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Constitutive model for bimodular elastic damage of concrete

Abstract

An elastic damage model for concrete has been proposed considering damage-induced bimodularity. A scalar damage parameter has been chosen to quantify the damage. Expressions for the material compliance tensor components have been derived from the assumed strain and complementary energy functions stated in terms of the principal stresses and strains. Incremental constitutive equations have been derived incorporating the elastic behavior due to stress increments as well as stiffness degradation. Within the current damage surface, the stiffness of the material with constant damage varies with applied stress variations. During loading beyond the current damage surface, the material experiences stiffness degradation due to increase in extent of damage suffered by it. Using the proposed elastic damage model, the material response has been predicted for different load histories.

Keywords

bimodular, damage parameter, damage function, damage evolution, concrete, constitutive model, stiffness degradation.

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1 INTRODUCTION

Concrete is a nonlinear inelastic quasi-brittle solid exhibiting energy dissipation, irreversible deformations and stiffness degradation upon short-term loading. Obviously, the elasto-plastic-damage framework is most appropriate for its constitutive modeling. Most of such available constitutive models of concrete [1], [6], [28], [35] and [34] are premised upon a theory of damage mechanics and a theory of elastoplasticity. These two theories are generally coupled together under state laws and evolution laws associated with damage and elastoplasticity under incremental loading. In the recent past, a number of very sophisticated constitutive models dealing with elastic damage uncoupled with plasticity have been proposed [4], [8], [9], [11], [16], [17] and [22]. The primary focus of the present paper is on the presentation of elastic-damage part of elastoplastic damage formulation. It has been assumed that the elastic damage does not result in any irreversible deformations, but only in energy dissipation and stiffness degradation.

Many of the available damage models of concrete employ the relevant concepts, principles and approaches followed in the field of continuum damage mechanics. In these phenomenological models, concrete is assumed to be a homogeneous isotropic linear elastic solid and the micromechanics of discrete cracks and their evolution upon loading are ignored. The most distinguishing characteristics of damage mechanics of concrete include different nature and extent of damage suffered by it under compression and tension, damage-induced anisotropy and partial stiffness recovery upon crack closure under compression. In continuum damage mechanics, this last aspect of behavior of damaged solids is termed as unilateral effect or damage deactivation under compression. Unilateral effect has been characterized as a difficult and open research field [9]. These aspects of observed behavior of concrete have been incorporated by using an approach involving spectral decomposition of stress/strain tensors into positive and negative components affecting material behavior differently. This approach involves decomposing stress and strain tensors into their positive and negative components by using fourth rank projection operators. Because of the asymmetry of the secant compliance/stiffness tensor derived in some earlier models [29] and [33], the approach has been criticized as resulting in spurious energy dissipation or generation in closed load cycles [9] and [7]. Following Lemaitre's approach, Gibbs elastic potential has been split in some models into deviatoric and hydrostatic parts respectively associated with a damage tensor and a damage scalar [11]. Of course, a few constitutive models have been proposed without invoking the spectral composition of the stress/strain tensor [22]. Still others have constructed their damage models based upon loading surface [7].

A thermodynamically-consistent elastic damage model has been proposed [24] by considering the material to be composed of an elastic phase and a no-tension phase. Volume fraction of the no-tension phase chosen as the damage measure and its special gradients with the corresponding thermodynamic forces are taken as state variables. Also, the indicator function distinguishing between the active/de-activated status of damage is based on the sense of the eigenvalues of the stress tensor.

Most of the earlier damage models [26] and [31] are incapable of incorporating all the experimentally observed behaviour such as load-induced damage anisotropy, damage-induced elastic anisotropy, unilateral effect, tension-compression asymmetry, non-negative damage evolution, anisotropic stiffness degradation, material failure, etc. The following attempts have been made to satisfy most of the above desirable characteristics of damage models [4] and [8] and to avoid the pathologies such as discontinuous stress strain relations, spurious energy generation/dissipation, etc., present in the earlier constitutive models.

Kuna-Ciskal and Skrzypek [17] and [32] have presented damage model of concrete based on modified Murakami and Kamiya model [27] of elastic material. The unilateral crack opening /closing effect has been incorporated in such a way that the continuity requirement during loading holds. Failure criterion is adopted based on checking positive definiteness of Hessian matrix of the free energy function. Damage has been modelled using a second order tensor and a scalar variable. These researchers obtained the incremental form of stress-strain relations using local approach to fracture. Challamel et al. [8] have provided strain based continuum damage model for quasi-brittle material describing damage as a second rank symmetric tensor. The model meets the continuity of stress-strain relations, consistent thermodynamic framework, unsymmetry in tension and compression, crack closure effects and anisotropic damage behaviour.

Badel et al. [4] have presented an anisotropic damage model of concrete where damage has been considered anisotropic in tension but isotropic in compression. The observed asymmetry between tension and compression has also been incorporated . Decomposing the value of strain in tension and compression, these researchers have provided expression for specific free energy which accounts for continuity of stress with respect to strain and damage variables and have also discussed the convexity of the function with respect to existence and uniqueness. Desmorat and Cantournet [11] have presented unified damage model which considers isotropic and anisotropic damage together with unilateral effects. The model is an extension of Lemaitre [20] approach for anisotropic damage and Ladeveze and Lemaitre's [18] decomposition of stress in tension and compression. Issues related to coupling of damage and plasticity as well as damage and elasticity have been discussed.

The Murakami-Kamiya model [27] has also been applied by Kolari [16] to simulate the observed splitting mode of failure of concrete under uniaxial compression. Proposed within the paradigm of thermodynamically consistent general standard material, most of these models are based upon Helmholtz or Gibb′s free energy functions and spectral decompositions of the stress/strain tensors using fourth rank projection operators. Using Green′s theorem, general constitutive equations for stress (or strain) tensor components are obtained and stated in terms of positive and negative strain (or stress) tensors, generally second rank damage tensors, etc. Different damage evolution laws for positive and negative stress/strain tensor histories are stated in terms of the thermodynamical potential and the indicator function.

Expression for only one tangent coefficient has been proposed [17]. Attempts have also been made to predict failure caused by damage ignoring the plasticity phenomenon. However, the predicted strengths in uniaxial compression, tension and biaxial stress are far from satisfactory [27] and [4].

Departing from the above paradigm, Proenca and Pituba have modeled the damaged concrete as a linear transversely isotropic bimodular hyperelastic solid [30]. Their elastic damage model is based upon the theory of transversely isotropic conewise linear elastic solids proposed by Curnier et al. [10] which exhibit different elastic moduli depending upon the sense of the volumetric elastic strain.

In view of the above cited literature, following conclusions can be drawn concerning the current state of the constitutive modelling of damage in concrete: Very ingenious form of thermodynamical functions and indicator functions have been proposed to simulate the various observed aspects of anisotropic damage evolution with unilateral effect. It is argued here that, in this theory, the anisotropy and nonlinearity associated with the bimodular nature of the damaged solid are ignored.

The objective of the present paper is to propose a phenomenological constitutive model for elastic damage of concrete. An attempt has been made in this paper to simulate the unilateral effect without invoking spectral decomposition of stress/strain tensors. It is claimed that the proposed model is capable of predicting various characteristic of concrete behaviour. The scope of the present investigation is limited to concrete undergoing only small deformations. A damage evolution law and constitutive equations for damaged concrete as well as for concrete undergoing damage under incremental loading have been presented. Using the proposed model, the response of the material subjected to diverse stress histories has been predicted and discussed. Theoretical significance of the approach followed has been delineated.

