\cdot\tau\ \cdot U(x,y)\,; \quad (82)$$

where f is the frequency.

Substituting,

$$\Delta\Delta U = \frac{\gamma \cdot t}{D} \cdot f^2 \cdot U + \frac{12 \cdot (1+\mu)\gamma}{5 \cdot E \cdot h} \cdot f^2 \cdot \Delta U \; ; \tag{83}$$

boundary conditions are satisfied if we take the solution in the form,

$$U_{mn} = sen\,\frac{m\pi x}{a} \cdot sen\,\frac{n\pi y}{b} \; ; \tag{84}$$

substituting again,

$$\left(\frac{m\pi}{a}\right)^4 + 2\cdot\left(\frac{m \cdot n \cdot \pi^2}{a \cdot b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 - \frac{\gamma \cdot t}{D} \cdot f^2{}_{mn} + \frac{12 \cdot (1+\mu)\gamma}{5 \cdot E \cdot h} \cdot f^2{}_{mn} \cdot \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] = 0 \; ; \tag{85}$$

and solving,

$$f_{mn} = \frac{\pi^2}{b^2} \cdot \left[n^2 + \left(m \cdot \frac{b}{a}\right)^2\right] \cdot \sqrt{\frac{E \cdot t^2}{1+\mu \cdot \gamma}} \cdot \frac{1}{\sqrt{24 \cdot 1 - \mu \; - 1.2 \cdot \frac{t^2 \cdot \pi^2}{b^2} \cdot \left[\left(m \frac{b}{a}\right)^2 + n^2\right]}} \; . \tag{86}$$

The term $\frac{\pi^2}{b^2} \cdot \left[n^2 + \left(m \cdot \frac{b}{a}\right)^2\right]$ is well known and corresponds to the one obtained in the work of Leissa (1973).

In order to compare the results for a square plate (Mindlin 1951), these results refer to the non-dimensional frequency parameter λ given by,

$$\lambda = f a^2 \sqrt{\frac{\rho h}{D}} \; . \tag{87}$$

Results are shown for different ratios thickness/length (0.01, 0.1 and 0.2) in Table 1 for the first and fourth modes of vibration.

Mode	h/a = 0.01	h/a = 0.1	h/a = 0.2
1	19.734	19.067	17.450
4	78.848	69.794	55.158
Sol.1 (Mindlin)	19.734	19.067	17.450
Sol.4 (Mindlin)	78.849	69.794	55.159

Table 1: Parameter λ of a simply supported square plate.

Excellent agreement with the theory, for simply supported plates, has been found.

Example 3: a simply-supported rectangular plate with bending moments at the edges (Figure 5). In order to solve the problem we take

$$w = \sum w_n(y) \cdot \text{sen}\,\frac{m \cdot \pi \cdot x}{a} \; ; \quad \theta_y = \sum T_y(y) \cdot \cos\frac{m \cdot \pi \cdot x}{a} \; ; \quad \theta_x = \sum T_x(y) \cdot \text{sen}\,\frac{m \cdot \pi \cdot x}{a} \tag{88}$$

with which $w = 0$ and $M_x = 0$ are satisfied, for $x = 0$ and $x = a$.

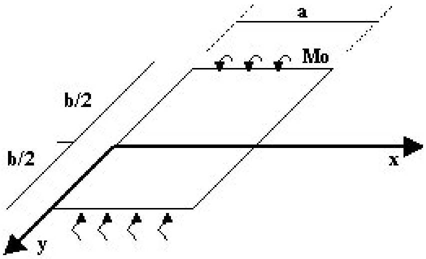

Figure 5: Bending of a rectangular plate by moments distributed along the pairs of opposite edges ($y = \pm b/2$).

Substituting in Eqs. (46)–(48), and taking into account that $w = 0$ for $y = \pm b/2$ and that w should be symmetrical in y for all of x, we obtain:

$$w = \sum C_{3m}\left[y \cdot sh\frac{m\cdot\pi\cdot y}{a} - \frac{b}{2}\cdot th\frac{m\cdot\pi\cdot b}{a} - ch\frac{m\cdot\pi\cdot y}{a}\right]\cdot \text{sen}\frac{m\cdot\pi\cdot x}{a} \tag{89}$$

