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Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation

Abstract

The dynamic response under a concentrated moving mass of an elastically supported non-prismatic Bernoulli-Euler beam resting on an elastic foundation with stiffness variation is investigated. For the solution of the fourth order partial differential equation with singular and variable coefficients, use is made of the technique based on the Generalized Galerkin's Method and the Struble's asymptotic technique. The numerical results are presented in plotted curves. The results show that the response amplitudes of the elastically supported non-prismatic Bernoulli-Euler beam decrease as the foundation modulli K increases. Also, the displacements of an elastically supported non-prismatic Bernoulli-Euler beam resting on a variable elastic foundation, for fixed value of K, decrease as the pre-stress N increases. The results again show that the critical speed for the moving mass problem is reached earlier than that for the moving force problem for the illustrative examples considered.

Keywords

pre-stress, resonance, Bernoulli-Euler, foundation modulli, moving mass, moving force.

S. T. Oni and T. O. Awodola*

Department of Mathematical Sciences, Federal University of Technology, Akure – Nigeria

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* Author email: oluthomas71@yahoo.com

1 INTRODUCTION

The study of dynamical behaviour of structures such as beams and plates, under the action of moving loads has attracted the attention of several researchers in Engineering, Applied Physics and Applied Mathematics. Notable among such researchers are Kolousek [6], Clastornik [2], Sadiku and Leipholz [14] and so on.

In a problem of beam under moving load like this, the effect of the mass of the load is of great importance since the position of the load changes continuously. Extensive work has been done on this class of dynamical problems when the structural members have uniform cross-sections. Work on practical problems involving non-uniform Bernoulli-Euler beam which is under the action of moving concentrated load is very rare in literature [3]. Such one-dimensional structures are of variable cross-sections and as such their properties vary with respect to the spatial coordinates along the span of the structure. Worthy of mention in this area of work is the work of Kolousek [6].

Recently, several researchers have made tremendous efforts in the study of dynamics of structures under moving loads, these include Oni [9], Gbadeyan and Oni [4], Adams [1], Savin [16], Jia-Jang [18]. In fact, Oni and Awodola [12] considered the flexural motions under moving loads of elastically supported beams resting on Winkler elastic foundation with stiffness variation. The technique was based on the generalized Galerkin's method and integral transformations and the beam was assumed to have uniform cross section. In all of these, considerations have been limited to cases of uniform beams. Where non-uniform beams are considered, they are considered only for classical boundary conditions.

Among the recent works is also the work of Oni [8] who considered the response of a non-uniform thin beam resting on a constant elastic foundation to several moving masses. For the solution of the problem, he used the versatile technique of Galerkin to reduce the complex governing fourth order partial differential equation with variable and singular coefficients to a set of ordinary differential equations. The set of ordinary differential equations was later simplified and solved using modified asymptotic method of Struble. This work, though impressive, was based only on beam with the classical simply supported end conditions. Other studies on non-uniform beam include Douglas et al [5], Oni and Awodola [10] and Oni and Omolofe [13]. I remark here that most of the studies in this area have been treated only for classical boundary conditions.

Nevertheless, for practical applications in many cases, it is more realistic to consider non-classical boundary conditions because the ideal boundary conditions can seldom be realized. A common example is the elastically supported end conditions. As a problem of this kind, Saito et al [15] presented a theoretical analysis of the steady state response of a plate strip constrained elastically along its edges against rotation and translation under the action of a moving transverse line load. The first five speeds of the applied load for which a resonance effect occurs in the system are plotted as functions of the edge constraint parameters, Wilson [17] studied the response of a cantilever plate strip restrained elastically against rotation and subjected to a moving normal line load. Also Muscolino et al [7] considered the response of beams resting on viscoelastically damped foundation to moving oscillators.

The results of these works on non-classical boundary conditions could seriously be misleading as only the force effect of the moving load is taken into consideration in their calculations while the inertia effect is neglected. Thus, the problem of the dynamic response under concentrated moving mass of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation is investigated in this paper.

2 THE GOVERNING EQUATION

The problem of the dynamic response of elastically supported non-uniform Bernoulli-Euler beam resting on variable Winkler elastic foundation and traversed by moving loads is governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 U(x,t)}{\partial x^2}\right]+\mu(x)\frac{\partial^2 U(x,t)}{\partial t^2}-N\frac{\partial^2 U(x,t)}{\partial x^2}$$
$$+M\delta(x-ct)\left[\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right]U(x,t)+k(x)U(x,t)=Mg\delta(x-ct) \tag{1}$$

where E is the Young's Modulus, $U(x,t)$ is the transverse displacement, $k(x)$ is the variable elastic foundation, μ(x) is the variable mass per unit length of the beam, $I(x)$ is the variable moment of inertia, N is the pre-stress and x, t are respectively spatial and time coordinates.

An example of variable elastic foundation of the form [3]

$$k(x)=K(4x-3x^2+x^3) \tag{2}$$

is adopted, where K is the foundation modulus.

Next, the example in Oni [8] shall be adopted and $I(x)$ and $\mu(x)$ take the forms

$$I(x)=I_0\left(1+\sin\frac{\pi x}{L}\right)^3 \tag{3}$$

and

$$\mu(x)=\mu_0\left(1+\sin\frac{\pi x}{L}\right) \tag{4}$$

where I_0 and μ_0 are constants.