2 DAMAGE FUNCTION AND DAMAGE EVOLUTION

Upon loading, concrete is assumed to suffer only isotropic damage quantified by a scalar damage parameter ω. Damage evolution is controlled by the following damage function :

$$f(I_1, J_2, \theta) = A\frac{J_2}{(f_c')^2} + \alpha\frac{\sqrt{J_2}}{f_c'} + B\frac{I_1}{f_c'} + C\frac{I_1^2}{(f_c')^2} - 1 = 0 \qquad (1)$$

where $\alpha = X\kappa\cos\theta + (1-\kappa)Y$; $\quad C = C_0(1-\kappa)$; $\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{\frac{3}{2}}}$

Here, the parameters A, B, C_0, X and Y are material constants. The stress invariants I_1 and J_2 are normalized by the compressive strength $f_c^{'}$ of concrete. The hardening function κ controls the evolution of the damage surfaces, in the stress space, from the initial damage surface ($\kappa = 0.3$) to the failure surface ($\kappa = 1$). The observed behaviour of concrete is the combined effect of elastoplasticity and damage. The elastoplastic loading surfaces are coincidental with the damage surfaces and the loading function, called damage function here, is calibrated by using available empirical data on concrete [2] and [3].

The empirical data formats ($\sigma_1, \sigma_2, \sigma_3$ and θ) used for calibration include (0, 0 , -$f_c^{'}$, 60°), ($0.1f_c^{'}$, 0 , 0, 0°) and (0, $-1.16f_c^{'}$, $-1.16f_c^{'}$, 0°) at the failure surface and (0, 0 , $-0.3f_c^{'}$, 60°) and ($0.09f_c^{'}$, 0, 0, 0°) at the initial damage surface. The values of the material parameters A, B, C_0, X and Y turn out to be 4.064147, 3.524653, 0.420382, 10.980986 and 13.698277 respectively [2] and [3].

Concrete does not suffer any damage within the initial damage surface. Loading beyond this initial damage surface results in an increase in the value of the hardening function κ. The damage evolution with hardening is traced by the $\omega - \kappa$ relation mediated through another variable σ. This latter variable σ is the absolute value of the stress in the uniaxial compressive stress test used for calibrating the damage evolution. The following damage evolution law is being proposed in this paper:

$\frac{\sigma}{f_c'} \leq 0.3, \kappa \leq 0.3$

$$\omega = 0 \qquad (2)$$

$0.3 \leq \frac{\sigma}{f_c'} \leq 1, 0.3 \leq \kappa \leq 1$

$$\omega = \omega_0\left(\frac{\frac{\sigma}{f_c'} - 0.3}{0.7}\right)^2 \qquad (3)$$

where ω_0 is the maximum value of the damage parameter at failure, here assumed to be equal to unity. The required $\kappa - \sigma$ relation is obtained from the damage function for the state of uniaxial compressive stress ($\sigma_{33} = -\sigma$). For this case one obtains $J_2 = \frac{\sigma^2}{3}$; $\sqrt{J_2} = \frac{\sigma}{\sqrt{3}}$; $I_1 = -\sigma$; $\theta = 60°$,

$$\begin{aligned} &\frac{A}{3}\left(\frac{\sigma}{f'_c}\right)^2 + \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}X\kappa + (1-\kappa)Y\right]\left(\frac{\sigma}{f'_c}\right) \\ &- B\frac{\sigma}{f'_c} + C_0(1-\kappa)\left(\frac{\sigma}{f'_c}\right)^2 - 1 = 0 \end{aligned} \tag{4}$$

The evolution of the damage surfaces in the biaxial and triaxial states of stress with increase in value of κ has been presented in Figure (1) and Figure (2). Each damage surface is characterized by a particular value of κ and so of the damage parameter ω. Within the current damage surface, the value of the damage parameter ω remains constant and corresponds to its maximum value reached in the past. The variation of ω with κ has been presented in Figure (3). Thus, for any load history, the value of κ, and so of σ and ω, can be obtained.

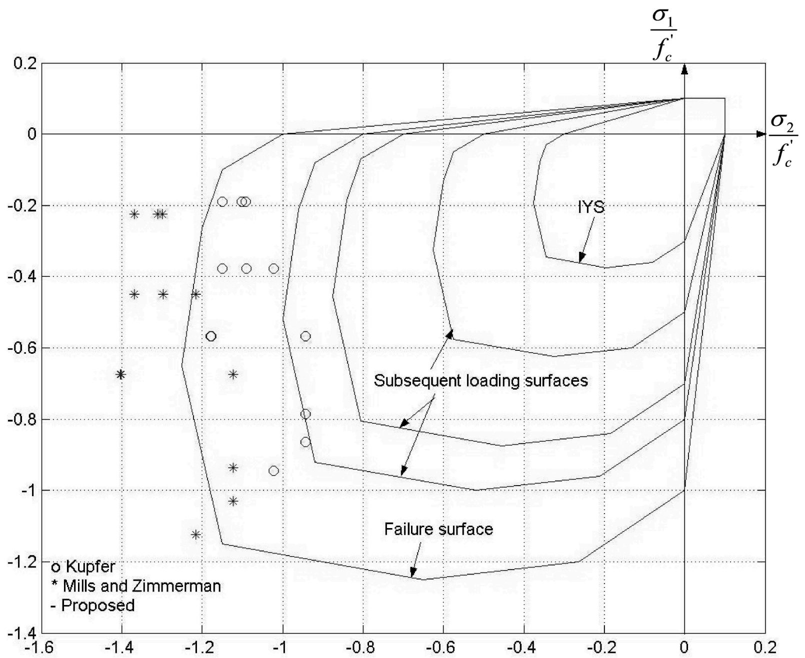

Figure 1 Evolution of damage surfaces in plane stress histories

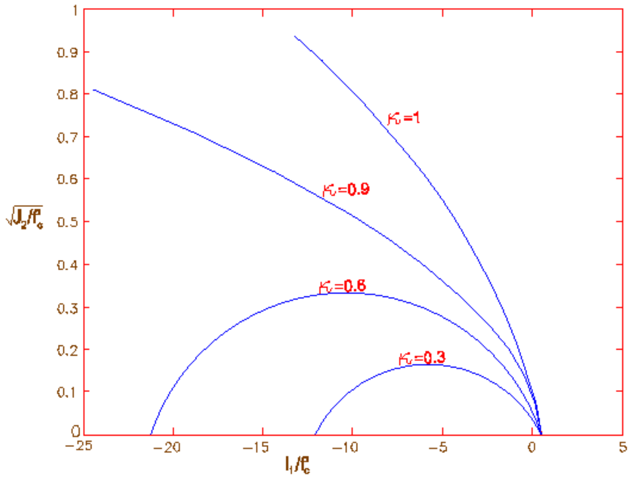

Figure 2 Evolution of damage surface

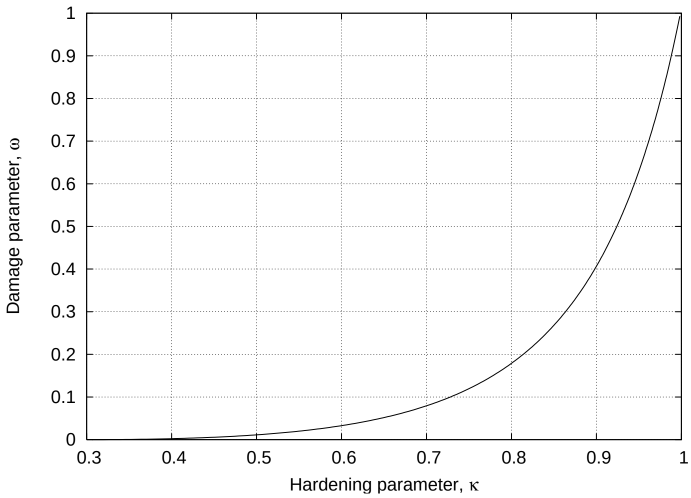

Figure 3 Variation of damage parameter with hardening function

3 BIMODULAR ELASTIC DAMAGED CONCRETE UNDER PRINCIPAL STRESSES

For any load history, the extent of damage suffered by concrete at a point is measured by the maximum value of the damage parameter ω. Damage results in stiffness degradation, but partial stiffness recovery in damaged concrete is assumed to occur under compression introduced by the current loading. Following Mazars and co-workers [26] and [25], two different scalar damage parameters are defined for compression and tension respectively as follows in terms of the damage parameter ω. For compression,

