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Flexural motions under accelerating loads of structurally prestressed beams with general boundary conditions

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Abstract

The transverse vibration of a prismatic Rayleigh beam resting on elastic foundation and continuously acted upon by concentrated masses moving with arbitrarily prescribed velocity is studied. A procedure involving generalized finite integral transform, the use of the expression of the Dirac delta function in series form, a modification of the Struble's asymptotic method and the use of the Fresnel sine and cosine functions is developed to treat this dynamical beam problem and analytical solutions for both the moving force and moving mass model which is valid for all variant of classical boundary conditions are obtained. The proposed analytical procedure is illustrated by examples of some practical engineering interest in which the effects of some important parameters such as boundary conditions, prestressed function, slenderness ratio, mass ratio and elastic foundation are investigated in depth. Resonance phenomenon of the vibrating system is carefully investigated and the condition under which this may occur is clearly scrutinized. The results presented in this paper will form basis for a further research work in this field.

Keywords

Rayleigh beam, resonance phenomenon, asymptotic method, concentrated masses, transverse vibration, slenderness ratio.

1 INTRODUCTION

Studies concerning the flexural vibrations of a structural elements carrying moving mass has been an area of active research for more than a century in many diverse areas such as civil, structural, mechanical and aerospace engineering. Initially, this class of problem was first applied in the design of railway bridges and the application was later extended to other transportation engineering such as the design of bridges, guideways, overhead cranes, cableways, rails, roadways, runways, tunnels, and pipelines with moving masses [2]. Evidently, extensive researches have been conducted by many researchers particularly on the analysis of continuous elastic system under the actions of moving sub-system [1, 3–5, 7, 8, 11, 12, 14–17, 19, 21]; this is of course due to its enormous practical significance. In general, most of the previous publications dealt with a beam model whose dynamic characteristics are described by Euler-Bernoulli beam equations. It is however well known that if the slenderness ratio is large, or vibration of higher modes is concerned, the use of classical Bernoulli-Euler beam theory cannot ensure sufficient accuracy. Thus, a beam theory which takes into account the effects of shearing deformation or rotatory inertia or both must be adopted for more accurate analysis [10].

Even with the inclusion of shear deformation and rotatory inertia into the equation of motion, a good number of these studies have considered a much simpler problem where the motion of the moving mass is described by a constant velocity type of motion. However, situation arises when the moving mass accelerates by a forward force or decelerates, reduces speed and come to rest at any desired position on the beam and causing the friction between the mass and the beam to increase considerably. Under such condition, the vibrating system exhibits dynamic behaviour which may be more complicated.

Previous studies where such a dynamical system was investigated are Wang [20] who studied the dynamical analysis of a finite inextensible beam with an attached accelerating mass. He employed the Galerkin procedure in conjunction with the method of numerical integration to tackle the partial differential equations which describe the transient vibrations of the beam-mass system. He concluded that the applied forward force amplifies the speed of the mass and the displacement of the beam. Though the theory developed here is versatile, its application is only limited to the case of beams executing flexural motions according to the simple Bernoulli-Euler theory of flexure. Nevertheless, it is easy to see that a typical element of an elastic system performs not only a translatory motion but also rotates [18]. Hilal and Ziddeh [6] investigated the vibration analysis of beams with general boundary conditions traversed by a single point force traveling with variable velocity. They obtained analytical solution to the beam problem and compared the results with same beam under the actions of a concentrated force traveling at constant velocity. Their method of solution is only suitable to handle an approximate model in which the vehicle-structure interaction is completely neglected; this type of beam model has been described by Guiseppe and Alessandro [13] as the crudest approximation known to the literature of assessing the dynamic response of an elastic system which supports moving concentrated masses. Lee [9] tackled the transverse vibration of a Timoshenko beam acted on by an accelerating mass. In his study, he presented numerical results for a prescribed constant acceleration or deceleration and the slenderness ratio of the beam. He figured-out that the separation of the mass from the beam may occur for a Timoshenko beam when the traveling speed of the mass is large due to large initial traveling speed or large prescribed acceleration. Nevertheless, his method of solution is incapable of handling moving load problems involving end conditions other than simple ones.