Substituting equations (2) and (3) and (4) into equation (1), one obtains

$$EI_0\frac{\partial^2}{\partial x^2}\left[\left(1+\sin\frac{\pi x}{L}\right)^3\frac{\partial^2 U(x,t)}{\partial x^2}\right]+\mu_0\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial t^2}-N\frac{\partial^2 U(x,t)}{\partial x^2}$$
$$+M\delta(x-ct)\left[\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right]U(x,t)+K(4x-3x^2+x^3)U(x,t)=Mg\delta(x-ct) \tag{5}$$

which, on further simplification, yields

$$\frac{EI_0}{4}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)\frac{\partial^4 U(x,t)}{\partial x^4}\right.$$
$$+\left(\frac{30\pi}{L}\cos\frac{\pi x}{L}+\frac{24\pi}{L}\sin\frac{2\pi x}{L}-\frac{6\pi}{L}\cos\frac{3\pi x}{L}\right)\frac{\partial^3 U(x,t)}{\partial x^3}$$
$$\left.+\left(\frac{24\pi^2}{L^2}\cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}\sin\frac{\pi x}{L}+\frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial x^2}\right]$$
$$+\mu_0\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial t^2}-N\frac{\partial^2 U(x,t)}{\partial x^2}+M\delta(x-ct)\left[\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right]U(x,t)$$
$$+K\left(4x-3x^2+x^3\right)U(x,t)=Mg\delta(x-ct) \tag{6}$$

We shall consider the case when the non-uniform Bernoulli-Euler beam has classical boundary conditions at the end $x = 0$ and is elastically supported at the other end $x = L$. We shall thereafter consider the Bernoulli-Euler beam elastically supported at both ends.

Thus, in the first instance, the associated boundary conditions at $x = 0$ can be any of

$$U(0,t) = 0 = U^{1}(0,t) \tag{7a}$$

$$U(0,t) = 0 = U^{11}(0,t) \tag{7b}$$

$$U^{111}(0,t) = 0 = U^{11}(0,t) \text{ and} \tag{7c}$$

$$U^{1}(0,t) = 0 = U^{111}(0,t) \tag{7d}$$

while for the end $x = L$, we have

$$U^{11}(L,t) - k_1 U^{1}(L,t) = 0 \text{ and} \tag{8a}$$

$$U^{111}(L,t) + k_2 U(L,t) = 0 \tag{8b}$$

where k_1 is the stiffness against rotation and k_2 is the stiffness against translation. It is clear that when $k_1 = 0$ and $k_2 = \infty$ we have the simply supported end, when $k_1 = \infty$ and $k_2 = \infty$ we have the Clamped end, when $k_1 = 0$ and $k_2 = 0$ we have the Free end and when $k_1 = \infty$ and $k_2 = 0$ we have the Sliding end.

A Bernoulli-Euler beam elastically supported at both ends is considered next, and the boundary conditions are:

$$U^{11}(0,t) - k_1 U^{1}(0,t) = 0 \text{ and } U^{111}(0,t) + k_2 U(0,t) = 0 \text{ at } x = 0 \tag{9a}$$

and

$$U^{11}(L,t) - k_1 U^{1}(L,t) = 0 and U^{111}(L,t) + k_2 U(L,t) = 0 \text{ at } x = L \tag{9b}$$

The initial conditions without any loss of generality are taken as

$$U(x,0) = 0 = \frac{\partial U(x,0)}{\partial t} \tag{10}$$

3 ANALYTICAL APPROXIMATE SOLUTION

Evidently, a closed form solution of the partial differential equation (6) does not exist. Thus, the Galerkin's method described in Oni and Awodola [11] is employed to reduce the equation to a sequence of ordinary differential equations. Thus a solution of the form

$$U_n(x,t) = \sum_{m=1}^{n} W_m(t) V_m(x), \tag{11}$$

where $V_m(x)$ is chosen such that the desired boundary conditions are satisfied, is sought.

Equation (11) when substituted into equation (6) yields

$$\begin{aligned}
&\sum_{m=1}^{n}\left\{\frac{EI_0}{4}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)V_m^{iv}(x)\right.\right.\\
&+\left(\frac{30\pi}{L}\cos\frac{\pi x}{L}+\frac{24\pi}{L}\sin\frac{2\pi x}{L}-\frac{6\pi}{L}\cos\frac{3\pi x}{L}\right)V_m^{111}(x)\\
&\left.+\left(\frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}+\frac{24\pi^2}{L^2}\cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}\sin\frac{\pi x}{L}\right)V_m^{11}(x)\right]W_m(t)\\
&+\mu_0\left(1+\sin\frac{\pi x}{L}\right)V_m(x)\ddot{\underset{m}{W}}(t)-NV_m^{11}(x)W_m(t)\\
&+M\delta(x-ct)\left[V_m(x)\ddot{\underset{m}{W}}(t)+2cV_m^1(x)\dot{\underset{m}{W}}(t)+c^2V_m^{11}(x)W_m(t)\right]\\
&\left.+K\left(4x-3x^2+x^3\right)V_m(x)W_m(t)-Mg\delta(x-ct)\right\}=0
\end{aligned} \tag{12}$$

In order to determine $W_m(t)$, it is required that the expression on the left hand side of equation (12) be orthogonal to the function $V_k(x)$. Thus,

$$\begin{aligned}
&\int_0^L\sum_{m=1}^{n}\left\{\frac{EI_0}{4}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)V_m^{iv}(x)\right.\right.\\
&+\left(\frac{30\pi}{L}\cos\frac{\pi x}{L}+\frac{24\pi}{L}\sin\frac{2\pi x}{L}-\frac{6\pi}{L}\cos\frac{3\pi x}{L}\right)V_m^{111}(x)\\
&\left.+\left(\frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}+\frac{24\pi^2}{L^2}\cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}\sin\frac{\pi x}{L}\right)V_m^{11}(x)\right]W_m(t)\\
&+\mu_0\left(1+\sin\frac{\pi x}{L}\right)V_m(x)\ddot{\underset{m}{W}}(t)-NV_m^{11}(x)W_m(t)\\
&+M\delta(x-ct)\left[V_m(x)\ddot{\underset{m}{W}}(t)+2cV_m^1(x)\dot{\underset{m}{W}}(t)+c^2V_m^{11}(x)W_m(t)\right]\\
&\left.+K\left(4x-3x^2+x^3\right)V_m(x)W_m(t)-Mg\delta(x-ct)\right\}V_k(x)dx=0
\end{aligned} \tag{13}$$

