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On the transverse motions of non-prismatic deep beam under the actions of variable magnitude moving loads

B. Omolofe*, S. T. Oni and J. M. Tolorunshagba

Department of Mathematical Sciences, Federal University Of Technology
Akure Ondo State – Nigeria

Abstract

This paper investigates the flexural motions of non-uniform deep beams resting on variable elastic foundation and traversed by harmonic variable magnitude moving loads. The versatile Galerkin's method and the integral transform techniques were employed to treat the coupled second order partial differential equations governing the motion of the vibrating system. Analytical solution was obtained for both the transverse displacement response and the rotation of the non-uniform deep beam. Analytical and Numerical results show that, as the value of foundation stiffness K increases the deflection profile of the non-uniform deep beam decreases. It is also found that the critical velocity of the dynamical systems increases with an increase in the values of foundation stiffness K, thereby reducing the risk of resonance.

Keywords: Flexural motions, Critical velocity, foundation stiffness, Deep beams, transverse response, Galerkin's method.

1 Introduction

The vibration analysis of beams or beam-like structural elements has been and continues to be the subject of numerous researchers, since it embraces a wide class of problems with immense importance in Engineering Science. However, studies on beam problems have largely been restricted to the case when the beam structure is uniform. In particular, works on non-uniform deep beams are not common.

In non-uniform structures, the flexural rigidity and mass per unit length of the beam become certain functions of the spatial coordinate x in the model equation. This renders the exact solution to the dynamical problem impossible as the governing partial differential equation now has variable coefficients.

Among the earliest researchers on the dynamic analysis of an elastic beam was Ayre et al [2] who studied the effect of the ratio of the weight of the load to the weight of a simply supported beam for a constantly moving mass load. They obtained the exact solution for the resulting partial differential equation by using the infinite series method. Kenny [10] similarly studied and

*Corresp. author email: babatope_omolofe@yahoo.com Received 18 Aug 2008; In revised form 19 Mar 2009

found the possible velocities for the propagation of free bending waves and studied their relation to the critical velocity of the beam. He also presented an analytical solution and resonance diagrams for a constant velocity of a rapidly moving load on an elastic foundation including the effect of a viscous damping. In a more recent development, Foda and Abduljabbar [7] worked on the dynamic Green formulation for the response of a beam structure to a moving mass while Park et al [12] studied the natural frequencies and open-loop responses of an elastic beam fixed on a moving cart and carrying and intermediate mass. In the same vein, Gbadeyan and Aiyesimi [9] considered the dynamic response of a simple beam continuously supported by visco-elastic foundation to a load moving at non-uniform speed. In all the aforementioned works, investigations were limited to the analysis of beam flexure of Bernoulli-Euler beams models. Specifically, the effects of shear deformation and rotatory inertia were neglected in the governing partial differential equations.

Among the few studies on the dynamic analysis of deep beams under moving load include the work of Djondjorov [6] who investigated the invariant properties of Timoshenko beam equations, Wang [14] who studied the vibration of multi-span Timoshenko beams to a moving force and Oni [11] who studied the transverse vibrations under moving loads of deep beams on a variable elastic foundation. In all their works, it is tacitly assumed that the beam has uniform cross sections. To the best of authors knowledge, the more practical cases of deep beam moving load problems in which the beam under consideration is of non-uniform cross-section has not been tackled.

This paper therefore, is concerned with the problem of the transverse motions of non-prismatic deep beams resting on variable elastic foundations and subjected to a harmonic magnitude moving load.

2 Mathematical Model

This paper considers the dynamic behaviour of a non-uniform deep beam resting on a variable elastic foundation when it is under the action of a moving load. The beam's properties such as moment of inertia I and the mass per unit length μ of the beam vary along the span L of the beam. The beam is assumed to maintain contact with the variable subgrade reaction modulus $K_0(x)$ and that there is no friction forces at the interface. The deflection V(x,t) from the equilibrium and the rotation U(x,t) of the beam under the action of a variable magnitude moving load is described by the system of partial differential equations

$$\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - K^*GA\left[\frac{\partial^2 V(x,t)}{\partial x^2} - \frac{\partial U(x,t)}{\partial x}\right] + K_0(x)V(x,t) = P(x,t) \tag{1}$$

and

$$E\frac{\partial}{\partial x}\left[I(x)\frac{\partial U(x,t)}{\partial x}\right] + K^*GA\left[\frac{\partial V(x,t)}{\partial x} - U(x,t)\right] - I(x)\rho\frac{\partial^2 U(x,t)}{\partial t^2} = 0 \tag{2}$$

where $\mu(x)$ is the mass m of the beam per unit length L, K* is a constant dependent on the shape of the cross-section, G is the modulus of elasticity in the shear, A is the cross-sectional area, P(x,t) is the harmonic moving force, E is the Young's modulus of the beam, $I(x)$ is the variable moment of inertia of the beam cross-section, ρ is the mass of the beam per unit volume and $K_0(x)$ is the variable elastic foundation.