$$\omega_c = \alpha_c \omega \tag{5}$$

and for tension

$$\omega_t = \alpha_t \omega \tag{6}$$

The corresponding values of Young's moduli of elasticity are

$$E_c = (1 - \omega_c) E_0 = [1 - \alpha_c \omega] E_0 \tag{7}$$

and

$$E_t = (1 - \omega_t) E_0 = [1 - \alpha_t \omega] E_0 \tag{8}$$

Based on the available experimental data under uniaxial compression, the value of α_c has been assumed as 0.2. The material response has been predicted by assuming α_t to be 0.677. Obviously, better estimation of α_c and α_t, as and when available, can easily be incorporated in the proposed model. The variation of E_c and E_t with κ is presented in Figure (4). The variation of E_c and E_t with damage is more or less compatible with available experimental data on concrete under uniaxial compression. The axial compressive and lateral tensile moduli at peak stress have been found to be respectively 60 % and 20 % of modulus of intact concrete [21]. Also, as assumed in this paper, under uniaxial tension, the elastic modulus has been predicted by others reduce to about one-third of its value for undamaged concrete [27].

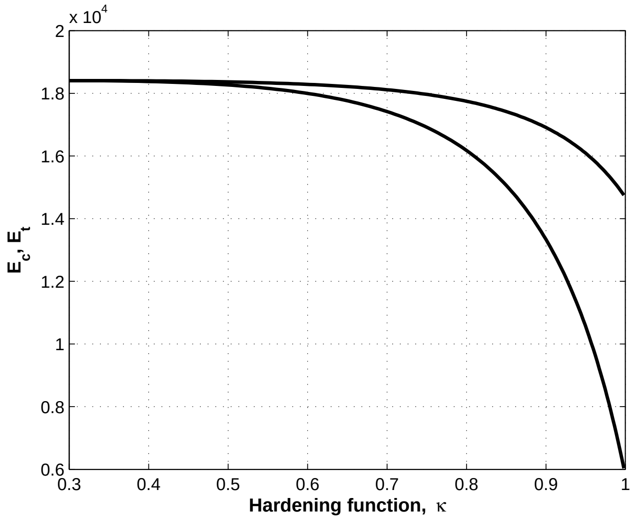

Figure 4 Variation of Young's modulus in compression and tension with hardening function

The operative Young's modulus of elasticity out of the two possible values E_c and E_t in any principal direction at a point in a bimodular damaged concrete is assumed to depend upon the sense of the corresponding principal strain. Thus, depending upon the sense of the three principal strains, four elastically-distinct cases can be identified for the linear hyperelastic material point. The expressions for the principal stresses σ_i and the strain energy density function W are stated in terms of principal strain ε_j as

$$\sigma_i = C_{ij}\varepsilon_j \tag{9a}$$

$$W = \frac{1}{2}C_{ij}\varepsilon_i\varepsilon_j \tag{9b}$$

where C_{ij} denote the relevant secant elasticity coefficients.

In the case of principal strains of the same sense, the material point is isotropic. Otherwise, it is transversely isotropic requiring only four independent elastic constants for relating principal stresses and strains. The elastic constants for the above four elastically distinct cases are identified as follows:

- Case I. $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \geq 0$
  $C_{11} = C_{22} = C_{33} = a_1, C_{23} = C_{31} = C_{12} = c_1$
- Case II. $\varepsilon_1 \geq \varepsilon_2 \geq 0 \geq \varepsilon_3$
  $C_{11} = C_{22} = a_2, C_{33} = b_2, C_{23} = C_{31} = d_2, C_{12} = c_2$
- Case III. $\varepsilon_1 \geq 0 \geq \varepsilon_2 \geq \varepsilon_3$
  $C_{11} = a_3, C_{22} = C_{33} = b_3, C_{23} = c_3, C_{31} = C_{12} = d_3$
- Case IV. $0 \geq \varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$
  $C_{11} = C_{22} = C_{33} = b_4, C_{23} = C_{31} = C_{12} = c_4$

The strain energy in aforementioned cases are as follows.

$$W_I = \frac{1}{2}a_1\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right] + c_1\left[\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1\right] \tag{10}$$

$$W_{II} = \frac{1}{2}a_2\left[\varepsilon_1^2 + \varepsilon_2^2\right] + \frac{1}{2}b_2\varepsilon_3^2 + c_2\varepsilon_1\varepsilon_2 + d_2\left[\varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1\right] \tag{11}$$

$$W_{III} = \frac{1}{2}a_3\varepsilon_1^2 + \frac{1}{2}b_3(\varepsilon_2^2 + \varepsilon_3^2) + c_3(\varepsilon_2\varepsilon_3) + d_3(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_1) \tag{12}$$

$$W_{IV} = \frac{1}{2}b_4\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right] + c_4\left(\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1\right) \tag{13}$$

Following Green and Mkirtichian [15], the principal stresses and strain energy are required to be continuous functions of the principal strains. This constitutive restriction is imposed even though the material exhibits sudden change in the elastic constants as and when any of the principal strains suffers change in sense.

In view of these restrictions, the following equalities between the elastic constants are established:

$$a_1 = a_2 = a_3 = a,\ b_2 = b_3 = b_4 = b$$

$$c_1 = c_2 = c_3 = c_4 = d_2 = d_3 = c$$

The following constitutive equation is adopted for the conventional isotropic linear elastic solids.

$$\sigma_i = \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mu\varepsilon_i \tag{14}$$

where λ and μ are Lame's constants.

Thus, the parameter c is equal to the constant λ in the conventional isotropic linear elastic solid. Its value is same for the open (or closed) cracks under tensile (or compressive) principal strains in Case I and Case IV respectively for all values of the damage parameter. Obviously, the value of c is expected to be independent of the value of damage parameter ω. Thus, c is same for the damaged as well as undamaged solid.

The Lame′s constants for the Case I (triaxial tension), Case IV (triaxial compression) and for the undamaged solid respectively are (λ_t, μ_t), (λ_c, μ_c) and (λ, μ). Thus, $c = \lambda_t = \lambda_c = \lambda$.

The Poisson′s ratios ν_t and ν_c are determined in terms of ν_0 from the above condition.

$$\frac{E_t\nu_t}{(1-2\nu_t)(1+\nu_t)} = \frac{E_c\nu_c}{(1-2\nu_c)(1+\nu_c)} = \frac{E_0\nu_0}{(1-2\nu_0)(1+\nu_0)} \tag{15}$$

Also

$a = \lambda + 2\mu_t$ $\quad$ $b = \lambda + 2\mu_c$ $\quad$ $2\mu_t = \frac{E_t}{1+\nu_t}$ $\quad$ $2\mu_c = \frac{E_c}{1+\nu_c}$

Thus, only three required independent elastic constants can be either (a, b, c) or (λ, μ_t, μ_c) or $(E_t, E_c, \nu_t$ or $\nu_c)$. Since E_c, E_t and ν_c and ν_t all are known in terms of E_0 and ν_0 for the specified value of the damage parameter ω, the three independent parameters characterizing the damaged elastic solid are identified as E_0, ν_0 and ω. Out of these, the elastic constants E_0 and ν_0 of the intact concrete are known, while the relevant value of the damage parameter ω is uniquely determined by the load history. It is to the credit of the proposed elastic damage model that no new empirical parameter need be determined to establish the constitutive identity of the damaged concrete. Even though their variation with damage parameter ω is linear, the elastic moduli E_t and E_c vary nonlinearly with hardening function κ as shown in Figure (4). Computations show that, at maximum damage ($\omega_0 = 1$), in partial compatibility with available data [27], the Poisson′s ratios ν_c and ν_t attain values of 0.23 and 0.35 respectively as shown in Figure (5). As the effect of Poisson′s ratio is small, the variation of the shear moduli μ_c and μ_t with ω, plotted in Figure (6), also turns out be almost linear.