When equation (13) is further simplified and rearranged, one obtains

$$\begin{aligned}
&\sum_{m=1}^{n}\left\{[\rho_1+\rho_2]\ddot{\underset{m}{W}}(t)+\frac{EI_0}{4\mu_0}\left[10\rho_3+15\rho_4-6\rho_5-\rho_6+\frac{6\pi}{L}(5\rho_7+4\rho_8-\rho_9)\right.\right.\\
&\left.+\frac{3\pi^2}{L^2}\left(3\rho_{10}+8\rho_{11}-5\rho_{12}-\frac{4L^2N}{3EI_0\pi^2}\rho_{13}\right)+\frac{4K}{EI_0}(4\rho_{14}-3\rho_{15}+\rho_{16})\right]W_m(t)\\
&\left.+\frac{M}{\mu_0}\left[\rho_{17}(t)\ddot{\underset{m}{W}}(t)+2c\rho_{18}(t)\dot{\underset{m}{W}}(t)+c^2\rho_{19}(t)W_m(t)\right]\right\}=\frac{Mg}{\mu_0}\rho_{20}(t)
\end{aligned} \tag{14}$$

where

$$\rho_1 = \int_0^L V_m(x)V_k(x)dx\,;\ \rho_2 = \int_0^L \sin\frac{\pi x}{L}V_m(x)V_k(x)dx\ ;$$
$$\rho_3 = \int_0^L V_m^{iv}(x)V_k(x)dx\ ;\ \rho_4 = \int_0^L \sin\frac{\pi x}{L}V_m^{iv}(x)V_k(x)dx\ ;$$
$$\rho_5 = \int_0^L \cos\frac{2\pi x}{L}V_m^{iv}(x)V_k(x)dx\ ;\ \rho_6 = \int_0^L \sin\frac{3\pi x}{L}V_m^{iv}(x)V_k(x)dx\ ;$$
$$\rho_7 = \int_0^L \cos\frac{\pi x}{L}V_m^{111}(x)V_k(x)dx\ ;\ \rho_8 = \int_0^L \sin\frac{2\pi x}{L}V_m^{111}(x)V_k(x)dx\ ;$$
$$\rho_9 = \int_0^L \cos\frac{3\pi x}{L}V_m^{111}(x)V_k(x)dx\ ;\ \rho_{10} = \int_0^L \sin\frac{3\pi x}{L}V_m^{11}(x)V_k(x)dx\ ;$$
$$\rho_{11} = \int_0^L \cos\frac{2\pi x}{L}V_m^{11}(x)V_k(x)dx\ ;\ \rho_{12} = \int_0^L \sin\frac{\pi x}{L}V_m^{11}(x)V_k(x)dx\ ;$$
$$\rho_{13} = \int_0^L V_m^{11}(x)V_k(x)dx\ ;\ \rho_{14} = \int_0^L xV_m(x)V_k(x)dx\ ;$$
$$\rho_{15} = \int_0^L x^2V_m(x)V_k(x)dx\ ;\ \rho_{16} = \int_0^L x^3V_m(x)V_k(x)dx\ ;$$
$$\rho_{17} = \int_0^L \delta(x-ct)\,V_m(x)V_k(x)dx\ ;\ \rho_{18} = \int_0^L \delta(x-ct)\,V_m^1(x)V_k(x)dx\ ;$$
$$\rho_{19} = \int_0^L \delta(x-ct)\,V_m^{11}(x)V_k(x)dx\ ;\ \rho_{20} = \int_0^L \delta(x-ct)\,V_k(x)dx\ ;$$

The Dirac-delta function as an even function can be expressed as

$$\delta(x-ct) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\cos\frac{n\pi x}{L} \tag{15}$$

Substituting (15) into equation (14), the equation, after some rearrangements takes the form

$$\begin{aligned}
&\sum_{m=1}^{n}\Bigg\{\beta_1(m,k)\,\ddot{W}_m(t) + \beta_2(m,k)\ W_m(t)\\
&+\frac{M}{L\mu_0}\Bigg[\left(\beta_{3A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\beta_{3B}(n,m,k)\right)\ddot{W}_m(t)\\
&+2c\left(\beta_{4A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\beta_{4B}(n,m,k)\right)\dot{W}_m(t)\\
&+c^2\left(\beta_{5A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\beta_{5B}(n,m,k)\right)W_m(t)\Bigg]\Bigg\} = \frac{Mg}{\mu_0}V_k(ct)
\end{aligned} \tag{16}$$

where

$$\beta_1(m,k) = \rho_1 + \rho_2\,;$$

$$\beta_2(m,k) = \frac{EI_0}{4\mu_0}\left[10\rho_3 + 15\rho_4 - 6\rho_5 - \rho_6 + \frac{6\pi}{L}(5\rho_7 + 4\rho_8 - \rho_9) + \frac{3\pi^2}{L^2}\left(3\rho_{10} + 8\rho_{11} - 5\rho_{12} - \frac{4L^2N}{3EI_0\pi^2}\rho_{13}\right) + \frac{4K}{EI_0}(4\rho_{14} - 3\rho_{15} + \rho_{16})\right]$$

$$\beta_{3A}(m,k) = \int_0^L V_m(x)\,V_k(x)dx\ ;$$

$$\beta_{3B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V_m(x)\,V_k(x)dx\ ;$$

$$\beta_{4A}(m,k) = \int_0^L V_m^1(x)\,V_k(x)dx\ ;$$

$$\beta_{4B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V_m^1(x)\,V_k(x)dx\ ;$$

$$\beta_{5A}(m,k) = \int_0^L V_m^{11}(x)\,V_k(x)dx\ ;$$

$$\beta_{5B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V_m^{11}(x)\,V_k(x)dx$$

