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Multi-model finite element scheme for static and free vibration analyses of composite laminated beams

Abstract

A transition element is developed for the local global analysis of laminated composite beams. It bridges one part of the domain modelled with a higher order theory and other with a 2D mixed layerwise theory (LWT) used at critical zone of the domain. The use of developed transition element makes the analysis for interlaminar stresses possible with significant accuracy. The mixed 2D model incorporates the transverse normal and shear stresses as nodal degrees of freedom (DOF) which inherently ensures continuity of these stresses. Non critical zones are modelled with higher order equivalent single layer (ESL) theory leading to the global mesh with multiple models applied simultaneously. Use of higher order ESL in non critical zones reduces the total number of elements required to map the domain. A substantial reduction in DOF as compared to a complete 2D mixed model is obvious. This computationally economical multiple modelling scheme using the transition element is applied to static and free vibration analyses of laminated composite beams. Results obtained are in good agreement with benchmarks available in literature.

Keywords

Mixed formulation; finite element method; laminated composite beams; transition element; local global analysis; Hamilton's variational principle; principle of minimum potential energy.

U.N. Band[a*]
Y.M. Desai[b]

[a,b]Department of Civil Engineering, Indian Institute of Technology Bombay, Powai, Mumbai - 400076, India.
[b]JK & MJ Mehta Chair Professor
[b]desai@civil.iitb.ac.in

Corresponding author:
[a*]umeshband@gmail.com

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1 INTRODUCTION

Laminated composites are finding varied engineering applications as these materials possess a good environmental resistance, strength and are light in weight. Failure modes of laminated composites include the delamination failure leading to separation of the layers and loss of integrity of the structure. Due to the heterogeneous material properties of the different layers of the composite laminate high interlaminar stresses develop at the interfaces. For the kinetic equilibrium in the thickness direction, these interlaminar stresses need to be continuous between the layers. Pipes and

Pagano (1970); Rybicki (1971) brought out that delamination failure can be attributed to the interlaminar transverse shear and normal stresses. For a sound design, an accurate evaluation of the interlaminar stresses becomes inevitable. Different analytical and finite element equivalent single layer (ESL) and layerwise (LW) models have been proposed for analysis of composites. The ESL models in the literature predict the global parameters with a reasonable accuracy but fail to predict the continuity of interlaminar stresses. The displacement based LW models also fail to predict continuity of transverse stresses. These are more accurate relative to ESL models for global parameters but need higher computational effort. For estimation of the transverse stresses, a separate stress function or integration of stress equilibrium equations is essential. LW mixed model with transverse stresses as nodal DOF are accurate for global parameters and also for local interlaminar stresses.

Accurate elastic analysis of interlaminar stresses in composite laminates was presented by Srinivas and Rao (1970); Pagano (1969, 1970); Pagano and Hatfield (1972). As an offshoot of the plate formulation, many researchers have presented theories for analysis of composite laminated beams. Kant (1982) presented a comprehensive higher order theory for analysis of thick plates. Kant et al. (1997) applied the fully cubic theory for the dynamic analysis of beams. Various forms of higher order sub-theories were evolved by eliminating some terms from the comprehensive cubic theory (Marur and Kant, 1996; Manjunatha and Kant, 1993a; 1993b). Reddy (1987) presented higher order ESL as well as displacement based LW formulations. Contributions were also made by Lo et al. (1978); Spilker (1982), and others in the form of higher order theories. The formulation of Spilker had a stress based hybrid element having separate through thickness distributions for stress and displacements. To overcome the limitation of displacement based ESL, Rao et al. (2001) proposed an analytical solution based on mixed formulation in which the transverse stresses are invoked as DOF ensuring their continuity. This model has been employed for static and dynamic analyses of laminated plates and beams. Shimpi and Ghugal (1999) presented a trigonometric shear deformable LW theory capable of maintaining the continuity of transverse stresses. Desai and Ramtekkar (2002) presented a mixed finite element LW model with transverse stresses also included in the set of nodal DOF. The issue of continuity is inherently resolved and the results are seen to be in good agreement with the exact solutions. The formulation was applied to static analysis of composite beams. Ramtekkar et al. (2002) extended the mixed formulation for free vibration analysis of beams. Adyodgu (2006) presented a pair of shear deformable beam theories by employing parabolic and exponential shape functions respectively in conjunction with the classical beam theory for free vibration analysis of angle ply beams. Marur and Kant (2007) presented a higher order theory for free vibration analysis of angle ply beams.

Literature indicates that LW approach is the most suitable finite element formulation for accurate estimation of the interlaminar stresses. Within the LW models reported in literature, it is observed that the mixed formulation (Desai and Ramtekkar, 2002) yields very accurate solution and also satisfies the continuity requirements. However, such formulation is computationally expensive. Thus, layerwise formulation may not be suitable at all locations in the domain of a laminate. Amongst the ESL formulations the higher order theory (Kant et al., 1997) with 8 DOF(HOSTB8) has been reported to be comprehensive and computationally economical for the estimation of global parameters.

In this work, the positive traits of the higher order ESL theory and the mixed LWT are used together to obtain global as well as the local transverse stress parameters with a good accuracy. Laminate is modelled using a stack of 2D elements having mixed DOF in the zone where the transverse stresses are critical. Remaining zone is modelled using higher order ESL (HOSTB8). A unique transition element is developed to bridge the interface of ESL and mixed LWT based elements.

2 THEORETICAL FORMULATION FOR BEAM ANALYSIS

Three models have been formulated for analysis of transversely loaded laminated composite beams consisting of several orthotropic layers each having different properties;

(a) **Model 1**: This model adopts a cubic displacement field in the thickness direction for displacements (U , W) and has 8DOF per node. The theory has been identified as HOSTB8. The model is based on the plane stress state of stresses and strains.

(b) **Model 2**: In this model, mixed finite element LWT which has two displacements (U , W) and the transverse stresses (τ_{xz}, σ_z) as the nodal DOF is used. The theory is based on elasticity relationships. Therefore, requirement of any additional parameters/stress variation functions are advantageously avoided.