To find θ_y and θ_x , we substitute in (51)and take into account (36) . The solution must also satisfy the conditions $\theta_y = 0$ for $x = \pm a/2$; $\theta_x = 0$ for $y = 0$, thus obtaining:

$$\theta_y = \sum\left\{2\cdot c_{4m}\cdot ch(\tau\cdot y) + \frac{m\cdot\pi}{a}\cdot c_{3m}\cdot\left[\frac{b}{2}\cdot th\frac{m\cdot\pi\cdot b}{2\cdot a}\cdot ch\frac{m\cdot\pi\cdot y}{a} - y\cdot sh\frac{m\cdot\pi\cdot y}{a} - \right.\right.$$
$$\left.\left. -2\cdot\frac{m\cdot\pi}{a}\cdot\frac{h^2}{5\cdot(1-\mu)}\cdot ch\frac{m\cdot\pi\cdot y}{a}\right]\right\}\cdot\cos\frac{m\cdot\pi\cdot x}{a} \tag{90}$$

$$\theta_x = -\sum\left\{2\cdot\tau\cdot c_{4m}\cdot sh(\tau\cdot y) + \frac{m\cdot\pi}{a}\cdot c_{3m}\cdot\left[\frac{b}{2}\cdot th\frac{m\cdot\pi\cdot b}{2\cdot a}\cdot sh\frac{m\cdot\pi\cdot y}{a}\cdot\frac{m\cdot\pi}{a} - sh\frac{m\cdot\pi\cdot y}{a} - \right.\right.$$
$$\left.\left. -y\cdot ch\frac{m\cdot\pi\cdot y}{a}\cdot\frac{m\cdot\pi}{a} - 2\cdot\frac{m^2\cdot\pi^2}{a^2}\cdot\frac{h^2}{5\cdot(1-\mu)}\cdot sh\frac{m\cdot\pi\cdot y}{a}\right]\right\}\cdot\frac{a}{m\cdot\pi}\cdot\text{sen}\frac{m\cdot\pi\cdot x}{a} \tag{91}$$

with

$$\tau = \sqrt{\frac{m^2\cdot\pi^2}{a^2} + \frac{5\cdot(1-\mu)}{h^2}} \tag{92}$$

Now it can be verified that the pending differential equation (50) is satisfied identically. Constants c_{3m} and c_{4m} are obtained by considering the equilibrium of the plate under pure bending, so that $N_{xy} = 0$, for $y = \pm b/2$ and that the moment M_y for $y = \pm b/2$ should coincide with the value of the external bending moment applied, which in turn is expressed by:

$$M_y \Big|_{y=b/2} = \frac{4 \cdot M_o}{\pi} \cdot \sum \frac{1}{m} \cdot \text{sen} \frac{m \cdot \pi \cdot x}{a} \tag{93}$$

In this way, we obtain:

$$c_{3m} = \frac{2 \cdot M_o \cdot a}{D \cdot \pi^2 \cdot m^2 \cdot \rho}, \qquad c_{4m} = \frac{\pi \cdot m}{a} \cdot \alpha \cdot c_{3m} \tag{94}$$

being:

$$\alpha = \frac{m^2 \cdot \pi^2}{a^2} \cdot \frac{h^2}{5 \cdot (1-\mu)} \cdot \frac{sh \frac{m \cdot \pi \cdot b}{2 \cdot a}}{\tau \cdot sh \frac{\tau \cdot b}{2}} \tag{95}$$

$$\beta = (1-\mu) \cdot \left(\frac{m \cdot \pi}{a} + \frac{5 \cdot a}{h^2 \cdot m \cdot \pi}\right)$$
$$\rho = \frac{\beta \cdot ch \frac{\tau \cdot b}{2} \cdot m^2 \cdot \pi^2 \cdot h^2 \cdot sh \frac{m \cdot \pi \cdot b}{2 \cdot a}}{\tau \cdot sh \frac{\tau \cdot b}{2} \cdot a^2 \cdot 5 \cdot (1-\mu)} - ch \frac{m \cdot \pi \cdot b}{2 \cdot a} \cdot \left(1 + \frac{m^2 \cdot \pi^2}{a^2} \cdot h^2\right) \tag{96}$$

These expressions are formally analogous to the ones obtained in the work of Panc (1975).