A solution valid for all cases of boundary conditions is sought. Consequently, $V_m(x)$ is chosen as the beam function given as

$$V_m(x) = \sin\frac{\theta_m x}{L} + A_m\cos\frac{\theta_m x}{L} + B_m\sinh\frac{\theta_m x}{L} + C_m\cosh\frac{\theta_m x}{L} \tag{18}$$

where the constants A_m, B_m, C_m, and the mode frequency θ_m are determined by using the desired ends support conditions. Thus neglecting the summation sign and substituting (18) into equation (16) yields

$$\begin{aligned}&\Big\{\beta_1(m,k)\ddot{W}_m(t) + \beta_2(m,k)\ W_m(t)\\ &+ \Gamma_a\Big[\Big(\beta_{3A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\beta_{3B}(n,m,k)\Big)\ddot{W}_m(t)\\ &+ 2c\Big(\beta_{4A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\beta_{4B}(n,m,k)\Big)\dot{W}_m(t)\\ &+ c^2\Big(\beta_{5A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\beta_{5B}(n,m,k)\Big)W_m(t)\Big]\Big\}\\ &= \frac{Mg}{\mu_0}\left[\sin\frac{\theta_k ct}{L} + A_k\cos\frac{\theta_k ct}{L} + B_k\sinh\frac{\theta_k ct}{L} + C_k\cosh\frac{\theta_k ct}{L}\right]\end{aligned} \tag{19}$$

where

$$\Gamma_a = \frac{M}{L\mu_0} \tag{20}$$

Evidently, an exact analytical solution to equation (19) is not possible. Thus a modification of Struble's technique discussed in Oni [8] is employed. Consequently, equation (19) is rearranged to take the form

$$\ddot{\underset{m}{W}}(t) + \frac{2c\Gamma_a\left(\beta_{4A}(m,k) + 2\beta_{4B}(m,k)\cos\frac{\pi ct}{L}\right)}{\beta_1(m,k) + \Gamma_a\left(\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right)}\dot{\underset{m}{W}}(t) + \frac{\left[\beta_2(m,k) + c^2\Gamma_a\left(\beta_{5A}(m,k) + 2\beta_{5B}(m,k)\cos\frac{\pi ct}{L}\right)\right]}{\beta_1(m,k) + \Gamma_a\left(\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right)}W_m(t) = \frac{\Gamma_a gL\left[\sin\frac{\theta_k ct}{L} + A_k\cos\frac{\theta_k ct}{L} + B_k\sinh\frac{\theta_k ct}{L} + C_k\cosh\frac{\theta_k ct}{L}\right]}{\beta_1(m,k) + \Gamma_a\left(\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right)} \tag{21}$$

The homogenous part of equation (21) is first considered and a modified frequency corresponding to the frequency of the free system due to the presence of moving mass is sought. An equivalent free system operator defined by the modified frequency then replaces equation (21). To do this, consider a parameter $\varepsilon < 1$ for any arbitrary mass ratio Γ_a defined as

$$\varepsilon = \frac{\Gamma_a}{1 + \Gamma_a} \tag{22}$$

It follows that

$$\Gamma_a = \varepsilon\left[1 + o(\varepsilon) + o(\varepsilon^2) + \ldots\right] \tag{23}$$

Consequently,

$$\frac{1}{\beta_1(m,k) + \varepsilon\left(\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right)} = \frac{1}{\beta_1(m,k)}\left[1 - \frac{1}{\beta_1(m,k)}\varepsilon\left(\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right) + o(\varepsilon^2) + \ldots\right] \tag{24}$$

where

$$\left|\frac{\varepsilon}{\beta_1(m,k)}\left(\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right)\right| < 1 \tag{25}$$

which implies that all the coefficients of $W_m(t)$ and its derivatives in equation (21) can be written in terms of the parameter ϵ. When ϵ is set to zero in equation (21), a situation corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible is obtained. In such a case, the solution is of the form

$$W_m(t) = C_m\cos(\gamma_m t - \phi_m) \tag{26}$$

where

$$C_m \text{ and } \phi_m \text{ are constants and } \gamma_m^2 = \frac{\beta_2(m,k)}{\beta_1(m,k)} \tag{27}$$

However, since for any arbitrary mass ratio Γ_a, we always have $\varepsilon < 1$, then Struble's technique requires that the solution of the homogenous part of equation (21) be given in an asymptotic form, namely

$$W_m(t) = \varphi(m,t)\cos[\gamma_m t - \Omega(m,t)] + \varepsilon W_1(t) + o(\varepsilon^2) \tag{28}$$

Substituting equation (28) and its derivatives into the homogenous part of equation (21) one obtains

$$\begin{aligned}
&-2\gamma_m \dot{\varphi}(m,t)\sin[\gamma_m t - \Omega(m,t)] + 2\gamma_m\varphi(m,t)\dot{\Omega}(m,t)\cos[\gamma_m t - \Omega(m,t)] - \varphi(m,t)\gamma_m^2\cos[\gamma_m t - \Omega(m,t)] \\
&+ \frac{2c\varepsilon}{\beta_1(m,k)}\left[\beta_{4A}(m,k) + 2\beta_{4B}(m,k)\cos\frac{\pi ct}{L}\right][-\varphi(m,t)\gamma_m\sin[\gamma_m t - \Omega(m,t)]] \\
&+ \left\{\frac{\beta_2(m,k)}{\beta_1(m,k)} - \frac{\varepsilon\beta_2(m,k)}{\beta_1^2(m,k)}\left[\beta_{3A}(m,k) + 2\beta_{3B}(m,k)\cos\frac{\pi ct}{L}\right]\right. \\
&\left.+\frac{c^2\varepsilon}{\beta_1(m,k)}\left[\beta_{5A}(m,k) + 2\beta_{5B}(m,k)\cos\frac{\pi ct}{L}\right]\right\}[\varphi(m,t)\cos[\gamma_m t - \Omega(m,t)]] = 0
\end{aligned} \tag{29}$$

where terms higher than $o(\varepsilon)$ have been neglected.