The conclusion that, while the value of one of Lame's constant λ is invariant under change of state of damaged solid, the value of other constant μ depends upon the state of strain signifying the particular case as well as upon the extent of damage suffered by the material. These facts are in accordance with those deduced by Green and Mkrtichian [15] even though the approach followed in the present paper is quite distinct.

The proposed elastic damage model is based upon a theory of bimodular isotropic linear elastic solids undergoing small deformations. This latter theory of bimodular solids, through motivated by the work of Green and Mkrtichian [15], is distinct from it. Similarities between these two theories include four elastically distinct cases depending upon the sense of principal

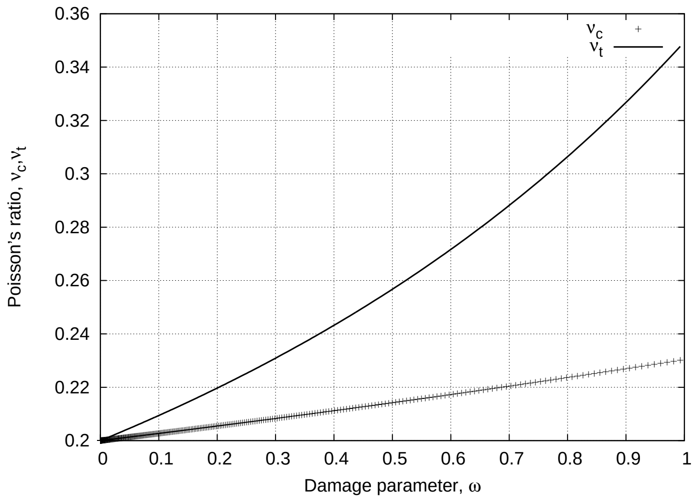

Figure 5 Variation of Poisson′s ratios with damage parameter

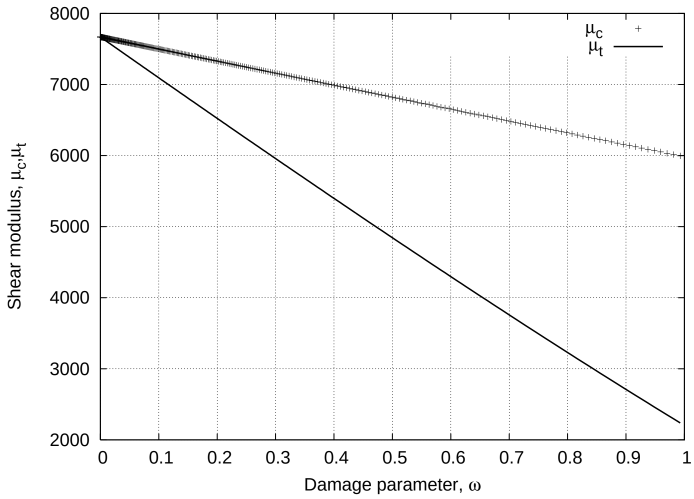

Figure 6 Variation of shear moduli with damage parameter

strains and the requirement of continuity of functional dependence of stresses and strain energy upon the strains. To bring out the difference between these two theories clearly, the expressions for strain energy and stress tensor proposed by Green and Mkrtichian for Case II are reproduced below:

$$W = \frac{1}{2}[tr\varepsilon]^2 + \mu_t tr\varepsilon^2 + (\mu_c - \mu_t)\varepsilon_3^2 \tag{16}$$

$$\sigma = \lambda[tr\varepsilon]I + 2\mu_t\varepsilon + 2(\mu_c - \mu_t)\varepsilon_3[a_3 a_3] \tag{17}$$

where σ and ε denote stress and strain tensors. Here, the symbol a_3 denotes the unit normal vector along principal strain ε_3. It should be remembered that ε_3, the minor principal strain, is the principal strain of distinct sense and its orientation represents the axis of rotational symmetry for Case II. Thus, Green and Mkrtichian express strain energy in terms of principal strains of distinct sense and the stress tensor in terms of its orientation. In contrast, in this paper, the strain energy is expressed in terms of strain tensor components and the expressions for stress tensor components do not require the determination of the orientation of the principal strain directions. Also, in contrast to Green and Mkrtichian, the elasticity and compliance tensors are stated here explicitly in reference to any general orientation of the coordinate system. Still, for the same state of strain, both of these theories of isotropic bimodular solids predict the same energy and state of stress.

4 ISOTROPIC HYPERELASTIC DAMAGED CONCRETE

The expression for the complementary energy function Ω for hyperelastic damaged concrete can be stated in terms of principal stresses σ_i as

$$\Omega = \frac{1}{2} D_{ij} \sigma_i \sigma_j \tag{18}$$

where $D_{ij} = C_{ij}^{-1}$ is the compliance matrix in reference to principal axes for the material. Typical variation of some components of compliance matrix with damage ω has been presented in Figures (7) and (8). The compliance tensor so obtained pertains to the principal coordinate axes. The constitutive equations as well as explicit expressions for the components of compliance matrix in reference to an arbitrary Cartesian coordinate system are derived below:

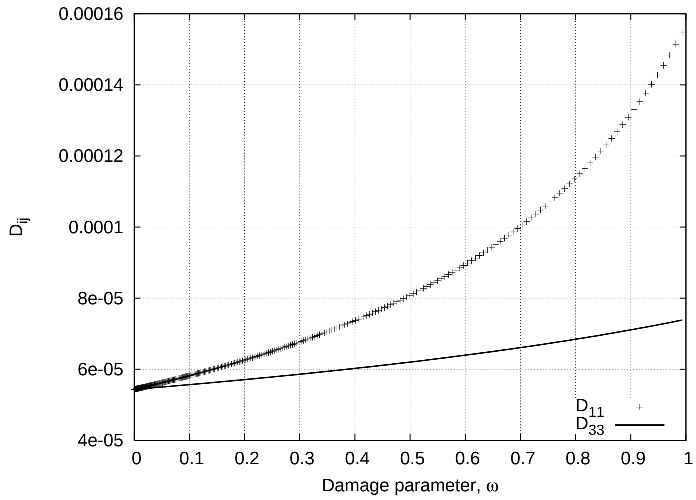

Figure 7 Variation of D_{11} and D_{33} with damage parameter

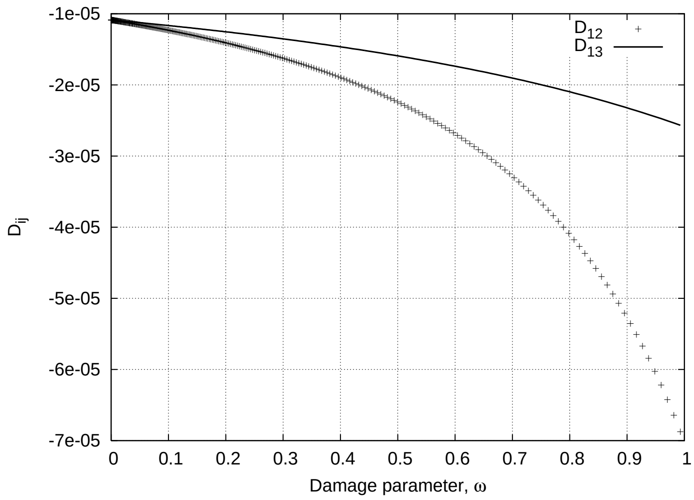

Figure 8 Variation of D_{12} and D_{13} with damage parameter

The expressions for the principal stresses in terms of the stress invariants are stated as

$$\sigma_r = \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta_r \qquad r = 1,2,3 \tag{19}$$

where $\theta_1 = \theta$; $\theta_2 = \theta - \frac{2\pi}{3}$; $\theta_3 = \theta + \frac{2\pi}{3}$

In view of these expressions, the expression for the complementary energy function takes the form

$$\Omega = \frac{1}{9}N_1 I_1^2 + \frac{2}{3\sqrt{3}}N_2 I_1\sqrt{J_2} + \frac{4}{3}N_3 J_2 \tag{20}$$

where N_1, N_2 and N_3 depend upon D_{ij} and Lode angle θ.