Thus, this work therefore, assesses the dynamic behaviour of a structurally prestressed uniform Rayleigh beam resting on elastic foundation and traversed by masses traveling at an arbitrarily prescribed velocity. Effects of some very important beam parameters on the motions of the vibrating systems are investigated.

2 THEORETICAL FORMULATION

Consider the flexural motion of a uniform finite Rayleigh beam resting on an elastic foundation and carrying a relatively large mass M. The mass M is assumed to touch the beam at time $t = 0$ and travel across it with a non-uniform velocity such that the motion of the contact point of the moving load is described by the function

$$f(t) = x_0 + ct + \frac{1}{2}at^2 \tag{1}$$

where x_0 is the point of application of force $P = Mg$ at the instance $t = 0$, c is the initial velocity and a is the constant acceleration of motion. Furthermore, the beam's properties such as moment of inertia$I$, and the mass per unit length μ of the beam do not vary along the span L of the beam.

The equation of motion with damping neglected is given by the fourth order partial differential equation

$$\begin{aligned} &EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial t^2} - \mu R_0\frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} + KV(x,t) \\ &+M\delta\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2(c+at)\frac{\partial^2 V(x,t)}{\partial x\partial t} + (c+at)^2\frac{\partial^2 V(x,t)}{\partial x^2} + a\frac{\partial V(x,t)}{\partial x}\right] \\ &= Mg\delta\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right] \end{aligned} \tag{2}$$

where x is the spacial coordinate, t is the time, $V(x,t)$ is the Transverse Displacement, EI is the flexural rigidity of the structure, μ is the mass per unit length of the beam, N is the axial force, R_0 is the rotatory inertia factor, K is the elastic foundation stiffness, M is the mass of the traversing concentrated load, g is the acceleration due to gravity and $\delta(\cdot)$ is the well known Dirac delta function.

The boundary conditions of the structure under consideration is arbitrary and the initial conditions without any loss of generality is taken as

$$V(x,0) = 0, \frac{\partial V(x,0)}{\partial t} = 0 \forall\ x \tag{3}$$

Since the load is assumed to be of mass M and the time t is assumed to be limited to that interval of time within which the mass is on the beam, that is

$$0 \le f(t) \le L \tag{4}$$

3 ANALYTICAL PROCEDURES

Equation (2) is a fourth order partial differential equation which in addition to being singular has variable coefficients. In this section, a general approach is developed in order to solve the initial value problem. The approach involves expressing the Dirac delta function as a

Fourier cosine series and then reducing the modified form of the fourth order partial differential equation above using the generalized finite integral transform. The resulting transformed differential equation having some variable coefficients is then simplified using the modified Struble's asymptotic technique.

3.1 The generalized finite integral transform

For the dynamical systems, the governing equation is a fourth order partial differential equation with variable and singular coefficients. The Generalized Finite Integral Transform (GFIT) is employed to remove the singularities in the governing equations and to reduce it to a sequence of second order ordinary differential equations with variable coefficients. This generalized finite integral transform is defined by

$$\overline{V}(m,t) = \int_0^L V(x,t)U_m(x)dx \tag{5}$$

with the inverse

$$V(x,t) = \sum_{m=1}^{\infty} \frac{\mu}{V_m}\overline{V}(m,t)U_m(x) \tag{6}$$

where

$$V_m = \int_0^L \mu U_m^2(x)dx \tag{7}$$

and $U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of functions for beam problems are beam mode shapes. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L} \tag{8}$$

is chosen as a suitable kernel of the integral transform (5) where, λ_m is the mode frequency, A_m, B_m, C_m are constants which are obtained by substituting (8) into the appropriate boundary conditions.