The variational equations are obtained by equating the coefficients of $\sin[\gamma_m t - \Omega(m,t)]$ and $\cos[\gamma_m t - \Omega(m,t)]$ terms on both sides of the equation (29). Thus, we note that

$$\cos\frac{\pi ct}{L}\sin[\gamma_m t - \Omega(m,t)] = \frac{1}{2}\sin\left[\frac{\pi ct}{L} + \gamma_m t - \Omega(m,t)\right] + \frac{1}{2}\sin\left[\gamma_m t - \Omega(m,t) - \frac{\pi ct}{L}\right]$$

and

$$\cos\frac{\pi ct}{L}\cos[\gamma_m t - \Omega(m,t)] = \frac{1}{2}\cos\left[\frac{\pi ct}{L} + \gamma_m t - \Omega(m,t)\right] + \frac{1}{2}\cos\left[\frac{\pi ct}{L} - \gamma_m t + \Omega(m,t)\right]$$

and neglecting those terms that do not contribute to the variational equations, equation (29) reduces to

$$\begin{aligned}
&2\gamma_m\varphi(m,t)\dot{\Omega}(m,t)\cos[\gamma_m t - \Omega(m,t)] - 2\gamma_m\dot{\varphi}(m,t)\sin[\gamma_m t - \Omega(m,t)] \\
&-\varphi(m,t)\gamma_m^2\cos[\gamma_m t - \Omega(m,t)] - \frac{2c\varepsilon}{\beta_1(m,k)}\varphi(m,t)\gamma_m\beta_{4A}(m,k)\sin[\gamma_m t - \Omega(m,t)] \\
&+\gamma_m^2\varphi(m,t)\cos[\gamma_m t - \Omega(m,t)] - \frac{\varepsilon\gamma_m^2}{\beta_1(m,k)}\beta_{3A}(m,k)\varphi(m,t)\cos[\gamma_m t - \Omega(m,t)] \\
&+\frac{c^2\varepsilon}{\beta_1(m,k)}\beta_{5A}(m,k)\varphi(m,t)\cos[\gamma_m t - \Omega(m,t)] = 0
\end{aligned} \tag{30}$$

The variational equations of the problem are obtained respectively as

$$2\gamma_m\varphi(m,t)\,\dot{\Omega}(m,t) - \frac{\varepsilon\gamma_m^2\beta_{3A}(m,k)\varphi(m,t)}{\beta_1(m,k)} + \frac{\varepsilon c^2\beta_{5A}(m,k)\varphi(m,t)}{\beta_1(m,k)} = 0 \tag{31}$$

and

$$\gamma_m\,\dot{\varphi}(m,t) + \frac{\varepsilon c\gamma_m\beta_{4A}(m,k)\varphi(m,t)}{\beta_1(m,k)} = 0 \tag{32}$$

Rearranging equations (31) and (32), we have

$$\dot{\Omega}(m,t) = \frac{\varepsilon[\gamma_m^2\beta_{3A}(m,k) - c^2\beta_{5A}(m,k)]}{2\gamma_m\beta_1(m,k)} \tag{33}$$

and

$$\dot{\varphi}(m,t) = \frac{-\varepsilon c\beta_{4A}(m,k)\varphi(m,t)}{\beta_1(m,k)} = 0 \tag{34}$$

Solving equations (33) and (34) respectively yields

$$\Omega(m,t) = \frac{\varepsilon[\gamma_m^2\beta_{3A}(m,k) - c^2\beta_{5A}(m,k)]\,t}{2\gamma_m\beta_1(m,k)} + \Omega_m \tag{35}$$

where Ω_m is a constant and

$$\varphi(m,t) = \phi_0 e^{(-\eta^0 t)} \tag{36}$$

where $\eta^0 = \frac{\varepsilon c\beta_{4A}(m,k)}{\beta_1(m,k)}$ and ϕ_0 is a constant.

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogenous system is

$$W_m(t) = \varphi(m,t)\cos[\omega_m t - \Omega_m] \tag{37}$$

where

$$\omega_m = \gamma_m - \frac{\varepsilon[\gamma_m^2\beta_{3A}(m,k) - c^2\beta_{5A}(m,k)]}{2\gamma_m\beta_1(m,k)} \tag{38}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass.

In view of (37), the homogenous part of the equation (21) can be written as

$$\frac{d^2W_m(t)}{dt^2} + \omega_m^2 W_m(t) = 0 \tag{39}$$

while the entire equation (21) takes the form

$$\frac{d^2W_m(t)}{dt^2} + \omega_m^2 W_m(t) = R_0\left[\sin\frac{\theta_k ct}{L} + A_k\cos\frac{\theta_k ct}{L} + B_k\sinh\frac{\theta_k ct}{L} + C_k\cosh\frac{\theta_k ct}{L}\right] \tag{40}$$

where $R_0 = \frac{\varepsilon L g}{\beta_1(m,k)}$.

Using Laplace transformation technique and the convolution theory, expression for $W_m(t)$ is obtained and in view of equation (11) taking (18) into consideration, one obtains

$$U_n(x,t) = \sum_{m=1}^{n} \frac{R_0}{\omega_m\left[\omega_m^4 - p_k^4\right]} \left\{ \left[\omega_m^2 + p_k^2\right] \left[A_k\omega_m\left(\cos(p_k t) - \cos(\omega_m t)\right) - \left(p_k \sin(\omega_m t) - \omega_m \sin(p_k t)\right)\right] \right.$$
$$\left. + \left[\omega_m^2 - p_k^2\right] \left[C_k\omega_m\left(\cosh(p_k t) - \cos(\omega_m t)\right) + B_k\left(\omega_m \sinh(p_k t) - p_k \sin(\omega_m t)\right)\right] \right\} \left[\sin\frac{\theta_m x}{L} + A_m \cos\frac{\theta_m x}{L}\right.$$
$$\left. + B_m \sinh\frac{\theta_m x}{L} + C_m \cosh\frac{\theta_m x}{L}\right] \tag{41}$$

where
$$p_k = \frac{\theta_k c}{L} \tag{42}$$

Equation (41) represents the response to a moving mass of an elastically supported non-prismatic Bernoulli-Euler beam resting on a variable elastic foundation. The corresponding moving force solution is

$$U_n(x,t) = \sum_{m=1}^{n} \frac{A_m^0}{\gamma_m\left[\gamma_m^4 - p_k^4\right]} \left\{ \left[\gamma_m^2 + p_k^2\right] \left[A_k\gamma_m\left(\cos(p_k t) - \cos(\gamma_m t)\right) - \left(p_k \sin(\gamma_m t) - \gamma_m \sin(p_k t)\right)\right] \right.$$
$$\left. + \left[\gamma_m^2 - p_k^2\right] \left[C_k\gamma_m\left(\cosh(p_k t) - \cos(\gamma_m t)\right) + B_k\left(\gamma_m \sinh(p_k t) - p_k \sin(\gamma_m t)\right)\right] \right\} \left[\sin\frac{\theta_m x}{L} + A_m \cos\frac{\theta_m x}{L}\right.$$
$$\left. + B_m \sinh\frac{\theta_m x}{L} + C_m \cosh\frac{\theta_m x}{L}\right] \tag{43}$$

where $A_m^0 = \frac{Mg}{\mu_0\beta_1(m,k)}$.