Using Green′s theorem, one obtains the following constitutive equation for hyperelastic damaged concrete

$$\varepsilon_{ij} = \frac{\partial\Omega}{\partial\sigma_{ij}} = \alpha_1\delta_{ij} + \alpha_2 S_{ij} + \alpha_0 T_{ij} \tag{21}$$

where $\frac{\partial I_1}{\partial\sigma_{ij}} = \delta_{ij}$ $\frac{\partial J_2}{\partial\sigma_{ij}} = S_{ij}$ $\frac{\partial\theta}{\partial\sigma_{ij}} = T_{ij}$

Here σ_{ij} and S_{ij} denote stress tensor and deviatoric stress tensor components. The coefficients α_1,α_2 and α_0, all scalar functions of the stress invariants (I_1, J_2, θ), are given as

$$\alpha_1 = \frac{\partial\Omega}{\partial I_1} = \left[\frac{2N_1}{9}I_1 + \frac{2N_2}{3\sqrt{3}}\sqrt{J_2}\right] \tag{22}$$

$$\alpha_2 = \frac{\partial\Omega}{\partial J_2} = \left[\frac{N_2}{3\sqrt{3}}I_1\frac{1}{\sqrt{J_2}} + \frac{4}{3}N_3\right] \tag{23}$$

$$\alpha_0 = \frac{\partial\Omega}{\partial\theta} = \left[\frac{2}{3\sqrt{3}}I_1\sqrt{J_2}N_4 + \frac{4}{3}J_2N_5\right] \tag{24}$$

Expressions for the tangent compliance tensor components, $D_{ijkl} = \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}$ in general coordinates are derived as follows:

$$\begin{aligned} D_{ijkl} = a\delta_{ij}\delta_{kl} + b\left[\delta_{ik}\delta_{jl} - \frac{\delta_{ij}\delta_{kl}}{3}\right] + c\left[\delta_{ij}\delta_{kl} + \delta_{kl}\delta_{ij}\right] \\ + d\left[\delta_{ij}T_{kl} + \delta_{kl}T_{ij}\right] + e(\delta_{ij}\delta_{kl}) + f\left[T_{ij}T_{kl}\right] \\ + (g+h)\left[S_{ij}T_{kl} + S_{kl}T_{ij}\right] + \alpha_0 \frac{\partial^2\theta}{\partial\sigma_{ij}\partial\sigma_{kl}} \end{aligned} \tag{25}$$

where
$a = \frac{2}{9}N_1;\quad b = \alpha_2;\quad c = \frac{1}{3\sqrt{3}}N_2\frac{1}{\sqrt{J_2}};\quad d = \frac{2}{3\sqrt{3}}N_4\sqrt{J_2};$
$e = \frac{-1}{6\sqrt{3}}N_2\frac{I_1}{J_2^{\frac{3}{2}}};\quad f = \frac{-2}{3\sqrt{3}}N_2 I_1\sqrt{J_2} + \frac{4}{3}N_5 J_2;$
$g = \frac{1}{3\sqrt{3}}N_4\frac{I_1}{\sqrt{J_2}};\quad h = \frac{4}{3}N_5$
Also

$$T_{ij} = \frac{3\sqrt{3}}{4\sin 3\theta}\frac{J_3}{J_2^{\frac{5}{2}}}S_{ij} - \frac{\sqrt{3}}{2J_2^{\frac{3}{2}}\sin 3\theta}S_{ir}S_{rj} + \frac{1}{\sqrt{3}\sin 3\theta}\frac{1}{\sqrt{J_2}}\delta_{ij} \tag{26}$$

where $S_{ir}S_{rj} = S_{i1}S_{1j} + S_{i2}S_{2j} + S_{i3}S_{3j}$
and

$$\begin{aligned} \frac{\partial T_{ij}}{\partial \sigma_{kl}} = \frac{-15\sqrt{3}}{8}\frac{1}{\sin 3\theta}J_3 S_{ij}\frac{1}{J_2^{\frac{7}{2}}}S_{kl} \\ + \frac{3\sqrt{3}}{4}\frac{1}{\sin 3\theta}\frac{S_{ij}}{J_2^{\frac{5}{2}}}(S_{kr}S_{rl} - \frac{2}{3}J_2\delta_{kl}) \\ + \frac{3\sqrt{3}}{4\sin 3\theta}\frac{J_3}{J_2^{\frac{5}{2}}}(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}) - \frac{9\sqrt{3}}{4}\frac{J_3}{J_2^{\frac{5}{2}}}S_{ij} \\ \cot 3\theta \csc 3\theta T_{kl} + \frac{3\sqrt{3}}{4\sin 3\theta}S_{ir}S_{rj}\frac{1}{J_2^{\frac{5}{2}}}S_{kl} + \frac{3\sqrt{3}}{2} \\ \frac{S_{ir}S_{rj}}{J_2^{\frac{3}{2}}}\cot 3\theta \csc 3\theta T_{kl} - \frac{\sqrt{3}}{2J_2^{\frac{3}{2}}\sin 3\theta}[S_{i1}(\delta_{1k}\delta_{jl} \\ - \frac{\delta_{1j}\delta_{kl}}{3}) + S_{1j}(\delta_{ik}\delta_{1l} - \frac{\delta_{i1}\delta_{kl}}{3}) + S_{i2}(\delta_{2k}\delta_{jl} \\ - \frac{\delta_{2j}\delta_{kl}}{3}) + S_{2j}(\delta_{ik}\delta_{2l} - \frac{\delta_{i2}\delta_{kl}}{3}) + S_{i3}(\delta_{3k}\delta_{jl} \\ - \frac{\delta_{3j}\delta_{kl}}{3}) + S_{3j}(\delta_{ik}\delta_{3l} - \frac{\delta_{3j}\delta_{kl}}{3})] - \frac{1}{2\sqrt{3}}\delta_{ij} \\ \frac{1}{sin3\theta}\frac{1}{J_2^{\frac{3}{2}}}S_{kl} - \frac{\sqrt{3}}{\sqrt{J_2}}\cot 3\theta csc3\theta T_{kl}\delta_{ij} \end{aligned} \tag{27}$$