As a check, it can be seen that for a thin plate (h << a, b) in which one of its dimensions is much less than the other (b << a, b), so that

$$th \frac{m \cdot \pi \cdot b}{2 \cdot a} \approx \frac{m \cdot \pi \cdot b}{2 \cdot a}, \; ch \frac{m \cdot \pi \cdot b}{2 \cdot a} \approx 1 \text{ y } \rho \approx -1 \quad \text{y} \;, \tag{97}$$

the vertical displacement in the centre in which y = 0 gives the solution is

$$w = \frac{M_o \cdot b^2}{2 \cdot D \cdot \pi} \sum_{1,3,\ldots} \frac{1}{m} \cdot \text{sen} \frac{m \cdot \pi \cdot x}{a} = \frac{1}{8} \cdot \frac{M_o \cdot b^2}{D} \tag{98}$$

i.e. an equation that coincides with the exact solution (Timoshenko 1959).

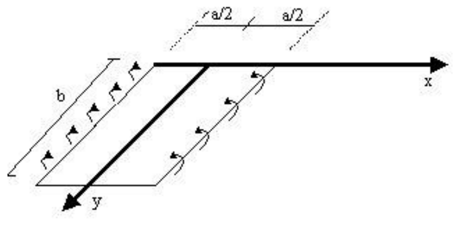

Figure 6: Bending of a rectangular plate by moments distributed along the edges ($x = \pm a/2$).

If the bending moments are applied at the other two edges (Figure 6), we shall adopt

$$M_x = \frac{4 \cdot M_o}{\pi} \cdot \sum \frac{1}{m} \cdot \mathrm{sen}\,\frac{m \cdot \pi \cdot y}{b} \qquad w^I = \sum w_m^I \cdot \mathrm{sen}\,\frac{m \cdot \pi \cdot y}{b} \tag{99}$$

$$\theta_y^I = \sum T_y(x) \cdot \mathrm{sen}\,\frac{m \cdot \pi \cdot y}{b}\,; \qquad \theta_x^I = \sum T_x(x) \cdot \cos\frac{m \cdot \pi \cdot y}{b} \tag{100}$$

arriving at another solution with similar characteristics to the previous one.

Example 4: in this example the case of a uniformly loaded clamped plate is studied (Figure 7). Using some results from the previous example, we are now in a position to tackle the solution of a uniformly loaded clamped plate, simply by making

$$\theta_x\Big]_{y=\pm b/2} = \theta_y\Big]_{x=\pm a/2} = 0\ . \tag{101}$$

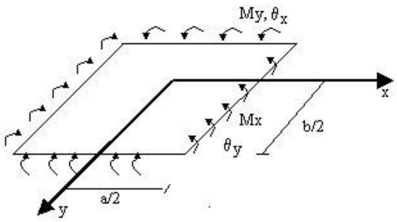

Figure 7: Pure bending of a rectangular plate by moments that are uniformly distributed along the edges.

The rotation in a simply supported plate under a uniformly distributed load is:

$$\theta_x^{SP}\Big]_{y=b/2} = \sum_{1,3,5,...}\sum_{1,3,5,..} (-1)^{\frac{m+2\cdot n-1}{2}} \cdot C_m^o \cdot \cos\frac{m \cdot \pi \cdot x}{a} \tag{102}$$

where C_m^o is known and is obtained from the expressions deduced in the fourth example studied. Substituting x and y for

$$x + \frac{a}{2}\,,\ y + \frac{b}{2} \tag{103}$$

gives

$$\left\{(\frac{m \cdot \pi}{a})^2 + (\frac{n \cdot \pi}{b})^2 + \frac{5 \cdot (1-\mu)}{h^2}\right\} \cdot C_m^o = \frac{5 \cdot (1-\mu)}{h^2} \cdot \frac{16 \cdot q_o}{\pi^6 \cdot D} \cdot \frac{n \cdot \pi}{b} \left\{\frac{1}{m \cdot n} \cdot \frac{1}{(\frac{m}{a})^2 + (\frac{n}{b})^2} \cdot \left[\frac{1}{(\frac{m}{a})^2 + (\frac{n}{b})^2} + \frac{h^2}{5 \cdot (1-\mu)}\right]\right\} \tag{104}$$