4 ILLUSTRATIVE EXAMPLES

For the illustration of the results in the foregoing analysis, we provide some examples;

1. Simple-Elastic non-uniform Bernoulli-Euler beam
2. Elastic-Elastic non-uniform Bernoulli-Euler beam

4.1 Simple-elastic non-uniform Bernoulli-Euler beam

In this case, the beam is simply supported at one end and elastically supported at the other end. Hence, the deflection and the bending moment vanish for the non-uniform Bernoulli-Euler beam simply supported at the end $x = 0$. Thus,

$$U(0,t) = 0 = U^{11}(0,t) \tag{44}$$

while at the other end $x = L$, the beam is elastically supported and we have

$$U^{11}(L,t) - k_1 U^1(L,t) = 0 = U^{111}(L,t) + k_2 U(L,t) \tag{45}$$

and hence for the normal modes

$$V_m(0) = 0 = V_m^{11}(0) \ \text{ at } \ x = 0 \tag{46}$$

and

$$V_m^{11}(L) - k_1 V_m^1(L) = 0 = V_m^{111}(L) + k_2 V_m(L) \ \text{ at } \ x = L \tag{47}$$

It can therefore be shown that

$$A_m = C_m = 0 \; and \; B_m = \frac{k_1 \cos\theta_m + \frac{\theta_m}{L}\sin\theta_m}{\frac{\theta_m}{L}\sinh\theta_m - k_1\cosh\theta_m} = \frac{\frac{\theta_m^3}{L^3}\cos\theta_m - k_2\sin\theta_m}{\frac{\theta_m^3}{L^3}\cosh\theta_m + k_2\sinh\theta_m} \tag{48}$$

Thus we have

$$\tan\theta_m = \tanh\theta_m \tag{49}$$

as the frequency equation for the dynamical problem, such that

$$\theta_1 = 3.927\,, \ \ \theta_2 = 7.069\,, \ \ \theta_3 = 10.210, \ldots \tag{50}$$

Using (48) and (50) in equations (41) and (43) one obtains the displacement response respectively to a moving mass and a moving force of simple-elastic ends non-uniform Bernoulli-Euler beam on a variable elastic foundation.

4.2 Elastic-elastic non-uniform Bernoulli-Euler beam

Here, the non-uniform Bernoulli-Euler beam on variable foundation is taken to be elastically supported both at both ends $x = 0$ and $x = L$, the conditions are expressed as

$$U^{11}(0,t) - k_1 U^1(0,t) = 0 = U^{111}(0,t) + k_2 U(0,t) \tag{51}$$

and

$$U^{11}(L,t) - k_1 U^1(L,t) = 0 = U^{111}(L,t) + k_2 U(L,t) \tag{52}$$

Similarly, for normal modes

$$V_m^{11}(0) - k_1 V_m^1(0) = 0 = V_m^{111}(0) + k_2 V_m(0) \tag{53}$$

and

$$V_m^{11}(L) - k_1 V_m^1(L) = 0 = V_m^{111}(L) + k_2 V_m(L) \tag{54}$$

Thus, following the same procedure, we have

$$C_m = \frac{\left[\frac{\theta_m}{L} - k_1 r_2\right]\sin\theta_m + \left[k_1 + \frac{r_2\theta_m}{L}\right]\cos\theta_m - \frac{r_1\theta_m}{L}\sinh\theta_m + k_1 r_1\cosh\theta_m}{k_1 r_1\sin\theta_m - \frac{r_1\theta_m}{L}\cos\theta_m + \left[\frac{r_3\theta_m}{L} - k_1\right]\sinh\theta_m + \left[\frac{\theta_m}{L} - k_1 r_3\right]\cosh\theta_m}$$
$$= \frac{-\left[\frac{r_2\theta_m^3}{L^3} + k_2\right]\sin\theta_m + \left[\frac{\theta_m^3}{L^3} - k_2 r_2\right]\cos\theta_m - k_2 r_1\sinh\theta_m - \frac{r_1\theta_m^3}{L^3}\cosh\theta_m}{\frac{r_1\theta_m^3}{L^3}\sin\theta_m + k_2 r_1\cos\theta_m + \left[\frac{\theta_m^3}{L^3} + k_2 r_3\right]\sinh\theta_m + \left[\frac{r_3\theta_m^3}{L^3} + k_2\right]\cosh\theta_m} \tag{55}$$

$$A_m = r_1 C_m + r_2 \ \ and\ \ B_m = r_3 C_m + r_1 \tag{56}$$

where

$$r_1 = \frac{\frac{\theta_m^4}{L^4} + k_1 k_2}{\frac{\theta_m^4}{L^4} - k_1 k_2}\ ; \quad r_2 = \frac{-\frac{2k_1\theta_m^3}{L^3}}{\frac{\theta_m^4}{L^4} - k_1 k_2}\ \ and\ r_3 = \frac{\frac{-2k_2\theta_m}{L}}{\frac{\theta_m^4}{L^4} - k_1 k_2}\ .$$

The frequency equation for the dynamical problem is obtained as

$$\tan\theta_m = \tanh\theta_m \tag{57}$$

such that

$$\theta_1 = 3.927\,,\ \ \theta_2 = 7.069\,,\ \ \theta_3 = 10.210,\ \ldots \tag{58}$$

Using equations (55), (56) and (58) in equations (41) and (43) one obtains the displacement response respectively to a moving mass and a moving force of non-uniform Bernoulli-Euler beam elastically supported at both ends and resting on a variable Winkler elastic foundation.