The expressions for N_1, N_2, N_3, N_4, N_5 and N_6 are as follows:

$$N_1 = \frac{1}{2}(D_{11} + D_{22} + D_{33}) + D_{23} + D_{13} + D_{12} \tag{28a}$$

$$N_2 = N_{2a}\cos\theta + N_{2b}\sin\theta \tag{28b}$$

$$N_{2a} = D_{11} - D_{23} - \frac{D_{22}}{2} + \frac{D_{31}}{2} + \frac{D_{12}}{2} - \frac{D_{33}}{2} \tag{28c}$$

$$N_{2b} = \frac{\sqrt{3}}{2}(D_{22} - D_{31} - D_{33} + D_{12}) \tag{28d}$$

$$N_3 = N_{3a} + N_{3b}\cos 2\theta - N_{3c} sin 2\theta \tag{28e}$$

$$N_{3a} = \frac{3}{8}(D_{22} + D_{33} - 2D_{23}) \tag{28f}$$

$$N_{3b} = \frac{1}{4}(2D_{11} - D_{22} - D_{33} - 2D_{12} + 4D_{23} - 2D_{31}) \tag{28g}$$

$$N_{3c} = \frac{\sqrt{3}}{8}(D_{22} - D_{33} - 2D_{12} + 2D_{31}) \tag{28h}$$

$$\frac{\partial N_2}{\partial \sigma_{ij}} = N_4 T_{ij}$$

$$N_4 = N_{2b}\cos\theta - N_{2a}\sin\theta \tag{28i}$$

$$\frac{\partial N_3}{\partial \sigma_{ij}} = N_5 T_{ij} \tag{28j}$$

$$N_5 = -2N_{3c}\cos 2\theta - 2N_{3b}\sin 2\theta \tag{28k}$$

$$\frac{\partial N_5}{\partial \sigma_{ij}} = N_6 T_{kl} \tag{28l}$$

$$N_6 = (4N_{3c}\sin 2\theta - 4N_{3b}\cos 2\theta) \tag{28m}$$

It can be observed that θ, I_1 and J_2 are homogeneous functions of order zero, one and two of the stress tensor components. Consequently, compliance tensor components D_{ijkl}, strain tensor components ε_{ij} and the complementary energy function Ω respectively are functions positively homogeneous of order zero, one and two of the stress tensor components. In view of Euler's theorem for such functions,

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}\sigma_{kl} = D_{ijkl}\sigma_{kl} = \varepsilon_{ij} \tag{29}$$

$$\frac{\partial \Omega}{\partial \sigma_{ij}}\sigma_{ij} = \varepsilon_{ij}\sigma_{ij} = 2\Omega \tag{30}$$

The first of the above equations implies the equality of the secant $\bar{D}_{ijkl}$ and tangent D_{ijkl} compliance tensor coefficients. The second equation proves the validity of the Clayperon's theorem. Conventionally, these statements are considered to be valid only for linear mechanical systems. Here, the validity of these statements has been proved even for a class of nonlinear mechanical systems. Also, under proportional stress variations, the damaged concrete has constant compliance tensor coefficients and behaves as a linear elastic solid. Damaged concrete is shown to exhibit nonlinear behaviour only under non-proportional stress variations resulting in rotation of principal stress and strain directions. However, even the non-proportional variation of purely normal stresses is not accompanied by material nonlinearity.

In terms of the secant compliance tensor $\bar{D}_{ijkl}$, the constitutive equation for the damaged concrete can be restated as

$$\varepsilon_{ij} = \bar{D}_{ijkl}\sigma_{kl} \tag{31}$$

On incremental loading, the strain increments are obtained as

$$d\varepsilon_{ij} = \bar{D}_{ijkl}d\sigma_{kl} + \sigma_{kl}d\bar{D}_{ijkl} \tag{32}$$

As per Dougill [12] and [13], the strain increments are contributed by stress increments as well as by stiffness degradation. Also, in terms of tangent compliance matrix D_{ijkl},

$$d\varepsilon_{ij} = D_{ijkl}d\sigma_{kl} \tag{33}$$

As $\bar{D}_{ijkl} = D_{ijkl}$, a comparison of Equations (32) and (33) implies that the variation of the compliance tensor coefficients under incremental loading does not result in additional strain increments. The validity of this conclusion is restricted to stress variations within the current damage surface wherein the value of the damage parameter remains constant. As shown in the next section, such is not the case when the material is loaded beyond the current damage surface.

5 INELASTIC CONCRETE UNDERGOING DAMAGE

At the current damage surface, the state of strain is determined by the current state of stress and the extent of damage suffered affecting the compliances. Thus,

$$\varepsilon_{ij} = \varepsilon_{ij}\left(\sigma_{kl}, \omega\right) \tag{34}$$

On application of stress increments resulting in additional damage, the strain increments introduced are obtained as

$$d\varepsilon_{ij} = \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}d\sigma_{kl} + \frac{\partial \varepsilon_{ij}}{\partial \omega}d\omega \tag{35}$$

As complementary energy, Ω, is the continuous function of σ_{ij} and ω, one can write

$$\frac{\partial \varepsilon_{ij}}{\partial \omega} = \frac{\partial}{\partial \omega}\left(\frac{\partial \Omega}{\partial \sigma_{ij}}\right) = \frac{\partial Z}{\partial \sigma_{ij}} \tag{36}$$

where $Z = \frac{\partial \Omega}{\partial \omega}$ is complementary energy release rate playing the role of complementary energy conjugate of the damage parameter ω.

The loading function for the damaged concrete is stated as

$$f = f(\sigma_{ij}, \omega) \tag{37}$$

Adopting an approach followed by many researchers, the consistency condition implies [31]

$$df = \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} + \frac{\partial f}{\partial \omega} d\omega = 0 \tag{38}$$

which gives

$$d\omega = -\frac{1}{\frac{\partial f}{\partial \omega}} \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} \tag{39}$$

In view of expressions (35) and (39), the incremental constitutive equation for concrete suffering incremental damage is stated as

$$d\varepsilon_{ij} = D^{ed}_{ijkl} d\sigma_{kl} \tag{40}$$

where the elastic-damage compliance tensor D^{ed}_{ijkl} turns out to be

$$D^{ed}_{ijkl} = D_{ijkl} - \frac{1}{\frac{\partial f}{\partial \omega}} \frac{\partial Z}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \tag{41}$$

Since the functions Z and f are distinct, the tensor D^{ed}_{ijkl} turns out to be asymmetric.

Also, using Equation (36), the component of the strain increment associated with damage increment is determined as follows:

$$d\varepsilon^{d}_{ij} = \frac{\partial \varepsilon_{ij}}{\partial \omega} d\omega = d\lambda_d \frac{\partial Z}{\partial \sigma_{ij}} \tag{42}$$

where the scalar multiplier is obtained as

$$d\lambda_d = d\omega = -\frac{1}{\frac{\partial f}{\partial \omega}} \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} \tag{43}$$

Thus, the function Z has been shown to play the role of the damage potential and the damage parameter increment $d\omega$ itself plays the role assigned to the scalar multiplier $d\lambda_d$ in the flow theory of plasticity. Since the damage potential $Z(\sigma_{ij}, \omega)$ happens to be distinct from the damage function $f(\sigma_{ij}, \omega)$, the material is said to obey non-associative 'flow' rule stated in Equation (42). This fact implies the asymmetry of the tangent elastic-damage compliance tensor as already established.

6 RESULTS AND PREDICTIONS OF THE PROPOSED ELASTIC DAMAGE MODEL

The mechanical behaviour of purely elastic gradually fracturing solids has been studied by Dougill [13] [12]. With the intention of comparing the efficacy of the proposed model with that of Dougill's, the mechanical behaviour of purely elastic concrete subjected to diverse proportional loading and unloading cycles has been presented in Figures (9, 10, 11, 12). The predicted loading, unloading and reloading behavior has been shown in these figures for different values of the hardening parameter such as $\kappa = 0.65, 0.85$ and 1. From the Figure (9), it can be observed that, like Dougill's gradually fracturing solid, there is no permanent strain/plastic strain in unloading curve in uniaxial compression. For every value of the hardening parameter, the unloading curves reach the zero strain point and again upon reloading, it traces the path of the previous load path. Similarly, equal and unequal biaxial compression, and triaxial compression followed by unloading from different damage surfaces have been presented in Figures (10, 11, 12) for different values of the hardening parameter.