3.2 Operational simplification

By applying the generalized finite integral transform (5), equation (2) can be written as

$$H_1\theta(0,L,t) + H_1\theta_A(t) - H_2\theta_B(t) + \overline{V_{tt}}(m,t) - R_0\theta_C(t) + H_3\overline{V}(m,t) + \theta_D(t) + \theta_E(t) + \theta_F(t) + \theta_G(t) = \frac{Mg}{\mu}U_m\left(x_0 + ct + \frac{1}{2}at^2\right) \tag{9}$$

where

$$H_1 = \frac{EI}{\mu}, H_2 = \frac{N}{\mu}, H_3 = \frac{K}{\mu} \tag{10}$$

$$\theta(0,L,t) = \left[\frac{\partial^3 V(x,t)}{\partial x^3}U_m(x) - \frac{\partial^2 V(x,t)}{\partial x^2}\frac{dU_m(x)}{dx} + \frac{\partial V(x,t)}{\partial x}\frac{d^2U_m(x)}{dx^2} - V(x,t)\frac{d^3U_m(x)}{dx^3}\right]_0^L \tag{11}$$

$$\theta_A(t) = \int_0^L V(x,t)\frac{d^4U_m(x)}{dx^4}dx \tag{12a}$$

$$\theta_B(t) = \int_0^L \frac{\partial^2 V(x,t)}{\partial x^2}U_m(x)dx \tag{12b}$$

$$\theta_C(t) = \int_0^L \frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}U_m(x)dx \tag{12c}$$

$$\theta_D(t) = \int_0^L \frac{M}{\mu}\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\frac{\partial^2 V(x,t)}{\partial t^2}U_m(x)dx \tag{12d}$$

$$\theta_E(t) = \int_0^L \frac{2M(c+at)}{\mu}\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\frac{\partial^2 V(x,t)}{\partial x\partial t}U_m(x)dx \tag{12e}$$

$$\theta_F(t) = \int_0^L \frac{M\,(c+at)^2}{\mu}\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\frac{\partial^2 V(x,t)}{\partial x^2}U_m(x)dx, \text{ and} \tag{12f}$$

$$\theta_G(t) = \int_0^L \frac{aM}{\mu}\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\frac{\partial V(x,t)}{\partial x}U_m(x)dx \tag{12g}$$

In order to evaluate the integrals (12a-12g), use is made of the property of the Dirac Delta function as an even function to express it in Fourier cosine series namely:

$$\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\cos\frac{n\pi x}{L} \tag{13}$$

Thus, in view of (6), using equation (13) in equation (9), after some simplification and rearrangements one obtains

$$
\begin{aligned}
&\overline{V}_{tt}(m,t)+\left[\omega_n^2+\frac{K}{\mu}\right]\overline{V}(m,t)-R_0\sum_{k=1}^{\infty}\overline{V}_{tt}(k,t)H_a(k,m)-\frac{N}{\mu}\sum_{k=1}^{\infty}\overline{V}(k,t)H_a(k,m)\\
&+\varepsilon^*\left\{\sum_{k=1}^{\infty}\overline{V}_{tt}(k,t)H_b(k,m)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}_{tt}(k,t)H_c(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right.\\
&+2(c+at)\sum_{k=1}^{\infty}\overline{V}_t(k,t)H_d(k,m)+4(c+at)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}_t(k,t)H_e(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\\
&+(c+at)^2\sum_{k=1}^{\infty}\overline{V}(k,t)H_f(k,m)+2(c+at)^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}(k,t)H_g(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\\
&\left.+a\sum_{k=1}^{\infty}\overline{V}(k,t)H_i(k,m)+2a\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}(k,t)H_j(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right\}=\\
&\frac{Mg}{\mu}\left[\sin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+A_m\cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+B_m\sinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\right.\\
&\left.C_m\cosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right]
\end{aligned}
\tag{14}
$$

where

$$H_a(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k''(x)U_m(x)dx \tag{15a}$$

$$H_b(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k(x)U_m(x)dx \tag{15b}$$

$$H_c(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k(x)U_m(x)\cos\frac{n\pi x}{L}dx \tag{15c}$$