(c) **Model 3**: This model is based on a local global finite element procedure to take advantage of computational efficiency of higher order ESL theory and accuracy of the LW mixed model.

2.1 Model 1: Development of ESL based theory (HOSTB8)

Displacements in two principal directions of the laminated beam as a fully cubic function of the thickness co-ordinate are

$$\begin{aligned} u(x,z,t) &= u_0(x,z,t) + z\theta_x(x,z,t) + z^2u_0^*(x,z,t) + z^3\theta_x^*(x,z,t) \\ w(x,z,t) &= w_0(x,z,t) + z\theta_z(x,z,t) + z^2w_0^*(x,z,t) + z^3\theta_z^*(x,z,t) \end{aligned} \tag{1}$$

The above displacement field eliminates any requirement of shear correction factor and chances of shear locking. Here (u_0 and w_0) are deformations in X , Z (laminate co-ordinate) directions at the mid-plane. (θ_x and θ_z) are rotations at mid-plane about principal directions of laminated beam and (u_0^* , θ_x^* , w_0^* and θ_z^*) are higher order terms from Taylor's series. By using material property, strain displacement relationship and the principle of minimum potential energy, the stiffness matrix for the laminate is developed. By using shape functions similar to those used for stiffness evaluation, the mass matrix is also developed. Detailed formulation can be seen in Kant et al. (1997). A four-node isoparametric line element of Lagrange family is used to discretize a laminated beam.

Integration is performed by employing 5x5 Gauss quadrature rule for the extension, bending, mass component and 3x3 Gauss rule for the shear part.

2.2 Model 2: Development of mixed LW model

A 6-node two-dimensional plane stress element based on mixed formulation is used by considering displacement fields $u(x,z)$, and $w(x,z)$ having quadratic variation along the length of beam and cubic variation in the transverse direction. The cubic variation has been adopted to invoke the transverse stresses as the nodal parameters in addition to the nodal deformations. The displacement field is expressed as

$$u_k(x,z) = \sum_{i=1}^{3} g_i a_{0ik} + z\sum_{i=1}^{3} g_i a_{1ik} + z^2\sum_{i=1}^{3} g_i a_{2ik} + z^3\sum_{i=1}^{3} g_i a_{3ik} \tag{2}$$

where

$$g_1 = \frac{\xi}{2}(\xi - 1)\ ,\quad g_2 = 1 - \xi^2\ ,\quad g_3 = \frac{\xi}{2}(1+\xi),\ \xi = x/L_x \tag{3}$$

$$k = 1,\ 2 \text{ and } u_1 = u;\ u_2 = w;$$

Further, a_{mik} ($m = 0,\ 1,\ 2,\ 3;\ i = 1,\ 2,\ 3$) are the generalized coordinates.

Variation of displacement fields has been assumed to be cubic through the thickness of element, although there are only two nodes along 'z' axis of an element. Derivative of displacement with respect to the thickness coordinate has also been included in the displacement field. Such a inclusion is required for invoking transverse stress components σ_z, and τ_{xz} as nodal DOF in the formulation. Further, it also ensures parabolic variation of the transverse stresses through the thickness of an element.

By making use of the elasticity relationship, displacement field $u_k(x,z)$ in Eq. (2) can be shown to be

$$u_k(x,z) = \sum_{n=1}^{6} g_i\left(f_q u_{kn} + f_p \hat{u}_{kn}\right) \tag{4}$$

Here, $i = 1,\ 2,\ 3$ for the nodes with $\xi = -1$, $\xi = 0$ and $\xi = 1$, respectively;
$q = 1,\ 2$ and $p = 3,\ 4$ for the nodes with $\eta = -1$ and $\eta = 1$, respectively for node numbers 1 to 6 and;

$$f_1 = \frac{1}{4}\left(2 - 3\eta + \eta^3\right);\ f_2 = \frac{1}{4}\left(2 + 3\eta - \eta^3\right);\ f_3 = \frac{L_z}{4}\left(1 - \eta - \eta^2 + \eta^3\right);\ f_4 = \frac{L_z}{4}\left(-1 - \eta + \eta^2 + \eta^3\right)$$

f_3 and f_4 correspond to derivative of displacements with respect to thickness co-ordinate, whereas, f_1 and f_2 correspond to the displacement DOF, u_{kn} ($k = 1,\ 2$ and $n = 1,\ 2,\ 3,\ ...6$) are nodal displacement variables, whereas $\hat{u}_{kn}$ ($= \partial u_{kn}/\partial z$) contains the nodal transverse stress variables. Application of principle of minimum potential energy is used to develop element property matrix. Detailed formulation can be seen in Desai and Ramtekkar (2002).

Numerical integration of system matrices (Stiffness property matrix, Mass property matrix and load vector) has been performed by using a 3x3 integration scheme in the length and 5x5 integration scheme in the thickness direction.

2.3 Model 3: Development of transition between ESL (HOSTB8) and 2D mixed LW model

Compatibility between two differently modelled sub-domains (by using Model 1 and Model 2) is enforced by degenerating a 2D mixed element through kinematic constraints compatible with deformations predicted by ESL HOSTB8 element. Cook et al. (2003) has presented a methodology to connect dissimilar elements.

A 2D-to-1D transition element has one or two edges of 2D element that are kinematically restrained to enforce compatibility with adjacent ESL elements. Such a edge is denoted as a transition edge in the sequel. The 2D element on the transition edge is conditioned for compatibility with the DOF of the ESL (HOSTB8) element to ensure continuity of the combined model. Such an element acts as a transition element to connect two independently modelled sub-domains. Transition is achieved by placing a stack of such transition elements used in different layers of a laminate at the transition edge.

A pair of incompatible mesh formulations is shown in Fig. 1 wherein a four-node ESL element with eight DOF per node (node numbers denoted with a prime) is connected to a stack of 2D mixed elements with four DOF per node (two translations and two transverse stresses).