The boundary conditions at the end x = 0 and x = L are given by

$$V(0,t) = 0;\ U(0,t) = 0$$

$$\frac{\partial V(L,t)}{\partial x} = 0; \frac{\partial U(L,t)}{\partial x} = 0 \tag{3}$$

and the initial conditions are

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \text{ and } U(x,0) = 0 = \frac{\partial U(x,0)}{\partial t} \tag{4}$$

For the variable moment of inertia I and the mass per unit length μ of the beam, we adopt the example in [8] and take $I(x)$ and $\mu(x)$ to be of the form

$$I(x) = I_0 \left(1 + \sin\frac{\pi x}{L}\right)^3 \tag{5}$$

and

$$\mu(x) = \mu_0 \left(1 + \sin\frac{\pi x}{L}\right) \tag{6}$$

Furthermore, the variable harmonic magnitude moving force $P(x,t)$ acting on the beam is given by

$$P(x,t) = P \cos\omega\, t\delta(x - vt) \tag{7a}$$

where ω is the frequency of the load and $\delta(\bullet)$ is the dirac-delta function.

In this paper, we adopt the example in [4] and define the variable elastic foundation K(x) as

$$K_0(x) = K(4x - 3x^2 + x^3) \tag{7b}$$

Where K is the foundation modulus constant

When equations (5), (6), (7a) and (7b) are substituted into equations (1) and (2), the result is a non-homogeneous system of partial differential equations with variable coefficients given by

$$\mu_0 \left(1 + \sin\frac{\pi x}{L}\right) \frac{\partial^2 V(x,t)}{\partial t^2} - K^* GA \left[\frac{\partial^2 V(x,t)}{\partial x^2} - \frac{\partial U(x,t)}{\partial x}\right] + K(4x - 3x^2 + x^3)V(x,t) = P \cos \omega t\delta(x - vt) \tag{8}$$

and

$$EI_0 \frac{\partial}{\partial x}\left[\left(1+\sin\frac{\pi x}{L}\right)^3 \frac{\partial U(x,t)}{\partial x}\right] + K^* GA\left[\frac{\partial V(x,t)}{\partial x} - U(x,t)\right] - I_0\left(1+\sin\frac{\pi x}{L}\right)^3 \rho \frac{\partial^2 U(x,t)}{\partial t^2} = 0 \qquad (9)$$

To the authors best of knowledge, a closed form solution to the simultaneous second order partial differential equations (8) and (9) does not exist. Consequently, an approximate analytical solution is desirable to obtain some vital information about the vibrating system.

3 Approximate Analytical Solution

In order to solve the beam problem above, we shall use the versatile solution technique called Galerkin's method often used in solving diverse problems involving mechanical vibrations [3, 5, 13]. This solution technique involves solving equations of the form

$$\Theta(V) - P = 0 \qquad (10)$$

where,

Θ is the differential operator, V is the structural displacement and P is the traverse load acting on the structure. To this effect, the solutions of the system of equations (8) and (9) are expressed as

$$V_j(x,t) = \sum_{j=1}^{n} P_j(t) Q_j(x) \qquad (11)$$

and

$$U_j(x,t) = \sum_{j=1}^{n} Y_j(t) X_j(x) \qquad (12)$$

where the functions $Q_j(x)$ and $X_j(x)$ are chosen to satisfy the pertinent boundary conditions. Thus, substituting equations (11) and (12) into the coupled simultaneous ordinary differential equations (8) and (9) we obtain

$$\sum_{j=1}^{n}\left\{\mu_0\left(1+\sin\frac{\pi x}{L}\right) Q_j(x)\ddot{P}_j(t) - K^* GA\left[P_j(t)Q_j''(x) - Y_j(x) - Y_j(t)X_j'\right] + K\left(4x - 3x^2 + x^3\right) P_j(t)Q_j(x)\right\}$$
$$-P\mathrm{Cos}\omega t\delta(x-ct) = 0 \qquad (13)$$

and

$$\sum_{j=1}^{n}\left\{\frac{EI_0}{4}\frac{\partial}{\partial x}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)X_j'(x)Y_j(t)\right]+K^*GA\left[P_j(t)Q_j'-Y_j(t)X_j(x)\right]\right.$$
$$\left.-\tfrac{I_0}{4}\rho\left(10+15\sin\tfrac{\pi x}{L}-6\cos\tfrac{2\pi x}{L}-\sin\tfrac{3\pi x}{L}\right)X_j(x)\ddot{Y}_j(t)\right\}=0 \tag{14}$$

To determine $P_j(t)$ and $Y_j(t)$, the expressions on the left hand sides of equations (13) and (14) are required to be orthogonal to the functions $Q_k(x)$ and $X_k(x)$ respectively. Thus,

$$\int_0^L\sum_{j=1}^{n}\left\{\left[\mu_0\left(1+\sin\frac{\pi x}{L}\right)Q_j(x)\ddot{P}_j(t)-K^*GA\left[P_j(t)Q_j''(x)-Y_j(t)X_j'\right]+K\left(4x-3x^2+x^3\right)P_j(t)Q_j(x)\right]\right.$$
$$\left.-P\cos\omega t\delta(x-ct)\right\}Q_k(x)dx=0 \tag{15}$$

and

$$\int_0^L\sum_{j=1}^{n}\left\{\frac{EI_0}{4}\frac{\partial}{\partial x}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)X_j'(x)Y_j(t)\right]+K^*GA\left[P_j(t)Q_j'-Y_j(t)X_j(x)\right]\right.$$
$$\left.-\tfrac{I_0}{4}\rho\left(10+15\sin\tfrac{\pi x}{L}-6\cos\tfrac{2\pi x}{L}-\sin\tfrac{3\pi x}{L}\right)X_j(x)\ddot{Y}_j(t)\right\}X_k(x)dx=0 \tag{16}$$