$$H_d(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k'(x)U_m(x)dx \tag{15d}$$

$$H_e(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k'(x)U_m(x)\cos\frac{n\pi x}{L}dx \tag{15e}$$

$$H_f(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k''(x)U_m(x)dx \tag{15f}$$

$$H_g(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k''(x)U_m(x)\cos\frac{n\pi x}{L}dx \tag{15g}$$

$$H_i(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k'(x)U_m(x)dx \tag{15h}$$

$$H_j(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k'(x)U_m(x)\cos\frac{n\pi x}{L}dx \tag{15i}$$

$$\omega_n^2 = \frac{\lambda_m^4}{L^4}\frac{EI}{\mu} \tag{15j}$$

$$\varepsilon^* = \frac{M}{\mu L} \tag{15k}$$

Equation (14) is the transformed equation governing the problem of a uniform Bernoulli-Euler beam on a constant elastic foundation. This coupled non-homogeneous Second order ordinary differential equation holds for all variants of the classical boundary conditions.

In what follows, two special cases of equation (14) are considered

3.3 Solution of the transformed governing equation

Case I: The Moving Force Problem. The differential equation describing the behaviour of a Rayleigh beam on an elastic foundation to a moving force moving at variable velocity may be obtained from equation (14) by setting $\varepsilon^* = 0$. It is an approximate model, which assumes the inertia effect of the moving mass negligible and only the force effect of the moving load is taken into consideration, thus in this case one obtains

$$\overline{V}_{tt}(m,t) + \left[\omega_n^2 + \frac{K}{\mu}\right]\overline{V}(m,t) - R_0\sum_{k=1}^{\infty}\overline{V}_{tt}(k,t)H_a(k,m) - \frac{N}{\mu}\sum_{k=1}^{\infty}\overline{V}(k,t)H_a(k,m) = \frac{Mg}{\mu}\left[\sin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) + A_m\cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) + B_m\sinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) + C_m\cosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{16}$$

Evidently, an exact analytical solution to equation (16) is not possible. Though the equation may readily yield to numerical technique, an analytical approximate method is desirable as solutions so obtained often shed light on vital information about the vibrating system. Thus, we are going to use a modification of the asymptotic method due to Struble's extensively discussed in [14]. To this effect, equation (16) is rearranged to take the form

$$\overline{V}_{tt}(m,t) + \frac{\gamma_{nf}^2}{[1-\varepsilon_0 LH_a(m,m)]}\overline{V}(m,t) - \frac{\varepsilon_0}{[1-\varepsilon_0 LH_a(m,m)]}\left[\sum_{\substack{k=1\\k\neq m}}^{\infty} LH_a(k,m)\overline{V}_{tt}(k,t) + N_0\sum_{\substack{k=1\\k\neq m}}^{\infty} H_a(k,m)\overline{V}(k,m)\right] = \frac{Mg}{\mu[1-\varepsilon_0 LH_a(m,m)]}\left[\sin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) + A_m\cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) + B_m\sinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) + C_m\cosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{17}$$

where

$$\varepsilon_0 = \frac{R^0}{L}, \ N_0 = \frac{LN}{R^0\mu}, \ \gamma_{nf}^2 = \omega_{nk}^2 - N_0 H_a(m,m), \ \omega_{nk}^2 = \omega_n^2 + \frac{K}{\mu} \ \text{and} \ H_a(m,m) = H_a(k,m)|_{k=m} \tag{18}$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the rotatory inertia. An equivalent free system operator defined by the modified frequency then replaces equation (17). Thus, we set the right-hand-side of (17) to zero and consider a parameter $\eta_0 < 1$ for any arbitrary ratio ε_0, defined as

$$\eta_0 = \frac{\varepsilon_0}{1+\varepsilon_0} \tag{19}$$

so that

$$\varepsilon_0 = \eta_0 + O(\eta_0^2) \tag{20}$$

Substituting equation (20) into the homogeneous part of equation (17) one obtains