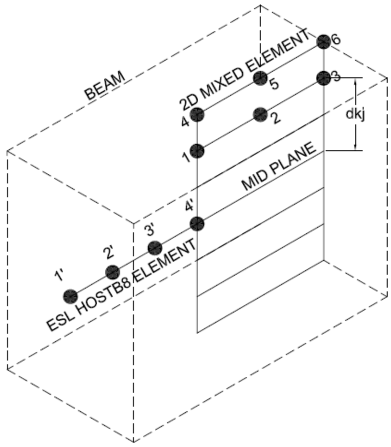


Figure 1: The configuration of connection between 2D mixed and HOSTB8 elements.

Kinematics of any point at a distance 'd_{kj}' from the reference plane of the laminate on the transition face is completely described by displacement field for the ESL. Because ESL and stack of 2D elements represent the same laminate, motion of the corner 2D node (node 1) (refer Fig. 1) is entirely prescribed by the two translations, two rotations and the higher order terms of its corresponding ESL node (node 4). Consequently, the DOF associated with nodes 1 and 4 are followers to the DOF associated with ESL leader node 4', and hence must be restrained. The node of the ESL on the transition edge form 2D elements' transition node. This node represents transition edge of 2D element stack. An indicative impression of the change in configuration of the 2D element on imposition of the restraint is shown in Fig. 2.

By using the displacement field of HOSTB8 in Eq. (3), kinematics $\{\hat{q}\}_{2D}^{k}$ of any 'k^{th}' node of 2D element on the transition surface and corresponding to the ESL leader 'j^{th}' node can be completely prescribed as

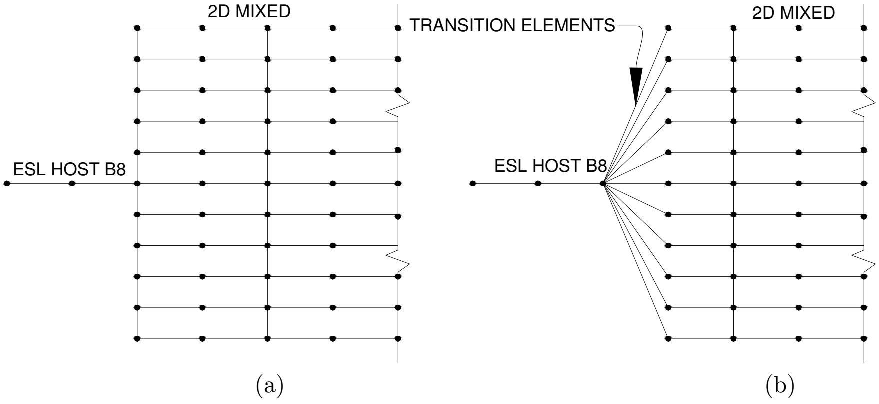


Figure 2: An indicative impression of unidirectional transition (a) before implementation of restraint (b) after implementation of restraint.

$$\begin{Bmatrix} \hat{u} \\ \hat{w} \end{Bmatrix}_{2D}^{k} = \begin{bmatrix} 1 & 0 & d_{kj} & 0 & d_{kj}^2 & 0 & d_{kj}^3 & 0 \\ 0 & 1 & 0 & d_{kj} & 0 & d_{kj}^2 & 0 & d_{kj}^3 \end{bmatrix} \{q\}_{1D}^{j} \tag{5}$$

$$\text{where } \{q\}_{1D} = \begin{bmatrix} u_0 & w_0 & \theta_x & \theta_z & u_0^* & w_0^* & \theta_x^* & \theta_z^* \end{bmatrix}^T$$

or

$$\{\hat{q}\}_{2D}^{k} = [R]_{kj} \{q\}_{1D}^{j} \tag{6}$$

By developing the restraint sub-matrices $[R]_{kj}$ for all pairs of ESL and 2D nodes, the transformation matrix $[R]$ for the entire element can be formulated by appropriately populating sub-matrices $[R]_{kj}$ corresponding to every pair. Finite element stiffness property, mass property matrices and internal force vector for the transition element are obtained by matrix transformations using the constructed stiffness property matrix, force vector ,mass matrix of 2D element and associated transformation matrix as follows,

$$\begin{aligned} [K_e]_{Tr} &= [R]^T [K_e]_{2D} [R] \\ \{F_e\}_{Tr} &= [R]^T \{F\}_{2D} \\ [M_e]_{Tr} &= [R]^T [M_e]_{2D} [R] \end{aligned} \tag{7}$$

Similar transformation can be performed for mass matrix of an element. The transformation in Eq. (9) degenerates the transition edge of the 2D element which becomes follower to the corresponding HOSTB8 leader node. All elements in the interior of the local transition edge are 6-node elements with all the nodes modelled with the mixed formulation. The stress DOF at the 2D nodes on the transition edge are condensed prior to imposition of the restraint for complete compatibility. Considering the stiffness matrices of the ESL elements, transition elements and the interior LW mixed elements the global equation can be obtained in the following form after assembly.

$$\left[K\right]^G = \sum_{i=1}^{m}\left[K_e^i\right] + \sum_{j=1}^{n}\left[K_e^j\right]_{tr} + \sum_{l=1}^{k}\left[K_e^l\right]$$
$$\left[M\right]^G = \sum_{i=1}^{m}\left[M_e^i\right] + \sum_{j=1}^{n}\left[M_e^j\right]_{tr} + \sum_{l=1}^{k}\left[M_e^l\right] \tag{8}$$

Here

$[K]^G$, $[M]^G$ are the global stiffness property and mass matrices, respectively;

$[K_e^i]$, $[M_e^i]$ are the element property and mass property matrices of i^{th} element, respectively, formed by using mixed LWT;

$[K_e^j]_{Tr}$, $[M_e^j]_{Tr}$ are the element property and mass matrices of j^{th} transition element, respectively;

$[K_e^l]$, $[M_e^l]$ are the element stiffness and mass matrices of l^{th} ESL element, respectively;

m , n and k in Eq. (10) represent number of LWT, transition and ESL elements.

The displacement vector of the transition element $\hat{q}_{tr}$ is composed of the DOF of the ESL node on the transition edge, and the DOF of the 2D nodes on the interior of element. The transition element that is developed by the application of the restraints consists of 5 nodes and 24DOF in case of an one edge transition.