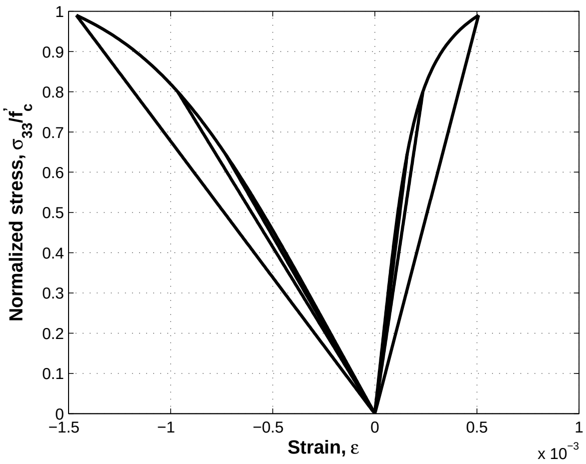

Figure 9 Loading, unloading and reloading in uniaxial compression $(0, 0, -1.0)$

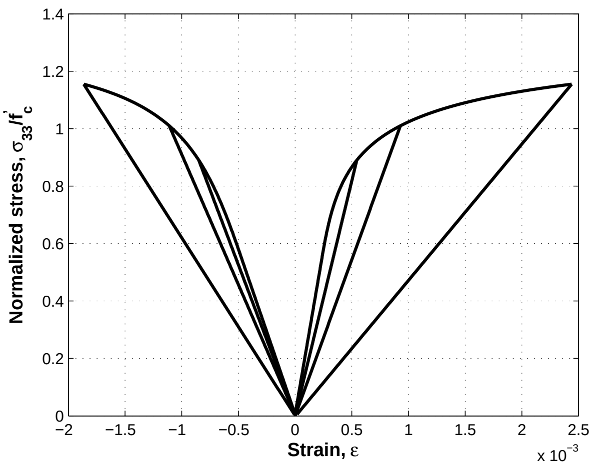

Figure 10 Loading, unloading and reloading in equal biaxial compression $(0, -1.0, -1.0)$

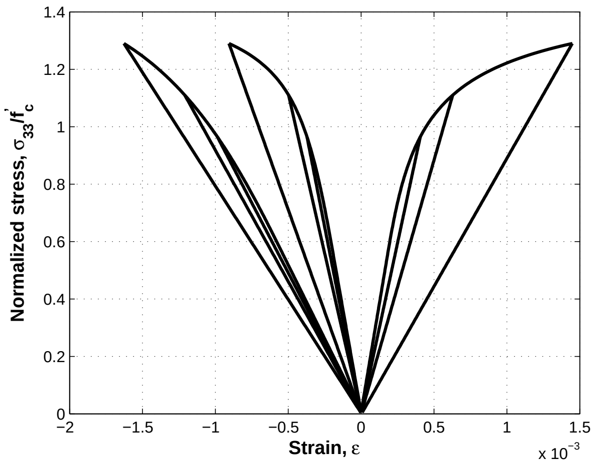

Figure 11 Loading, unloading and reloading in unequal biaxial compression $(0, -0.52, -1.0)$

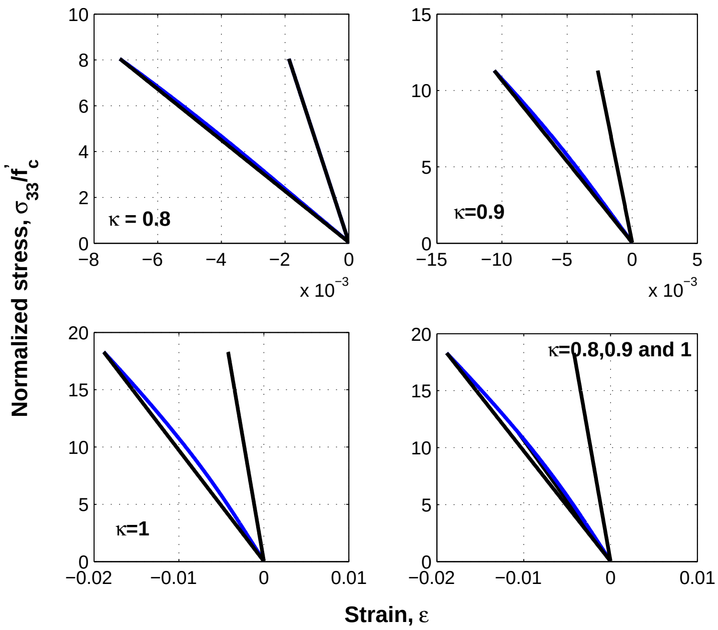

Figure 12 Loading, unloading and reloading in triaxial compression $(-0.52, -0.52, -1.0)$

In each loading, unloading and reloading case the concrete behaves in a similar fashion to the theory of progressively fracturing solids proposed by Dougill. It can be observed that the stiffness degradation introduced by damage renders the material nonlinear during loading even for proportional increase in stresses. Upon unloading, concrete is shown to exhibit energy dissipation but without any irreversible strains. Also, the material can be observed to exhibit stiffness degradation upon loading. In other words, the material stiffness upon unloading and reloading is obtained to be lesser than that of the intact undamaged concrete. Stiffness degradation suffered by the material is higher upon unloading from the current fracturing surface with higher value of damage associated with higher value of hardening function. From Figure 12, concrete can be observed to exhibit little stiffness degradation in triaxial compression. These predictions of the proposed model imply its similarity to Dougill's model for gradually fracturing solids [13] and [12].

Likewise, for all these stress histories, the material response upon unloading and reloading remains linear elastic. Such happens to be the case only because, in these examples, unloading has been achieved along proportional stress path. The proposed model is indeed capable of predicting nonlinear elastic behaviour within the current damage surface. It has been shown that the damaged concrete is nonlinear elastic solid. Obviously, the principle of superposition is not expected to be valid. The material response under non-proportional stress variations has been shown in Figure (13). A particular state of stress denoted by point A $\left(\frac{\sigma_{33}}{f'_c} = 0.5\right)$ is reached through proportional load path. The stress increments for the path segment AB (point B:$\left(\frac{\sigma_{33}}{f'_c} = 2.6\right)$) have been chosen to be different from those for the segment OA. It can be observed for the proportional load path OA, the strains are proportional to the stresses. Such a proportional stress-strain variation is absent in the non-proportional load path AB. However, as for each path segment, the respective stress increments are kept same, the stress-strain curves OA and AB are straight lines.

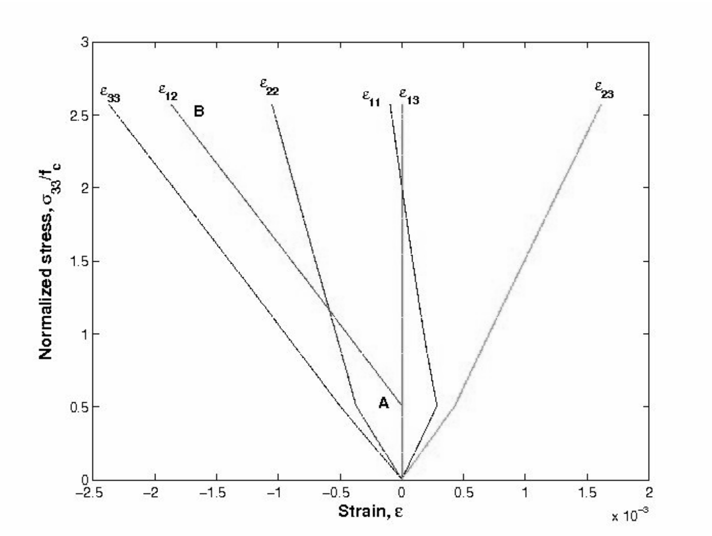

Figure 13 Response to monotonic loading along non-proportional stress paths

7 DISCUSSION AND INTERPRETATION

The extent and pattern of damage as well as the consequence stiffness degradation suffered by quasi-brittle material like concrete depends on the load history. It is well known that tensile stresses inflict more damage on concrete than do compressive stresses of the same magnitude. In the present paper, the extent of damage is quantified by the value of the scalar damage parameter ω. For any history of loading, the relevant value of the scalar damage parameter depends upon the maximum value of the hardening function κ which, in turn, is uniquely determined by the stress history via damage function. Thus, the extent of damage suffered by concrete is a continuous function of the monotonically increasing stresses. It is predicted here that concrete does not suffer any damage until the initial damage surface is reached. This occurs under a uniaxial tensile stress, pure shear stress, uniaxial compressive stress, equal biaxial compressive stress and hydrostatic pressure of magnitude $0.0902f_c^{'}$, $0.0783f_c^{'}$, $-0.2972f_c^{'}$, $-0.7276f_c^{'}$ and $-4.0837f_c^{'}$ respectively for concrete of grade $f_c^{'}$.