Equations (15) and (16) after some rearrangements and simplifications yield

$$A_1(j,k)\ddot{P}_j(t)+A_2(j,k)P_j(t)+A_3(j,k)Y_j(t)=P\cos\omega tQ_k(ct) \tag{17}$$

and

$$B_1(j,k)\ddot{Y}_j(t)+B_2(j,k)P_j(t)+B_3(j,k)Y_j(t)=0 \tag{18}$$

where

$$A_1(j,k)=\mu_0\int_0^L\left(1+\sin\frac{\pi x}{L}\right)Q_j(x)Q_k(x)dx \tag{19a}$$

$$A_2(j,k)=\int_0^L\left[-K^*GAQ_j''(x)+K_0\left(4x-3x^2+x^3\right)Q_j(x)\right]Q_k(x)dx \tag{19b}$$

$$A_3(j,k)=K^*GA\int_0^L X_j'(x)Q_k(x)dx \tag{19c}$$

$$B_1(j,k)=-\frac{I_0}{4}\rho\int_0^L\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)X_j(x)X_kdx \tag{19d}$$

$$B_2(j,k)=K^*GA\int_0^L QX_j(x)X_k(x)dx \tag{19e}$$

and

$$B_3(j,k) = \frac{EI_0}{4}\int_0^L \frac{\partial}{\partial x}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)X_j'(x)\right]X_k(x)dx - \int_0^L K^*GAX_j(x)X_k(x)dx \tag{19f}$$

Since our beam has simple supports at both ends x = 0 and x = L, we therefore choose the functions $Q_j(x)$ and $X_j(x)$ to be

$$Q_j(x) = \sin\frac{j\pi x}{L} \tag{20}$$

and

$$X_j(x) = \cos\frac{j\pi x}{L} \tag{21}$$

Thus, in view of (20) and (21), integrals (19a) to (19f) are evaluated to yield

$$A_1(j,k) = \mu_0\left[\frac{L}{2} - \frac{2L\left(1-j^2-k^2\right)}{\pi\left[1-(j+k)^2\right]\left[1-(j-k)^2\right]}\right] \tag{22a}$$

$$A_2(j,k) = KL^2\left(1-\frac{L}{2}+\frac{L^2}{8}\right)+\frac{K^*GFj^2\pi^2}{2L} \tag{22b}$$

$$A_3(j,k) = -\frac{K^*GAj\pi}{2} \tag{22c}$$

$$B_1(j,k) = -\frac{I_0\rho}{4}\left[5L+\frac{120L\left(1-j^2-k^2\right)}{4\pi\left[(1+k)^2-j^2\right]\left[(1-k)^2-j^2\right]}-3L-\frac{24L\left(9-j^2-k^2\right)}{4\pi\left[(3+k)^2-j^2\right]\left[(3-k)^2-j^2\right]}\right] \tag{22d}$$

$$B_2\,(j,k) = \frac{K^*GAL}{2} \tag{22e}$$

and

$$\begin{aligned} B_3\,(j,k) \quad &= \frac{EI_0}{4}\left[-\frac{10j^2\pi^2}{2L}-\frac{120j^2\pi\left(1-j^2-k^2\right)}{4L\left[(1+k)^2-j^2\right]\left[(1-k)^2-j^2\right]}-\frac{60j\pi}{4L}\left(\frac{(j+k)}{\left[(j+k)^2-1\right]}+\frac{(j-k)}{\left[(j-k)^2-1\right]}\right)\right. \\ &+\frac{3\pi^2j^2}{L}-\frac{12\pi^2j}{L}+\frac{24j^2\pi\left(3-j^2-k^2\right)}{4L\left[(3+k)^2-j^2\right]\left[(3-k)^2-j^2\right]} \\ &\left.+\frac{12j\pi}{4L}\left(\frac{(j+k)}{\left[(j+k)^2-9\right]}+\frac{(j-k)}{\left[(j-k)^2-9\right]}\right)\right]-\frac{LK^*GF}{2} \end{aligned} \tag{22f}$$

In what follows we subject the system of ordinary differential equations (17) and (18) to a Laplace transform defined as

$$\left(\tilde{\cdot}\right) = \int_0^{\infty} \left(\cdot\right) e^{-st} dt \tag{23}$$

where s is the Laplace parameter. Applying the initial condition (2.4), we thus obtain the following algebraic simultaneous equations

$$\left(A_1\left(j,k\right) s^2 + A_2\left(j,k\right)\right) P_j(s) + A_3\left(j,k\right) Y_j(s) = \frac{P_0}{2}\left[\frac{\omega + \frac{k\pi v}{L}}{\sin\left(\omega + \frac{k\pi v}{L}\right)} - \frac{\omega - \frac{k\pi v}{L}}{\sin\left(\omega - \frac{k\pi v}{L}\right)}\right] \tag{24}$$

and

$$\left(B_1\left(j,k\right) s^2 + B_3\left(j,k\right)\right) Y_j(s) + B_2\left(j,k\right) P_j(s) = 0 \tag{25}$$