2.4 Analysis of plate under static loading and vibration analysis

Standard finite element procedures are used to assemble and formulate global system matrices. For the static analysis the load deformation relationship is

$$\left[K\right]^G\left\{D\right\} = \left\{F\right\} \tag{9}$$

Gauss elimination method is used for determination of DOF in Eq. (9).

By using Hamilton's variational principle as brought out in Bathe (1997) and solution to the equation of motion of laminate, the characteristic equation of the eigenvalue problem is formed

$$\left(\left[K\right]^G - \omega^2\left[M\right]^G\right)\left\{\hat{D}\right\} = 0 \tag{10}$$

Here global mass matrix $\left[M\right]^G$ and stiffness property matrix $\left[K\right]^G$ are as defined in Eq. (8). $\left\{F\right\}$, $\left\{D\right\}$ are global nodal load and DOF vectors respectively. $\left\{\hat{D}\right\}$ is the modal vector and ω is natural frequency of laminate.

Solution of Eq. (10) after imposition of boundary conditions yield natural frequency ω and corresponding modal eigenvector $\left\{\hat{D}\right\}$. Sub-space iteration technique is used in the current work for solution of eigenvalue problem.

3 ILLUSTRATIVE EXAMPLES AND DISCUSSIONS

Following two mesh patterns are possible for a symmetrically loaded simply supported laminated beams;

(a) Peak τ_{xz} is estimated through **Mesh 1** by using a stack of 2D elements with mixed DOF at and in the vicinity of $\left(x = 0\right)$ and the remaining part of the laminate meshed with ESL elements.

(b) Peak σ_z is estimated through **Mesh 2** by using a stack of 2D elements with mixed DOF at and in the vicinity of $\left(x = a/2\right)$ and the remaining part of the laminate meshed with ESL elements.

For the static and free vibration analyses, symmetrical loading on the beam and simply supported end conditions are assumed. An example with different boundary conditions (Example 6) for an four layered angle ply beam is also considered for free vibration analysis. However, the same meshing patterns are used for discretization. Implementation of this novel hybrid finite element mesh is done on full length of beam. Free vibration analysis is also performed for two meshing arrangements to further validate the soundness and efficacy of the combined model. Stress estimation in static analysis for transverse stresses has been done by using selective meshing pattern according to suitability. Results obtained for static and free vibration analyses are compared with those from higher order ESL theories and LW mixed and displacement based finite element formulations available in literature.

A combined mesh size of (10)/(2x30) implies that full length of beam is discretized with 10 elements and out of which a portion of 2 elements is modelled with 30 2D elements with mixed DOF in the thickness direction. The remaining domain is modelled with ESL (HOSTB8) elements. Thus, there will be 68 elements (8#HOSTB8+60#2D) in the entire domain. If the same laminate is modelled using only 2D elements, the total number of elements for a mesh (10x30) will be 300. Proposed combined mesh reduces total number of DOF due to reduction in number of elements required to model the domain and also ensures continuity of transverse stresses.

Beams with different layups, span to thickness ratios (S), and material properties with various end conditions have been considered for static and free vibration analyses. Boundary conditions are listed in Table 1. Material properties and normalisation factors used for static and free vibration analyses are mentioned with the results.

3.1 Static analysis

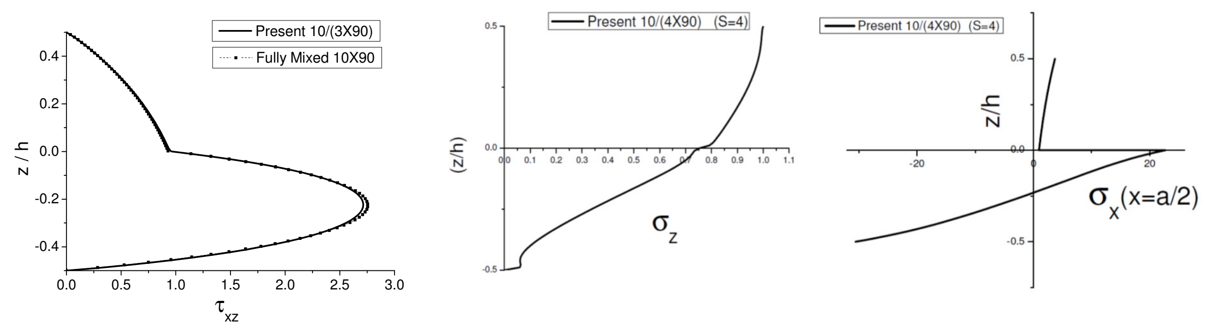
3.1.1 Example 1

A simply supported cross ply ($0^{\circ}/90^{\circ}$) beam with layers of equal thickness and subjected to unidirectional sinusoidal loading is considered. The beam is analysed using both combined mesh patterns for stresses (τ_{xz} and σ_z), longitudinal and transverse displacements (u and w). The transverse shear and transverse normal stresses has been selectively estimated by using Mesh 1 and Mesh 2 respectively. The results for S = 4 and 10 are compared in Table 2. Exact elasticity solution by Pagano (1969), and FE results obtained by Desai and Ramtekkar (2002), Liou and Sun (1987), Lu and Liu (1992), Manjunatha and Kant (1993b), and Shimpi and Ghugal (1999) have been presented for proper comparison. Results from the present analysis have been found to be in agreement with the established results. The variation of normalized longitudinal normal stress ($\bar{\sigma}_x$), transverse normal and shear stresses ($\bar{\sigma}_z$ and $\bar{\tau}_{xz}$) through the thickness of the beam with S = 4, have been shown in Fig. 3. Proximity of present results with the benchmarks validate the suitability of the combined model. Essential requirement of continuity of transverse stresses is also fulfilled with a substantial reduction in the computational effort.