The rotation due to the moment M_y is derived from Eq. (90), by substituting x for $x + \frac{a}{2}$, so that

$$\text{sen}\frac{m\cdot\pi\cdot(x+\frac{a}{2})}{a}=(-1)^{m-1/2}\cdot\cos\frac{m\cdot\pi\cdot x}{a}, \tag{105}$$

and thus

$$\theta_x^{My}\Big]_{y=b/2}=-\sum_{1,3,5,...}c_{3m}\cdot\Bigg[\alpha\cdot 2\cdot\tau\cdot sh\frac{\tau b}{2}+\frac{b}{2}\cdot th\frac{m\cdot\pi\cdot b}{2\cdot a}\cdot sh\frac{m\cdot\pi\cdot b}{2\cdot a}\cdot\frac{m\cdot\pi}{a}-sh\frac{m\cdot\pi\cdot b}{2\cdot a} \\ \frac{b}{2}\cdot ch\frac{m\cdot\pi\cdot b}{2\cdot a}\cdot\frac{m\cdot\pi}{a}-\frac{2\cdot m^2\cdot\pi^2}{a^2}\cdot\frac{h^2}{5\cdot(1-\mu)}\cdot sh\frac{m\cdot\pi\cdot b}{2\cdot a}\Bigg]\cdot(-1)^{m-1/2}\cdot\cos\frac{m\cdot\pi\cdot x}{a} \tag{106}$$

Finally, rotation, due to the bending moments M_x which were previously omitted so as not to obscure the expression with more formulae, yields (after substituting y by $y+b/2$)

$$\theta_x^{Mx}\Big]_{y=b/2}=\sum_{1,3,5,...}\frac{m\cdot\pi}{b}\cdot c^I{}_{3m}\cdot\Bigg[\alpha^I\cdot 2\cdot ch(\tau^I x)+\cdot\frac{a}{2}th\frac{m\cdot\pi\cdot a}{2\cdot b}\cdot ch\frac{m\cdot\pi\cdot x}{b}- \\ -x\cdot sh\frac{m\cdot\pi\cdot x}{b}-2\cdot\frac{m\cdot\pi}{b}\cdot\frac{h^2}{5\cdot(1-\mu)}\cdot ch\frac{m\cdot\pi\cdot x}{b}\Bigg] \tag{107}$$

However, the expression in square brackets is even in x and can therefore be expanded by

$$\sum_{1,3,5}A_i\cdot\cos\frac{i\cdot\pi\cdot x}{a} \tag{108}$$

which would leave us with

$$\theta_x^{Mx}\Big]_{y=b/2}=\sum_{1,3,5}\sum_{1,3,5}\frac{m\cdot\pi}{b}\cdot c^I_{3m}\cdot A_i\cdot\cos\frac{i\cdot\pi\cdot x}{a} \tag{109}$$

being

$$A_i=\frac{2}{a}\cdot\int_{-a/2}^{a/2}(\alpha^I\cdot 2\cdot ch(\tau^I\cdot x)+\frac{a}{2}th\frac{m\pi a}{2b}\cdot ch\frac{m\pi x}{b}-x\cdot sh\frac{m\pi x}{b}-2\cdot\frac{m\pi}{b}\cdot\frac{h^2}{5(1-\mu)}\cdot ch\frac{m\pi x}{b})\cdot\cos\frac{i\pi x}{a}\cdot dx \tag{110}$$

After placing

$$\theta_x^{SP}\Big]_{y=\pm b/2}+\theta_x^{My}\Big]_{y=\pm b/2}+\theta_x^{Mx}\Big]_{y=\pm b/2}=0, \tag{111}$$

taking into account that the equation is valid for any value of x, we obtain an infinite system of linear equations to determine the unknown coefficients c_{3m} and c^I_{3m} . The condition of zero rotation on the other two edges $\theta_y\Big]_{x=a/2}$ leads us to a similar system with the same unknown quantities, which are determined in each specific case by these two systems using the successive approximation method.

6 CONCLUSIONS

The most important conclusions to have been reported in this paper are:

1) We have derived a refined system of differential equations valid for the study of plates including first-order shear deformation effects, proposing the equilibrium of the plate element in its deformed geometry and taking into account the kinematic assumptions and constitutive equations of the Bolle–Reissner plate theory.

2) This constitutes a more refined generalization than the one presented by Reissner (1945) for moderately thick plates and the other presented by Timoshenko (1959). If it is expressed in terms of the Marcus moment (moment sum) and displacements, it is formally analogous to the equations proposed by Reismann (1980).

3) Analytical solutions for this system of differential equations were obtained for simply supported plates in static and dynamic analysis with excellent results without using any stress function. Analytical solutions for clamped plates were also deduced. Equations presented here are valid for moderately thick and thin plates and, in some cases, coincide exactly with Reissner's theory and other shear deformation plate theories.

4) The structure of the system of differential equations obtained permits one to solve them easily by numerical methods, like the finite difference method, in case of complex boundary conditions and where analytical solutions are difficult to obtain. Results will be presented in future work.

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