Solving the simultaneous equations (24) and (25) one obtains

$$P_j(s) = \frac{P_0\left(\frac{\omega + \frac{k\pi v}{L}}{s^2 + \left(\omega + \frac{k\pi v}{L}\right)^2} - \frac{\omega - \frac{k\pi v}{L}}{s^2 + \left(\omega - \frac{k\pi v}{L}\right)^2}\right)\left(B_1\left(j,k\right) s^2 + B_3\left(j,k\right)\right)}{2\Delta\left(j,k\right)} \tag{26}$$

and

$$Y_j(s) = \frac{P_0\left(\frac{\omega + \frac{k\pi v}{L}}{s^2 + \left(\omega + \frac{k\pi v}{L}\right)^2} - \frac{\omega - \frac{k\pi v}{L}}{s^2 + \left(\omega - \frac{k\pi v}{L}\right)^2}\right) B_2\left(j,k\right)}{2\Delta\left(j,k\right)} \tag{27}$$

where

$$\Delta\left(j,k\right) = A_1\left(j,k\right) B_1\left(j,k\right) s^4 + \left(A_1\left(j,k\right) B_3\left(j,k\right) + A_2\left(j,k\right) B_1\left(j,k\right)\right) s^2 - A_3\left(j,k\right) B_2\left(j,k\right) \tag{28}$$

equation (28) is rearranged to take the form

$$\Delta\left(j,k\right) = C_1\left(s^2 + \alpha^2\right)\left(s^2 + \beta^2\right) \tag{29}$$

where

$$\alpha^2 = \frac{C_2}{2C_1} - \sqrt{\frac{C_2^2}{4C_1^2} - \frac{C_3}{C_1}}, \qquad \beta^2 = \frac{C_2}{2C_1} + \sqrt{\frac{C_2^2}{4C_1^2} - \frac{C_3}{C_1}} \tag{30}$$

and

$$\begin{aligned} C_1 &= A_1\left(j,k\right) B_1\left(j,k\right), \\ C_2 &= A_1\left(j,k\right) B_3\left(j,k\right) + A_2\left(j,k\right) B_1\left(j,k\right), \\ C_3 &= A_2\left(j,k\right) B_3\left(j,k\right) - A_3\left(j,k\right) B_2\left(j,k\right) \end{aligned} \tag{31}$$

in view of (29) , equations (26) and (27) after some rearrangements and simplifications yield

$$\begin{aligned} P_j\left(s\right) \;=\;& \left[\tfrac{P_0 B_1(j,k)\beta}{2C_1(\beta^2-\alpha^2)} - \tfrac{P_0 B_3(j,k)}{2C_1\beta(\beta^2-\alpha^2)}\right]\left(\tfrac{\omega_1}{s^2+\omega_1^2}\right)\left(\tfrac{\beta}{s^2+\beta^2}\right) \\ -& \left[\tfrac{P_0 B_1(j,k)\alpha}{2C_1(\beta^2-\alpha^2)} - \tfrac{P_0 B_3(j,k)}{2C_1\alpha(\beta^2-\alpha^2)}\right]\left(\tfrac{\omega_1}{s^2+\omega_1^2}\right)\left(\tfrac{\alpha}{s^2+\alpha^2}\right) \\ -& \left[\tfrac{P_0 B_1(j,k)\beta}{2C_1(\beta^2-\alpha^2)} - \tfrac{P_0 B_3(j,k)}{2C_1\beta(\beta^2-\alpha^2)}\right]\left(\tfrac{\omega_2}{s^2+\omega_2^2}\right)\left(\tfrac{\beta}{s^2+\beta^2}\right) \\ -& \left[\tfrac{P_0 B_1(j,k)\alpha}{2C_1(\beta^2-\alpha^2)} - \tfrac{P_0 B_3(j,k)}{2C_1\alpha(\beta^2-\alpha^2)}\right]\left(\tfrac{\omega_2}{s^2+\omega_2^2}\right)\left(\tfrac{\alpha}{s^2+\alpha^2}\right) \end{aligned} \tag{32}$$

and

$$Y_j\left(s\right) = -\frac{P_0 B_2\left(j,k\right)}{2C_1\left(\beta^2-\alpha^2\right)}\left[\frac{\omega_1}{s^2+\omega_1^2}\cdot\frac{1}{s^2-\beta^2} - \frac{\omega_1}{s^2+\omega_1^2}\cdot\frac{1}{s^2-\alpha^2} - \frac{\omega_2}{s^2+\omega_2^2}\cdot\frac{1}{s^2-\beta^2} - \frac{\omega_2}{s^2-\omega_2^2}\cdot\frac{1}{s^2-\alpha^2}\right] \tag{33}$$

where

$$\omega_1 = \omega + \frac{k\pi v}{L}, \qquad \omega_2 = \omega - \frac{k\pi v}{L} \tag{34}$$

to obtain the Laplace inversion of equations (32) and (33), use is made of the following representations

$$g_1\left(s\right) = \frac{\omega_1}{s^2+\omega_1^2}, \qquad f_1\left(s\right) = \frac{\beta}{s^2+\beta^2} \tag{35}$$

$$g_2\left(s\right) = \frac{\omega_2}{s^2+\omega_2^2}, \qquad f_2\left(s\right) = \frac{\alpha}{s^2+\alpha^2} \tag{36}$$

so that the Laplace inversion of the equation (32) is the convolution of f_i 's and g_i 's defined as

$$f_i * g_j = \textstyle\int_0^t f_t\left(t-u\right) g_j\left(u\right) du \text{ where } \quad \begin{array}{l} i = 1,2,3\ldots\ldots \\ j = 1,2,3\ldots\ldots \end{array} \tag{37}$$