	Description	Location	Degree-of-freedom			
			u	W	τ_{xz}	σ_z
1.	Simple support (S) (Mixed LW Elements)	$X=0$ $Z=+h/2$ $Z=-h/2$	- - -	0 - -	- 0 0	- $q_0{}^*$ 0
2.	Clamped support (C) (Mixed LW Elements)	$X=0$ $Z=+h/2$ $Z=-h/2$	- - -	0 - -	- 0 0	- $q_0{}^*$ 0
3.	Free end (F) (Mixed LW Elements)	$X=0$ $Z=+h/2$ $Z=-h/2$	- - -	- - -	- 0 0	- $q_0{}^*$ 0
4.	Simply supported beams (SS) (with ESL HOSTB8 Elements at ends in Mesh 2)	$X=0$ Simple support $w_0=w_0^*=\theta_z=\theta_z^*=0$ $X=a$ Simple support $u_0=u_0^*=w_0=w_0^*=\theta_z=\theta_z^*=0$				
5.	Clamped support (C) (ESL HOSTB8 Elements)	$u_0=\theta_x=u_0^*=\theta_x^*=$ $w_0=w_0^*=\theta_z=\theta_z^*=0$				
6.	Free end (F) (ESL HOSTB8 Elements)	$u_0=\theta_x=u_0^*=\theta_x^*=$ $w_0=w_0^*=\theta_z=\theta_z^*=$ unrestrained				

Note: '–' indicates no boundary condition imposed on that degree-of-freedom at that location
'*' $q_0=0$ for free vibration analysis

Table 1: Boundary conditions for composite beams.



(a) (b) (c)

Figure 3: Through thickness variation of stresses in a 0°/90° laminated beam subjected to sinusoidal load (S = 4); (a) $\tau_{xz}(0,z)$, (b) $\sigma_z(a/2,z)$, (c) $\sigma_x(a/2,z)$

S	Source	$\bar{\sigma}_X\left(a/2,h/2\right)$	$\bar{\sigma}_X\left(a/2,-h/2\right)$	$\bar{\tau}_{XZ}$ (max.)	$\bar{W}\left(a/2,0\right)$
4	Pagano (1969)	3.8359	-29.9745	2.7300	4.7675
	Desai and Ramtekkar (2002)	3.8247	-29.9383	2.7500	4.7636
	Liou and Sun (1987)	-	-	-	4.5950
	Lu and Liu (1992)	3.5714	-30.0000	-	4.7773
	Manjunatha & Kant (1993b) HOSTB7	3.7500	-26.9700	2.823	4.2839
	Manjunatha & Kant (1993b) HOSTB8	3.7680	-26.9200	2.822	4.2903
	Shimpi and Ghugal (1999)	3.9650	-30.2980	-	4.7431
	Present Analysis Mesh 1(10/(3X90))	3.678	-29.08	2.748	4.700
	Present Analysis Mesh 2(10/(4X90))	3.651	-31.11	-	4.844
10	Pagano (1969)	-	-	-	2.9568
	Desai and Ramtekkar (2002)	19.7709	-175.995	7.5670	2.9540
	Liou and Sun (1987)	-	-	-	2.9520
	Lu andLiu (1992)	20.0000	-175.000	-	3.0000
	Manjunatha & Kant (1993b) HOSTB7	19.6700	-173.100	7.2850	2.8947
	Manjunatha & Kant (1993b) HOSTB8	19.7300	-173.000	7.2840	2.8965
	Shimpi and Ghugal (1999)	19.8999	-176.870	-	2.9743
	Present Analysis Mesh 1(10/(3X90))	19.53	-175.8	7.650	2.956
	Present Analysis Mesh 2(10/(4X90))	19.77	-179.3	-	2.956

Note: (i) '–' represents result not available.

Table 2: Comparison of the maximum transverse displacement, the in-plane normal and the transverse shear stresses for simply supported laminated beam under sinusoial loading (lamination scheme: 0˚/90˚)
($E_1 = 172.4$ GPa, $E_2 = E_3 = 6.89$ GPa, $G_{12} = G_{13} = 3.45$ GPa, $G_{23} = 1.378$ GPa, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$)

$$\left(z' = \frac{z}{h} \quad ; \quad \bar{W} = w\,\frac{100E_2h^3}{q_0a^4} \quad ; \quad \left(\bar{\sigma}_x\right) = \frac{\left(\sigma_x\right)h^2}{q_0a^2} \quad ; \quad \left(\bar{\tau}_{xz}\right) = \frac{\left(\tau_{xz}\right)h}{q_0a} \right)$$

3.1.2 Example 2

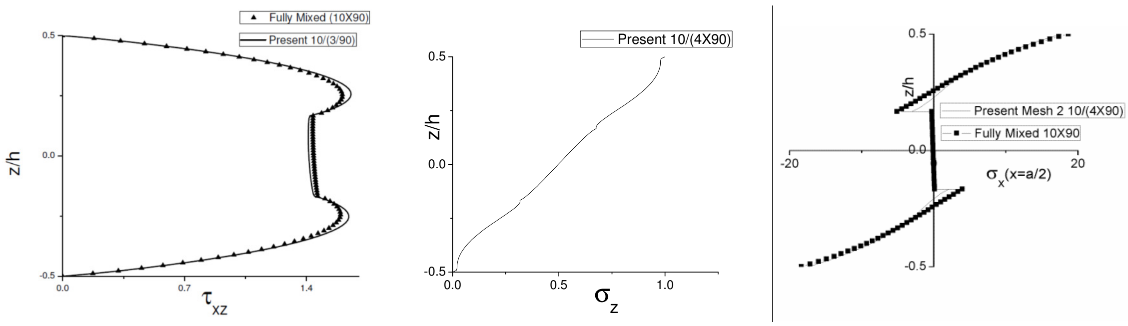
Static analysis of a three-layered cross-ply (0°/90°/0°) simply supported beam with equal thickness of each layer, has been considered in this example. The beam is subjected to uni-directional sinusoidal loading applied at top edge. The transverse shear and transverse normal stresses has been selectively estimated by using Mesh 1 and Mesh 2 respectively. Normalized maximum transverse displacement ($\bar{w}$), in-plane normal stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{xz}$) for the simply supported laminated beam (with S = 4 and 10) have been presented in Table 3. Results obtained through present analysis are compared with elasticity solution by Pagano (1969) and various analytical/ FE solutions given by Desai and Ramtekkar (2002); Lo et al. (1978); Spilker (1982); Engblom and Ochoa (1985); Toledano and Murakami (1987); Manjunatha and Kant (1993b). Present results shows good agreement with the elasticity solutions, which underlines the usefulness of the present model. Variation of normalized longitudinal normal stress ($\bar{\sigma}_x$) and transverse stresses (σ_z and $\bar{\tau}_{xz}$) through thickness of beam with S = 4 is presented in Fig. 3. It is observed that the results from the present multi-model scheme are in very good agreement with the elasticity solution (Pagano, 1969). It is also noted that the displacement based models (Lo et al., 1978; Engblom and Ochoa, 1985), have not been able to predict the in-plane and the transverse stresses. The continuity of the transverse stresses is ensured and the integration of equilibrium equations is advantageously avoided.