5 DISCUSSION OF THE ANALYTICAL SOLUTIONS

We shall examine the phenomenon of resonance in this section. Equation (43) reveals clearly that the non-uniform Bernoulli-Euler beam on a variable foundation and traversed by a moving force encounters a resonance effect when

$$\gamma_m = \frac{\theta_k c}{L} \tag{59}$$

while equation (41) shows that the same beam under the action of a moving mass reaches the state of resonance whenever

$$\omega_m = \frac{\theta_k c}{L} \tag{60}$$

where $\omega_m = \gamma_m - \frac{\varepsilon[\gamma_m^2\beta_{3A}(m,k) - c^2\beta_{5A}(m,k)]}{2\gamma_m\beta_1(m,k)}$.

Thus,

$$\omega_m = \frac{\gamma_m\left[\beta_1(m,k) - \frac{\varepsilon}{2}\left(\beta_{3A}(m,k) - \frac{c^2\beta_{5A}(m,k)}{\gamma_m^2}\right)\right]}{\beta_1(m,k)} = \frac{\theta_k c}{L} \tag{61}$$

Clearly $1 - \frac{\varepsilon}{2}\left(\frac{\gamma_m^2 \beta_{3A}(m,k) - c^2 \beta_{5A}(m,k)}{\gamma_m^2 \beta_1(m,k)}\right) < 1$ for all m.

Consequently, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, the resonance is reached earlier in the moving mass system than in the moving force system.

6 NUMERICAL CALCULATION AND DISCUSSIONS OF RESULTS

To illustrate the foregoing analysis, the non-uniform Bernoulli-Euler beam of length 12.192m is considered. Furthermore, the moving load is assumed to travel at the constant velocity of 8.123m/s, EI and M/Lμ are chosen to be 6.068x10^{6}m^3/s^2 and 0.25 respectively. The results are as shown on the various graphs below for the classes of boundary conditions so far considered.

6.1 Simple-elastic ends

Figures 1 and 2 present the effect of pre-stress (N) on the transverse deflection of the non-uniform beam, simply supported at one end and elastically supported at the other end, in both cases of moving force and moving mass respectively. The graphs show that an increase in the pre-stress decreases the deflection of the beam.

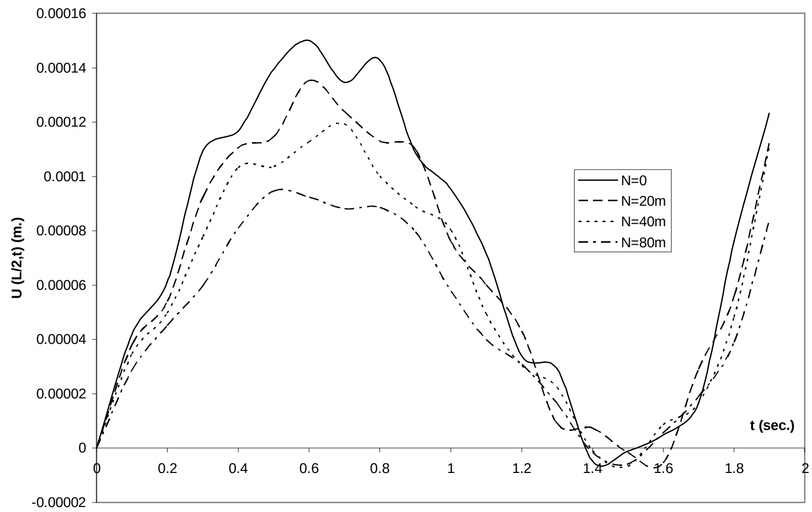

Figure 1 Deflection of moving force for Simple-Elastic non-uniform Bernoulli-Euler beam on variable elastic foundation for various values of pre-stress N.

For the purpose of comparison, the displacement curves of the moving force and moving mass for the beam, with one end simply supported and the other end elastically supported are illustrated in Figure 3. It is seen that the response amplitude of a moving mass is greater than that of a moving force problem.

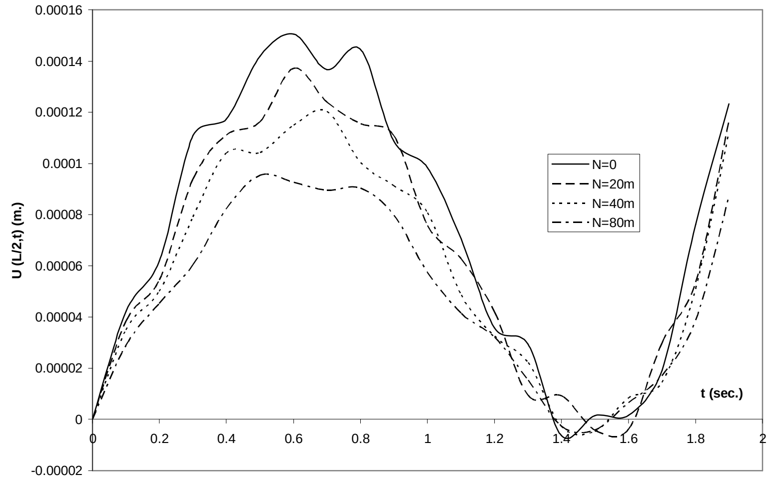

Figure 2 Displacement response of moving mass for Simple-Elastic non-uniform Bernoulli-Euler beam on variable elastic foundation for various values of pre-stress N.

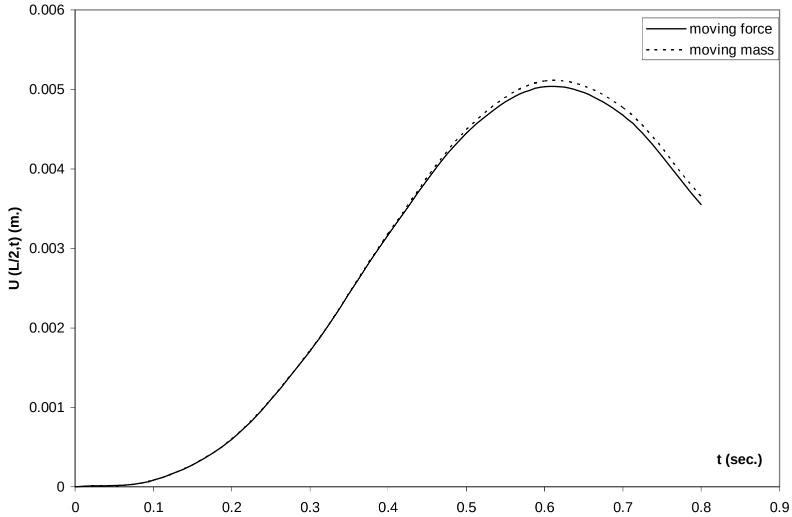

Figure 3 Comparison of moving force and moving mass for Simple-Elastic non-uniform Bernoulli-Euler beam on variable elastic foundation.