Likewise for the first four stages of stress, the maximum damage ω_0 is predicted to occur at peak stresses of magnitude $0.1f_c^{'}$, $0.1008f_c^{'}$, $-f_c^{'}$ and $-1.16f_c^{'}$ respectively, whereas such a stage is never reached under hydrostatic pressure. The explanation for the damage suffered by concrete under applied hydrostatic pressure lies in complex non-hydrostatic state of micro-stresses in the porous heterogeneous material like concrete. Thus, in compatibility with the empirical data on concrete [22], stress histories with dominant tension are predicted to cause more damage in it. As stated earlier, the scope of the paper is restricted to pre-peak behaviour of concrete. Some researchers [4] and [22] have questioned the validity of employing the continuum damage approach to the post-peak stage.

When subjected to general load history, concrete is known to suffer anisotropic damage in the form of oriented micro-cracks. As such, damaged concrete is an anisotropic elastic solid in response to general applied states of stress resulting in microcraks activated or deactivated in all directions. In the present paper, concrete is assumed to suffer only isotropic damage under all load histories. As such, the scope of validity of the proposed constitutive model is restricted only to hydrostatic stress histories causing isotropic damage. However, as argued below, the proposed model can still be applied for more general stress histories.

For the case of proportional monotonic loading, the distinction between the stress history and current state of stress is obliterated. This is because of the fact that, in such cases, the current state of stress fully characterizes the stress history as well. Let it be reiterated that the proposed model predicts isotropic damaged concrete when subjected to spherical tensile or compressive states of stress. The model predicts isotropic damaged concrete when the principal stresses differ slightly from each other whereas non-spherical states of stress are known to transform concrete into an anisotropic solid. However, the error in the model predictions does not increase with increase in the difference between the major and minor principal stresses. For understanding this claim, one has to recognize the bimodular nature of the isotropically damaged concrete.

The isotropically damaged concrete is predisposed to possess two values of Young's moduli of elasticity. Which one of these is excited in any particular principal acoustic axis depends upon the sense of the corresponding principal strain. As the difference between the principal stresses in the above case increases, one of the principal strains assumes a sense different from the other two. This renders even the isotropically damaged concrete transversely isotropic. The axis of rotational symmetry is oriented along either the major or minor principal stress which differs more from the intermediate principal stress whereas the plane of isotropy is constituted by the remaining two principal stresses differing from slightly each other. For such monotonic proportional stress histories, the damaged concrete has been observed to behave as orthotropic solids. However, the Young's moduli of elasticity in the directions of principal stresses with slightly different magnitudes do not differ much.

Thus, even though, in this paper, concrete has been assumed to suffer only isotropic damage for all stress histories, the elastic response of concrete subjected to monotonic proportional loading as predicted by the proposed model is quite satisfactory. Most of the experimental data on concrete pertains only to such stress histories. Such happens to be the case even for concrete in structures subjected to single dominant load combinations. Concrete is rarely subjected to triaxial tensile strains. It has been observed that concrete subjected to triaxial compressive stress histories suffers damage with little anisotropy [4]. Thus, from practical considerations, modelling of damaged concrete in Case I and Case IV as an isotropic linear elastic solid is justified to a certain extent. In the remaining two cases characterized by principal strains of mixed sense, damaged concrete is predicted to exhibit transverse isotropy in place of actual general anisotropy. Using the above quoted arguments by Badel et al. [4], the stiffness in the directions of principal strains of same sense is expected to be more or less equal, thus practically reducing the observed orthotropy to transverse isotropy as predicted by the proposed model.

In this paper, the isotropically damaged concrete has been modelled as a bimodular isotropic elastic solid. Material isotropy implies the co-axiality of the principal stresses and strain axes for all the states of stress [23] and [14]. For principal strains of the same sense, the material behaves as an isotropic linear elastic solid, albeit with different with different elastic constants in triaxial tension and compression. Stiffness recovery under compression is only partial due to tortuousity of microcracks resulting in softening even in the direction parallel to microcracks [29].

It has been provided in this paper, that damaged concrete subjected to a state of stress resulting in principal strains of mixed sense is a non-linear elastic solid. For such a case, the material compliance tensor components and strain tensor components are functions non-negatively homogeneous of order zero and one of the stress tensor components. These conclusions are confirmed in constitutive models premised upon entirely different approach. For example, the effective secant stiffness matrix components derived by Kuna-Ciskal and Skrzypek for the plane stress case can be seen to be functions of the ratios of the strain tensor components [17]. For similar plane stress case, investigated by Challamel et al. [8], the stress tensor components can be observed to be first order homogeneous functions of the strain tensor components.

In this paper, it has been shown that damage affects only the diagonal components of the material stiffness tensor C_{ij}. This conclusion is confirmed by other researchers as well [17]. This fact is intimately linked to the desired continuity of the stress-strain relation as and when one of the principal strains undergoes change of sense [27]. It is argued here that such qualitative evidence of compatibility between the proposed model and some well-established models, rather than mere empirical validation, constitutes a stronger proof of the basic soundness of the proposed model.

When all the principal strains are not of the same sense, the damaged concrete exhibits transverse isotropy [19]. Coincidence of the axis of rotational symmetry with the orientation of the axis of principal strain of distinct sense results in reduction of number of independent elastic constants to four. This is in contrast to the five independent elastic constants required to characterize the conventional transversely isotropic solid. Further reduction to three independent elastic constants is achieved by requiring the continuity of strain energy and principal stresses with change in the sense of the principal strains.

As a matter of fact, no elastic constants other than those of the undamaged concrete are required to describe the behaviour of the damaged concrete. Such happens to be case because the applied stress history can be used along with the proposed damage function to obtain the value of the scalar damage parameter. Thus, in contrast to some other constitutive model requiring many material parameters [4], it is to credit of the proposed constitute model for elastic damage of concrete that no further empirical calibration is required.

To recapitulate, the post-peak softening behaviour of the elastic solid undergoing damage lies outside the scope of the proposed local elastic damage model. Softening is shown to result in shear localisation, size effect, non-uniqueness of solutions of quasi-static boundary value problems, non-existence of solution of initial value problems such as wave propagation, mesh dependence of computed behaviour, etc. Non-local damage models have been found to be quite promising for investigating such important aspects of material behaviour [5]. Also, a constitutive model based on gradient of damage measure as a state variable has been shown to provide a satisfactory description of the strain softening phenomenon [24]. Clearly, there is a scope for extending the proposed elastic damage model to incorporate such softening behaviour.

8 CONCLUSIONS

A model for damage-induced bimodularity for concrete has been developed. A scalar damage variable has been employed to quantify the extent of isotropic damage suffered by concrete. Stress-strain relations for damaged concrete within damage surface have been presented. The proposed elastic damage model resembles the Dougill's theory of gradually fracturing solids. It is premised on, but is distinct from, the Green-Mkrtichian theory of isotropic bimodular solids. The validity of the proposed elastic damage model is restricted only to isotropically damaged bimodular concrete undergoing small deformation. Still, the model predicts more damage in tension than in compression. Also, for the same extent of damage caused by some stress history, the stiffness degradation under tensile stress as applied on the damaged solid is more than that for the compressive stresses. For states of stress resulting in principal strains of mixed sense, the damaged concrete has been shown to be transversely isotropic. Constitutive equations for the damaged elastic concrete as well as for concrete undergoing damage has been explicitly stated. The proposed model has been critically evaluated.

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