S	Source	$\bar{\sigma}_X\left(a/2,h/2\right)$	$\bar{\sigma}_X\left(a/2,-h/2\right)$	$\bar{\tau}_{XZ}$ (max)	$\bar{W}\left(a/2,0\right)$
4	Pagano (1969)	18.6102	-18.0023	1.5974	2.8600
	Desai and Ramtekkar (2002)	18.7523	-18.0497	1.6000	2.8400
	Lo et al. (1978)	-	-	1.5555	-
	Spilker (1982)	-	-	1.5636	2.8410
	Engblom and Ochoa (1985)	10.0090	-10.1081	1.7734	-
	Toledano and Murakami (1987)	-	-	-	2.8810
	Manjunatha & Kant (1993b) HOSTB3	13.8900	-13.8900	1.6630	1.9705
	Manjunatha & Kant (1993b) HOSTB4	13.940	-13.960	1.662	1.960
	Present Analysis Mesh 1(10/(3X90))	18.58	-17.95	1.662	2.758
	Present Analysis Mesh 2(10/(4X90))	17.74	-17.37	-	2.828
10	Pagano (1969)	73.6000	-73.2000	4.2346	0.9568
	Desai and Ramtekkar (2002)	73.4453	-73.4042	4.2510	0.9336
	Spilker (1982)	-	-	4.5292	0.9312
	Engblom and Ochoa (1985)	63.7344	-63.4025	4.4590	-
	Manjunatha & Kant (1993b) HOSTB3	67.4000	-67.4000	4.3950	0.7491
	Manjunatha & Kant (1993b) HOSTB4	67.410	-67.420	4.395	0.7479
	Present Analysis Mesh 1(10/(3X90))	72.520	-72.480	4.264	0.8986
	Present Analysis Mesh 2(10/(4X90))	72.98	-73.520	-	0.8856

Note: (i) '–' represents result not available

Table 3: Comparison of maximum transverse displacement, in-plane normal and transverse shear stresses for simply supported laminated beam under sinusoial loading (lamination scheme: 0˚/90˚/0˚)
($E_1 = 172.4$ GPa, $E_2 = E_3 = 6.89$ GPa, $G_{12} = G_{13} = 3.45$ GPa, $G_{23} = 1.378$ GPa, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$)

$$\left(z' = \frac{z}{h} \quad ; \quad \bar{W} = w\,\frac{100E_2h^3}{q_0a^4} \quad ; \quad \left(\bar{\sigma}_x\right) = \frac{\left(\sigma_x\right)h^2}{q_0a^2} \quad ; \quad \left(\bar{\tau}_{xz}\right) = \frac{\left(\tau_{xz}\right)h}{q_0a} \right)$$



(a) (b) (c)

Figure 4: Through thickness variation of stresses in a 0°/90°/0° laminated beam subjected to sinusoidal load (S = 4)
(a) $\tau_{xz}\left(0,z\right)$, (b) $\sigma_z\left(a/2,z\right)$, (c) $\sigma_x\left(a/2,z\right)$

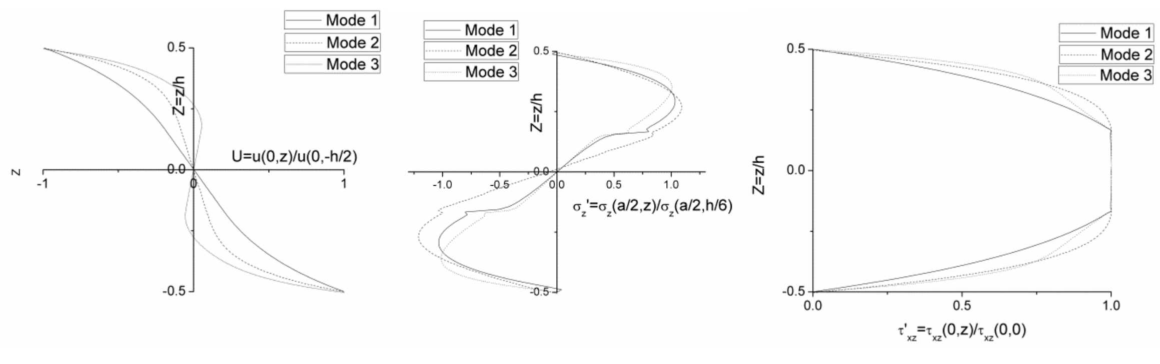
3.2 Free vibration analysis

3.2.1 Example 3

A six-layer symmetric cross-ply (0°/0°/90°/90°/0°/0°) thick beam under simple support is considered in this example. The non-dimensional frequencies in different modes obtained through the present model are compared in Table 4 with FOBT, HOBT by Marur and Kant (1996), mixed theory by Rao et al. (2001) and FEM solution by Ramtekkar et al. (2002). The through thickness variation of non dimensional inplane displacements, transverse normal and shear stresses for first three modes is shown Fig. 5. It can be observed that the results from the present studies are in good agreement with HOBT and the mixed theory with a substantial reduction in the total DOF.