Thus the Laplace inversion of (32) is given by

$$\begin{aligned} P_j\left(t\right) =& \left[\frac{P_0 B_1\left(j,k\right)\beta}{2C_1\left(\beta^2-\alpha^2\right)} - \frac{P_0 B_3\left(j,k\right)}{2\beta C_1\left(\beta^2-\alpha^2\right)}\right] H_1 - \left[\frac{P_0 B_1\left(j,k\right)\alpha}{2C_1\left(\beta^2-\alpha^2\right)} - \frac{P_0 B_3\left(j,k\right)}{2\alpha C_1\left(\beta^2-\alpha^2\right)}\right] H_2 \\ & - \left[\frac{P_0 B_1\left(j,k\right)\beta}{2C_1\left(\beta^2-\alpha^2\right)} - \frac{P_0 B_3\left(j,k\right)}{2\beta C_1\left(\beta^2-\alpha^2\right)}\right] H_3 + \left[\frac{P_0 B_1\left(j,k\right)\alpha}{2C_1\left(\beta^2-\alpha^2\right)} - \frac{P_0 B_3\left(j,k\right)}{2\alpha C_1\left(\beta^2-\alpha^2\right)}\right] H_4 \end{aligned} \tag{38}$$

where

$$H_1 \int_0^t \sin \beta (t-u) \sin \omega_1 u du, \qquad H_2 = \int_0^t \sin \alpha (t-u) \sin \omega_1 u du$$
$$H_3 \int_0^t \sin \beta (t-u) \sin \omega_2 u du, \qquad H_4 = \int_0^t \sin \alpha (t-u) \sin \omega_2 u du \tag{39}$$

In what follows, we shall evaluate integrals (39) above and to do so, we note the following trigonometric identities namely

$$\begin{aligned}\int_0^t \sin B (t-u) \sin A u du = & \frac{B \sin Bt}{B^2 - A^2} \left[\sin BtCosAt + \frac{A}{B} (\cos At \cos Bt - 1) \right] \\ & - \frac{B \cos Bt}{B^2 - A^2} \left[\sin At \cos Bt - \frac{A}{B} \sin Bt \cos At \right]\end{aligned} \tag{40}$$

In view of (40), integrals (39) are thus evaluated and we have

$$\begin{aligned}
H_1 &= \frac{\beta \sin \beta t}{\beta^2 - \omega_1^2} \left[\sin \beta t \cos \omega_1 t + \frac{\omega_1}{\beta} (\cos \omega_1 t \cos \beta t - 1) \right] - \frac{\beta \cos \beta t}{\beta^2 - \omega_1^2} \left[\sin \beta t \cos \omega_1 t - \frac{\omega_1}{\beta} \sin \beta t \cos \omega_1 t \right] \\
H_2 &= \frac{\alpha \sin \alpha t}{\alpha^2 - \omega_1^2} \left[\sin \alpha t \cos \omega_1 t + \frac{\omega_1}{\alpha} (\cos \omega_1 t \cos \alpha t - 1) \right] - \frac{\alpha \cos \alpha t}{\alpha^2 - \omega_1^2} \left[\sin \alpha t \cos \omega_1 t - \frac{\omega_1}{\alpha} \sin \alpha t \cos \omega_1 t \right] \\
H_3 &= \frac{\beta \sin \beta t}{\beta^2 - \omega_2^2} \left[\sin \beta t \cos \omega_2 t + \frac{\omega_2}{\beta} (\cos \omega_2 t \cos \beta t - 1) \right] - \frac{\beta \cos \beta t}{\beta^2 - \omega_2^2} \left[\sin \beta t \cos \omega_2 t - \frac{\omega_2}{\beta} \sin \beta t \cos \omega_2 t \right] \\
H_4 &= \frac{\alpha \sin \alpha t}{\alpha^2 - \omega_2^2} \left[\sin \alpha t \cos \omega_2 t + \frac{\omega_2}{\alpha} (\cos \omega_2 t \cos \alpha t - 1) \right] - \frac{\alpha \cos \alpha t}{\alpha^2 - \omega_2^2} \left[\sin \alpha t \cos \omega_2 t - \frac{\omega_2}{\alpha} \sin \alpha t \cos \omega_2 t \right]
\end{aligned} \tag{41}$$

Thus in view of equation (11) taking into account (38) one obtains

$$\begin{aligned}
V_j(x,t) = \sum_{j=1}^{n} \Bigg\{ & \left[\frac{P_0 B_1(j,k) \beta}{2C_1(\beta^2 - \alpha^2)} - \frac{P_0 B_3(j,k)}{2\beta C_1(\beta^2 - \alpha^2)} \right] H_1 - \left[\frac{P_0 B_1(j,k) \alpha}{2C_1(\beta^2 - \alpha^2)} - \frac{P_0 B_3(j,k)}{2\alpha C_1(\beta^2 - \alpha^2)} \right] H_2 \\
& - \left[\frac{P_0 B_1(j,k) \beta}{2C_1(\beta^2 - \alpha^2)} - \frac{P_0 B_3(j,k)}{2\beta C_1(\beta^2 - \alpha^2)} \right] H_3 + \left[\frac{P_0 B_1(j,k) \alpha}{2C_1(\beta^2 - \alpha^2)} - \frac{P_0 B_3(j,k)}{2\alpha C_1(\beta^2 - \alpha^2)} \right] H_4 \Bigg\} \times \sin \frac{j\pi x}{L}
\end{aligned} \tag{42}$$

which represents the transverse displacement response of the non-uniform deep beam under the action of variable magnitude harmonic moving load.