6.2 Elastic-elastic ends

It is observed in Figures 4 and 5 that as the value of the foundation modulus K increase the deflection amplitude of the elastic-elastic Bernoulli-Euler beam decreases for both cases of moving force and moving mass respectively.

Table 1 compares the displacement response of the moving force and moving mass for a Bernoulli-Euler beam, elastically supported at both ends, for fixed values of K and N. It is shown that the displacement response of the moving mass problem is greater than that of the moving force problem.

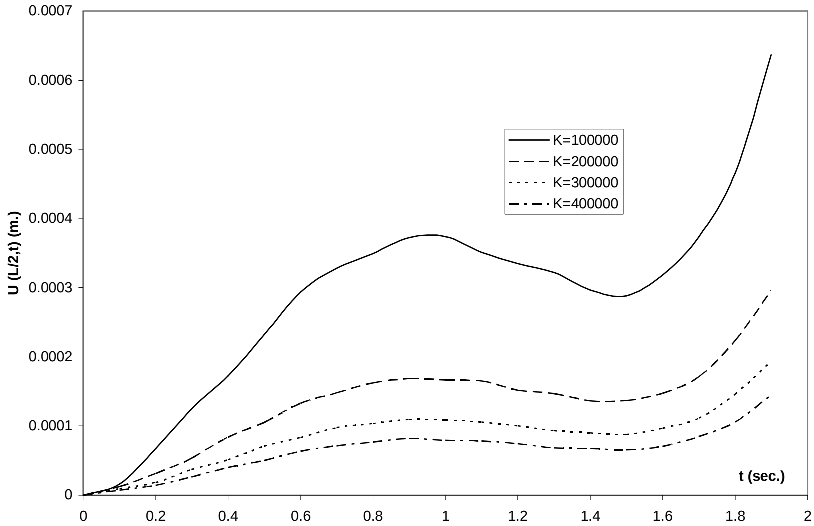

Figure 4 Deflection of moving force for Elastic-Elastic non-uniform Bernoulli-Euler beam on variable elastic foundation for various values of foundation modulus K.

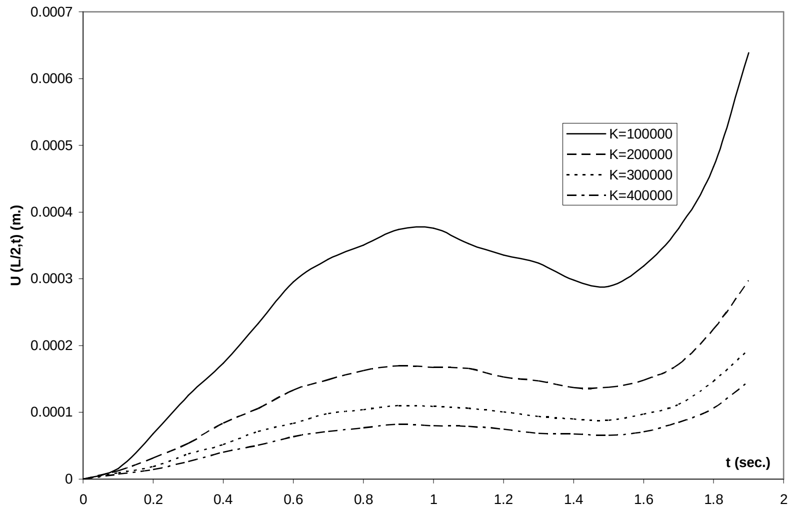

Figure 5 Displacement response of moving mass for Elastic-Elastic non-uniform Bernoulli-Euler beam on variable elastic foundation for various values of foundation modulus K.

Table 1 Displacement response of the moving force and moving mass for a Bernoulli-Euler beam, elastically supported at both ends, for fixed values of K and N.

S/N	T(sec.)	MOVING FORCE	MOVING MASS
1	0	0	0
2	0.1	1.60E-05	1.60E-05
3	0.2	6.76E-05	6.78E-05
4	0.3	1.25E-04	1.26E-04
5	0.4	1.73E-04	1.74E-04
6	0.5	2.32E-04	2.33E-04
7	0.6	2.94E-04	2.95E-04
8	0.7	3.28E-04	3.30E-04
9	0.8	3.49E-04	3.51E-04
10	0.9	3.73E-04	3.74E-04
11	1	3.74E-04	3.76E-04
12	1.1	3.51E-04	3.53E-04
13	1.2	3.35E-04	3.36E-04
14	1.3	3.22E-04	3.23E-04
15	1.4	2.97E-04	2.98E-04

7 CONCLUSION

The problem of the dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable Winkler elastic foundation has been solved. The elegant Galerkin's method is used to reduce the governing fourth order partial differential equations and a modification of the Struble's technique is employed for the solutions of the resulting Galerkin's equations. The numerical analyses carried out show that as the foundation modulli K increases, the response amplitudes of the elastically supported non-uniform Bernoulli-Euler beam decrease and that for fixed value of K, the displacements of an elastically supported non-uniform Bernoulli-Euler beam resting on a variable Winkler elastic foundation decrease as the pre-stress N increases.

Furthermore, for fixed K and N, the response amplitude for the moving mass problem is greater than that for the moving force problem for the illustrative examples so far considered. Also, as the pre-stress increases, the critical speed of the elastically supported non-uniform Bernoulli-Euler beam increases.

Finally, in the illustrative examples so far considered, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Consequently, the moving force solution cannot be a save approximation to the moving mass solution.

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