Mode	Marur and Kant(1996)		Rao et al. (2001)	Ramtekkar et.al (2002)	**Present Mesh 1 10/(2X90)**	**Present Mesh 2 10/(2X90)**
	FOBT	HOBT				
1	1.639	1.6540	1.655	1.657	1.656	1.655
2	3.810	3.9160	1.879	3.910	3.918	3.919
3	5.912	6.1800	3.908	6.138	6.180	6.187
4	7.988	8.4460	6.146	8.323	6.422	8.440
5	10.100	10.7110	8.400	10.440	8.447	10.725
6	11.181	12.9690	10.694	12.469	10.716	11.109
7	12.188	15.2220	11.107	14.385	11.109	12.531
8	12.953	17.4690	12.616	16.161	12.987	12.970
9	14.392	19.7120	13.061	17.771	15.261	15.290
10	16.732	-	15.510	-	17.524	17.494
11	19.088	-	17.404	-	17.808	18.875
12	19.205	-	18.837	-	18.900	19.980
13	-	-	20.168	-	19.765	22.104
14	25.910	-	21.639	-	22.093	22.988
15	-	-	-	-	24.338	24.857

Table 4: Comparison of non-dimensional natural frequencies $\varpi = \left(\omega \times a^2/h\right) \times \sqrt{\rho/E_1}$ of (0/0/90/90/0/0) simply supported beam
($E_1 = 525$ GPa, $E_2 = 21$ GPa, $G_{12} = 10.50$ GPa, $\nu_{12} = 0.3$, Density (ρ)= 800 Ns^2/m^4, Length = 762 mm, Breadth = 25.4 mm, Depth = 152.4 mm)



(a) (b) (c)

Figure 5: Through thickness variation of (a) Normal displacement, (b) transverse normal stress, (c) transverse shear stress, for first three modes of vibration.

3.2.2 Example 4

A six-layer un-symmetric cross-ply (0°/90°/0°/90°/0°/90°) thick beam under simple support condition is considered in this example. The non-dimensional natural frequencies ($\bar{\omega}$) for different modes have been presented in Table 5. Comparison of frequencies with the available results from FOBT, HOBT (4a, 4b, 5) by Marur and Kant (1996), mixed theory by Rao et al. (2001) and FEM solution by Ramtekkar et al. (2002) has been done. The results show good agreement with the earlier reported results in the literature.

Mode	Ramtekkar et al. (2002)	Marur and Kant (1996)			HOBT5	**Present Mesh 1**	**Present Mesh 2 10/(4X90)**
		FOBT	HOBT4a	HOBT4b			
1	1.376	1.432	1.483	1.434	1.416	1.384	1.415
2	2.791	3.597	3.806	3.614	3.531	3.509	3.522
3	3.480	5.750	6.153	5.870	5.675	5.431	5.667
4	5.577	7.856	8.457	8.114	7.795	5.687	7.766
5	7.696	9.994	10.809	10.462	10.021	7.759	9.934
6	9.887	10.932	10.935	10.762	10.668	9.897	10.646
7	11.107	11.181	12.248	11.110	11.110	11.109	11.109
8	11.597	12.104	13.132	12.807	12.285	12.067	12.109
9	12.189	14.319	15.575	15.253	14.633	14.254	14.324
10	14.062	15.868	16.468	15.839	15.663	15.1055	15.632
11	15.871	16.673	18.166	17.873	17.244	16.463	16.467
12	16.178	19.147	20.889	20.611	19.981	16.686	18.924
13	19.294	-	21.709	21.430	21.108	18.677	21.011
14	20.314	21.855	-	-	-	20.991	21.022
15	-	-	-	-	-	22.069	22.746

Table 5: Comparison of non-dimensional natural frequencies $\bar{\omega} = \left(\omega \times a^2/h\right) \times \sqrt{\rho/E_1}$ of a simply supported thick un-symmetrically laminated composite beam (0°/90°/0°/90°/0°/90°)
($E_1 = 525$ GPa, $E_2 = 21$ GPa, $G_{12} = 10.50$ GPa, $\nu_{12} = 0.3$, Density (ρ)= 800 Ns^2/m^4, Length = 762 mm, Breadth = 25.4 mm, Depth = 152.4 mm)

3.2.3 Example 5

A symmetric cross-ply (0°/90°/90°/0°) thin beam under simple support condition is considered for free vibration analysis. The non-dimensional natural frequencies ($\bar{\omega}$) obtained from the present investigation are compared in Table 6, with the first order beam theory FOBT (4a) by Marur and Kant (1996), HOBT by Kant et al. (1998) and the mixed theory by Rao et al. (2001) and FEM solution by Ramtekkar et al. (2002). Results have been observed to be in very good agreement with the HOBT and the mixed theory.

3.2.4 Example 6

A four layered angle ply beam with layup (θ°/-θ°/-θ°/θ°) is considered in this example. To study the influence of ply angle, fundamental natural frequencies of laminated beam are determined

Mode	Marur and Kant (1996) FOBT	Kant et al. (1998) HOBT	Rao et al. (2001)	Ramtekkar et al. (2002)	**Present Mesh 1**	**Present Mesh 2**
1	2.512	2.516	2.513	2.516	2.519	2.518
2	8.589	8.669	8.660	8.673	8.682	8.683
3	16.045	16.320	16.330	11.439	16.378	16.3803
4	23.795	24.371	24.436	16.383	17.199	24.529
5	-	-	-	24.613	24.537	32.846
6	-	-	-	33.087	32.829	34.272
7	-	-	-	39.493	41.138	41.009
8	-	-	-	41.827	49.328	49.911
9	-	-	-	50.869	51.110	57.709
10	-	-	-	57.907	57.692	67.783
11	-	-	-	70.367	66.216	67.979
12	-	-	-	73.415	75.743	74.927
13	-	-	-	85.748	83.753	87.663
14	-	-	-	99.837	86.072	94.738
15	-	-	-	100.101	97.077	98.978
16	-	-	-	116.524	108.294	108.598
17	-	-	-	119.522	114.234	118.252
18	-	-	-	125.950	118.106	119.548
19	-	-	-	127.136	119.547	124.985
20	-	-	-	-	125.742	127.233