Similarly,

$$Y_j(t) = -\frac{P_0 B_2(j,k)}{2C_1(\beta^2 - \alpha^2)} \left[\frac{1}{\beta} H_1 - \frac{1}{\alpha} H_2 - \frac{1}{\beta} H_3 + \frac{1}{\alpha} H_4 \right] \tag{43}$$

which on inversion yields

$$U_j\left(x,t\right)=\sum_{j=1}^{n}\ -\left\{\frac{P_0B_2\left(j,k\right)}{2C_1\left(\beta^2-\alpha^2\right)}\left[\frac{1}{\beta}\left(H_1-H_3\right)-\frac{1}{\alpha}\left(H_2-H_4\right)\right]\right\}\times\ \cos\frac{j\pi x}{L} \tag{44}$$

which is the rotation of the non-uniform deep beam under the action of variable magnitude harmonic moving load. Using the method of proof presented in [1], it is can be shown that the series solutions (42) and (44) converge rapidly.

4 Discussion On The Closed Form Solution

In this section, resonance phenomenon of our vibrating system is investigated, because the transverse displacement of elastic deep beam may grow without bound. Equation (42) clearly shows that the non-uniform elastic deep beam resting on variable elastic foundation will experience resonance effects whenever

$$4\left[A_2\left(j,k\right)B_3\left(j,k\right)-A_3\left(j,k\right)B_2\left(j,k\right)\right]=\left[A_1\left(j,k\right)B_3\left(j,k\right)+A_2\left(j,k\right)B_1\left(j,k\right)\right]^2 \tag{45}$$

$$\beta^2=\omega_1^2,\quad \beta^2=\omega_2^2 \tag{46}$$

$$\alpha^2=\omega_1^2,\quad \alpha^2=\omega_2^2 \tag{47}$$

It is also observed that as the foundation modulli increases the critical speed of the dynamical system increases thereby reducing the risk of resonant effects.

5 Comments On The Numerical Results

The theory presented in this paper is illustrated numerically. The velocity of the moving load and the length of the beam are respectively v=8.128m/s and L=12.192. The values of foundation moduli K are varied between 10000 N/m^3 and 150000 N/m^3 for $\omega=\frac{2\pi}{3}$.

Figure 1 displays the deflection profile of an elastic deep beams resting on variable elastic foundation and subjected to variable magnitude moving load. The figure shows that as the value of foundation stiffness K increases the deflection of the beam at various time t decreases. In Figures 2 and 3, the graphs of the critical speeds for resonant conditions (46) and (47) have been plotted against the foundation stiffness K. The graphs show that as the K_o increases, the critical speeds of the dynamical system increases.

Figure 4 depicts the deflection profile of non-uniform deep beam resting on elastic foundation and subjected to moving load for various load positions. It is deduced from this figure that the load position or the point of contact of the load on the structure affect the response amplitude

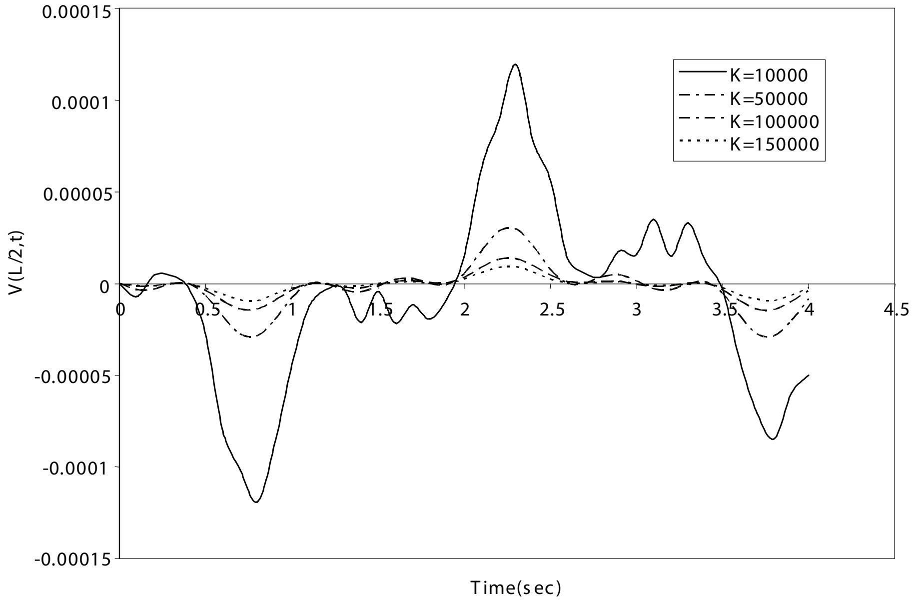

Figure 1: Deflection profile of a non-uniform deep beam under a moving load for various values of foundation stiffeness K.

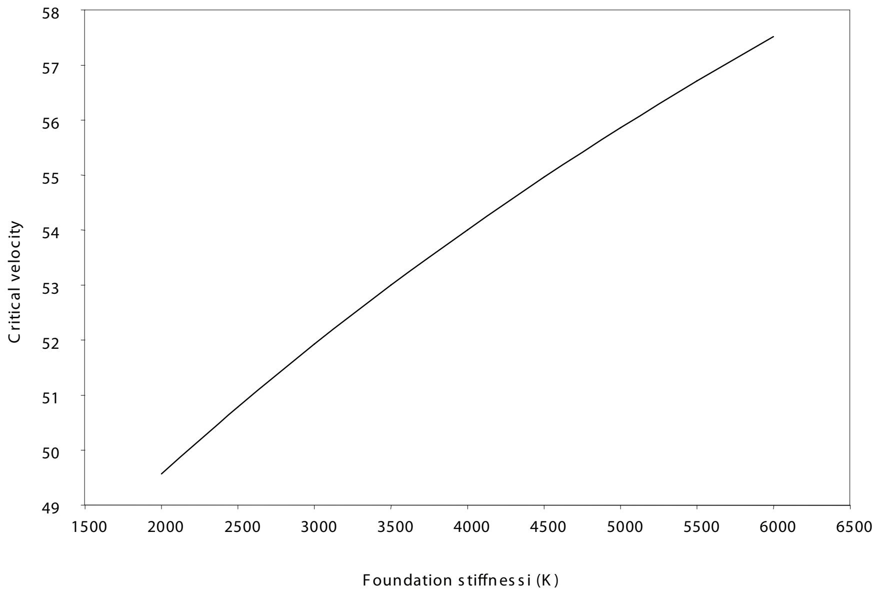

Figure 2: Critical velocity versus Foundation moduli K for $\alpha^2 = \omega_1^2$

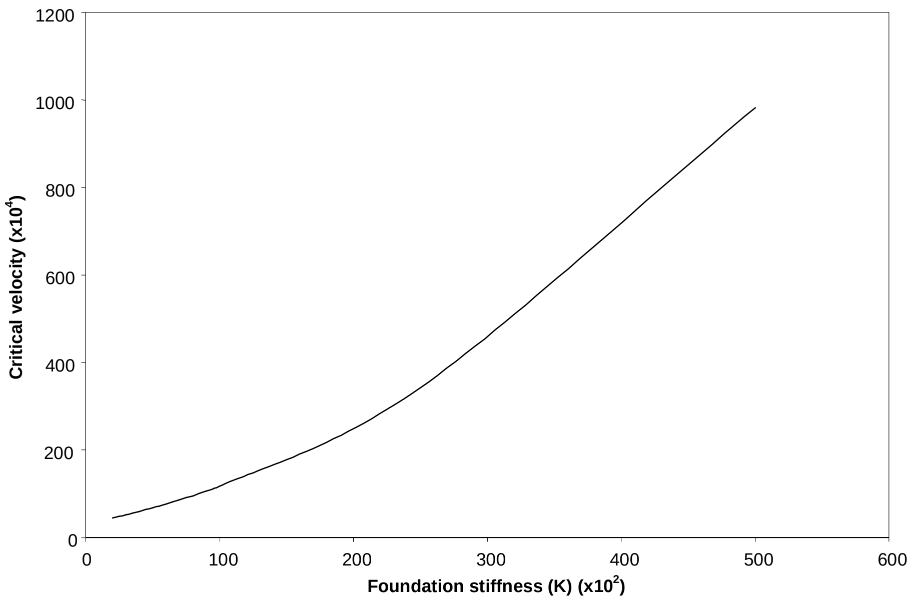

Figure 3: Critical velocity versus Foundation moduli K for $\beta^2 = \omega_1^2$

of the beam significantly. Figure 5 shows that as we increase the natural frequency w of the moving load, the transverse displacement response of the non-uniform deep beam under the action of the fast moving load reduces. Figure 6 clearly shows that the slight differences in the values of the natural frequency of the traveling load produces slight differences in the critical velocities of the vibrating system.

6 Concluding Remarks

In this paper, a procedure involving the Galerkin's method and integral transform technique has been used to solve the problem of a non-uniform deep beam when it is subjected to a harmonic variable magnitude moving load. The objective is to study the behaviour of the dynamical system. In particular, analytical solution in series form is obtained for the deflection and the rotation of the elastic deep beam and the effects of foundation stiffness K, the natural frequency w and the various load positions on the vibrating system are investigated. Analytical solution and Numerical result in plotted curves show that, as the value of foundation stiffness K increases the deflection profile of the non-uniform deep beam decreases. It is equally observed from figures 2 and 3 that the critical velocities of the dynamical systems increase with an increase in the values of foundation stiffness K. Thus, in general, higher values of foundation stiffness K reduce the risk of resonance in a dynamical system involving non-uniform beam under the action of a moving load.