Table 6: Comparison of non-dimensional natural frequencies $\bar{\omega} = \left(\omega \times a^2/h\right) \times \sqrt{\rho/E_1}$ of a simply supported thick un-symmetrically laminated composite beam (0°/90°/90°/0°)
($E_1 = 1.448\text{x}10^8$ kN/mm^2, $E_2 = 9.65\text{x}10^6$ kN/mm^2, $G_{12} = 4.14\text{x}10^6$ kN/mm^2,, $\nu_{12} = 0.3$, Density (ρ)= 1839.23 Ns^2/m^4, Length = 15 m, Breadth = Depth = 1 m)

BC	Source	Ply angle 'θ °'						
		0°	15°	30°	45°	60°	75°	90°
SS	Adyodgu (2006)	2.651	1.896	1.141	0.804	0.736	0.725	0.729
	Present Mesh 1 (10/(3x60))	2.615	1.386	0.749	0.721	0.731	0.705	0.731
	Present Mesh 2 (10/(2x60))	2.614	1.386	0.749	0.720	0.730	0.705	0.730
CC	Adyodgu (2006)	4.973	4.294	2.195	1.929	1.669	1.612	1.619
	Present Mesh 1 (10/(3x60))	4.676	2.906	1.662	1.604	1.628	1.577	1.632
	Present Mesh 2 (10/(2x60))	4.670	2.903	1.659	1.601	1.624	1.573	1.628
CF	Adyodgu (2006)	0.981	0.676	0.414	0.288	0.262	0.258	0.260
	Present Mesh 1 (10/(3x60))	0.973	0.500	0.268	0.258	0.261	0.252	0.261
	Present Mesh 2 (10/(2x60))	0.973	0.500	0.268	0.257	0.261	0.252	0.261
FC	**Present Mesh 1**	0.973	0.500	0.268	0.258	0.261	0.252	0.261

Table 7: Variation of non-dimensional fundamental frequency $\bar{\omega} = \left(\omega \times a^2/h\right) \times \sqrt{\rho/E_1}$ of a four layer angle ply laminated beam (θ °/ $-\theta$ °/ $-\theta$ °/ θ °) under different boundary conditions (BC).
($E_1 = 144.8$ GPa, $E_2 = E_3 = 9.65$ GPa, $G_{12} = G_{13} = 4.14$ GPa, $G_{23} = 3.45$ GPa, $\nu_{12} = \nu_{13} = \nu_{23} = 0.3$, Density (ρ)= 1389.23 Ns^2/m^4, Length = 381 mm, Breadth = 25.4 mm, Depth = 25.4 mm)

for $\theta = 0°$, 15°, 30°, 45°, 60°, 75°, 90°. Simply supported at both ends (SS), clamped at both ends (CC) and clamped free (CF) boundary conditions are considered. CF condition with Mesh 1 meshing pattern has a stack of 2D mixed elements at the clamped end and remaining domain is modelled with HOSTB8 elements. An end condition, free clamped (FC) is also considered with Mesh 1 meshing pattern having the stack of 2D mixed elements at the free end. The non-dimensional fundamental frequencies are compared with those presented by Adyodgu (2006) in Table 7. Variation of the fundamental frequency with the ply angle for boundary condition (SS) is shown in Fig 6. Except for $\theta = 15°$ and 30°, natural frequencies for all other orientation of plies are in close proximity of those presented by Adyodgu (2006). The consistency between results of the two meshing patterns under all boundary conditions strengthens the efficacy of the combined mesh model.

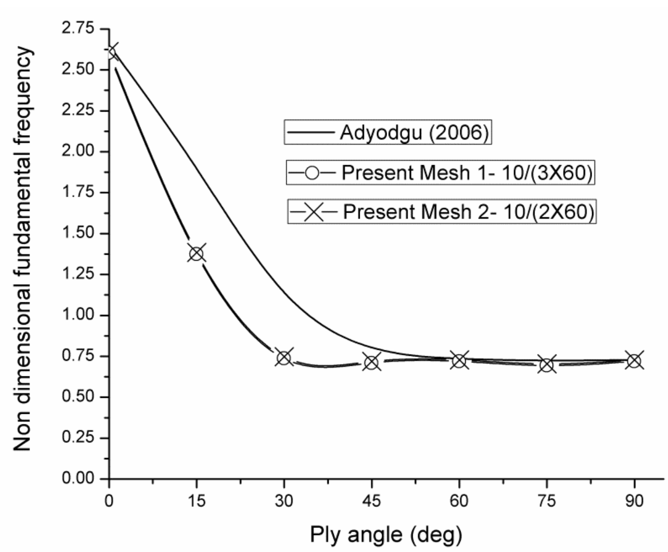


Figure 6: Variation of non-dimensional fundamental frequency of a four layer angle ply SS beam ($\theta°/-\theta°/-\theta°/\theta°$) with ply angle.

The results of above examples illustrate that the present multi-modelling scheme is apt for complete analysis of laminated beams. The accuracy requirements are fulfilled without sacrificing the computational efficiency. Continuity of transverse stresses is also ensured. It is observed that Ramtekkar et al. (2002) presented few frequencies which were not reported earlier in the literature. Present combined mesh model misses out on few of those additional frequencies reported in Ramtekkar et al. (2002) but on the other hand some new frequencies are encountered. The genuineness of these new frequencies is under investigation by the authors.

4 CONCLUSION

A finite element formulation is presented which combines the higher order ESL sub-domain to 2D mixed LW sub-domain using a transition element. It is observed that a reasonably good accuracy is achieved in the estimation of the deformations, stresses.and natural frequencies. Parallely a substantial reduction in the global DOF can be observed as compared to full 2D LWT model due to reduction in number of elements rquired for the entire domain. This makes the procedure computationally economical. Increase in the size of the local LW mixed sub-domain would make the mesh to tend towards full LW modeling and the methodology may loose its advantage. An accuracy above 90% in comparison with elasticity solution is seen to be achieved by using LW mixed model on 25% to 40% of the entire domain. The present combined formulation overcome the shortcoming of ESL in prediction of continuity of the transverse stresses. The method suggested is confirming to the elasticity principles as both the models used on the sub-domains are derived from the elasticity equations and two dimensional state of stress and strain. The results given by the developed model show reasonably good accuracy as compared with the benchmarks from the literature.

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