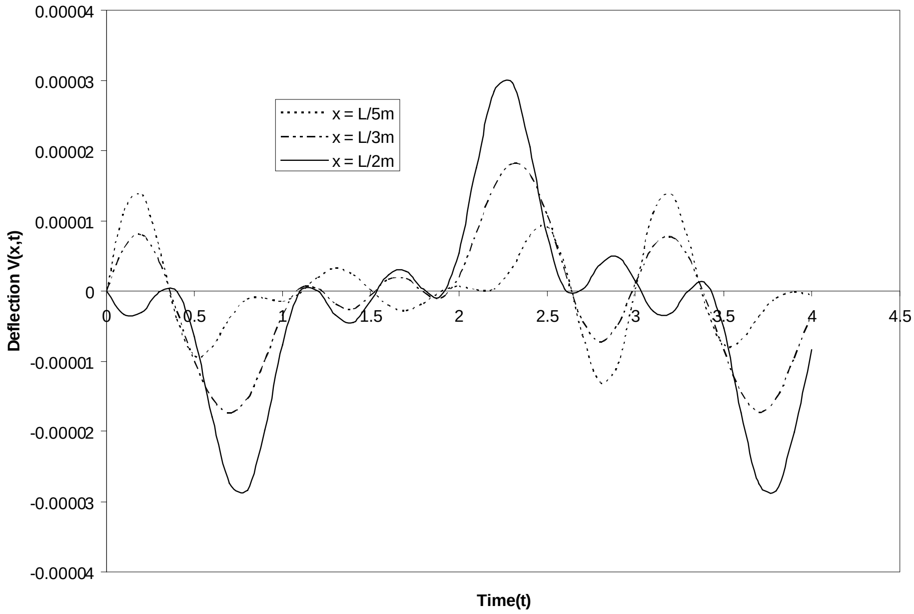

Figure 4: Deflection profile of non-uniform deep beam under a moving load at various load positions x for fixed value of foundation stiffeness K=50000

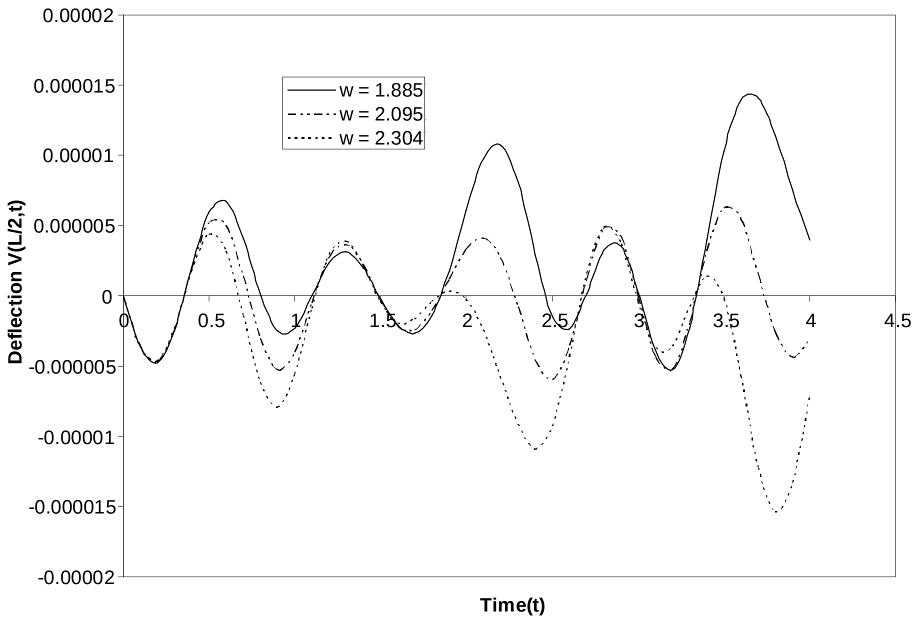

Figure 5: Deflection profile of non-uniform deep beam under moving load for various values of the load natural frequencies and for fixed value of foundation stiffness K = 50000

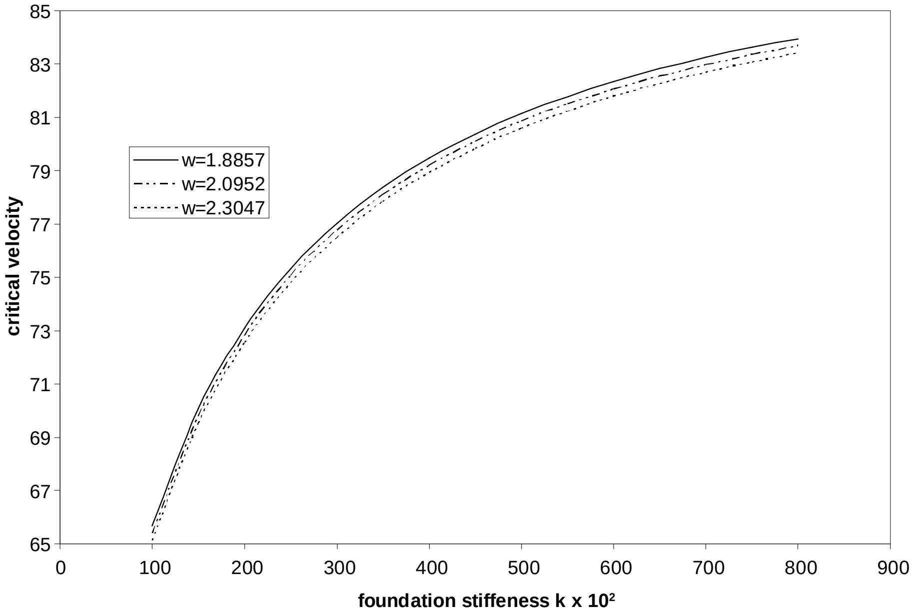

Figure 6: Comparison of the graph of critical velocity versus foundation stiffeness k for various values of natural frequency w

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