$$\begin{aligned} &\overline{V}_{tt}(m,t) + \gamma_{nf}^2(1+\eta_0 L H_a(m,m))\overline{V}(m,t) - \\ &\eta_0(1+\eta_0 L H_a(m,m))\left[\sum_{\substack{k=1\\k\neq m}}^{\infty} L H_a(k,m)\overline{V}_{tt}(k,t) + N_0 \sum_{\substack{k=1\\k\neq m}}^{\infty} H_a(k,m)\overline{V}(k,m)\right] = 0 \end{aligned} \tag{21}$$

When η is set to zero in equation (17) a situation corresponding to the case in which the rotatory inertia effect is regarded as negligible is obtained, then the solution of equation (17) can be written as

$$\overline{V}_{nf}(m,t) = C_{nf}\cos[\omega_{nf}t - \psi_{nf}] \tag{22}$$

where C_{nf}, ω_{nf} and ψ_{nf} are constants.

Furthermore as $\eta_0 < 1$ Struble's technique requires that the asymptotic solutions of the homogeneous part of the equation (17) be of the form

$$\overline{V}(m,t) = \Lambda(m,t)\cos[\omega_{nf}t - \phi(m,t)] + \eta_0\bar{V}(1,t) + O(\eta_0^2) \tag{23}$$

where $\Lambda(m,t)$ and $\phi(m,t)$ are slowly varying functions of time.

To obtain the modified frequency, equation (23) and its derivatives are substituted into equation (21) and neglecting terms which do not contribute to variational equations, one obtains.

$$\begin{aligned} &2\Lambda(m,t)\gamma_{nf}\dot{\phi}(m,t)\cos[\gamma_{nf}t - \phi(m,t)] - 2\dot{\Lambda}(m,t)\gamma_{nf}\sin[\gamma_{nf}t - \phi(m,t)] \\ &-\eta_0\gamma_{nf}^2 L H_a(m,m)\Lambda(m,t)\cos[\gamma_{nf}t - \phi(m,t)] = 0 \end{aligned} \tag{24}$$

retaining terms to $O(\eta_0)$ only.

The variational equations are obtained by equating the coefficients of $\sin[\gamma_{nf}t - \phi(m,t)]$ and $\cos[\gamma_{nf}t - \phi(m,t)]$ on both sides of the equation (24). Thus,

$$-2\dot{\Lambda}(m,t)\gamma_{nf} = 0 \tag{25}$$

and

$$2\Lambda(m,t)\gamma_{nf}\dot{\phi}(m,t) - \eta_0\gamma_{nf}^2 LH_a(m,m)\Lambda(m,t) = 0 \tag{26}$$

Solving equations (25) and (26) respectively gives

$$\Lambda(m,t) = C_{mf} \tag{27}$$

and

$$\phi(m,t) = -\frac{\eta_0 L\gamma_{nf}H_a(m,m)}{2\omega_{nf}}t + \psi_{mf} \tag{28}$$

where C_{mf}^0 and ψ_{mf} are constants.

Therefore, when the effect of the rotatory inertia is considered, the first approximation to the homogeneous system is

$$\overline{V}(m,t) = C_{mf}^0 Cos[\gamma_{mf}t - \psi_{mf}] \tag{29}$$

where

$$\gamma_{mf} = \frac{\gamma_{nf}}{2}[2 + \eta_0 LH_a(m,m)] \tag{30}$$

represents the modified natural frequency due to the effect of the rotatory inertia R_0. It is observed that when $\eta_0 = 0$, we recover the frequency of the moving force problem when the rotatory inertia effect of the beam is neglected. Thus, to solve the non-homogeneous equation (17), the differential operator which acts on $\overline{V}(m,t)$ and $\overline{V}(k,t)$ is replaced by the equivalent free System operator defined by the modified frequency γ_{mf}, thus using equation (30) the homogeneous part of equation (17) can be written as

$$\frac{d^2\overline{V}(m,t)}{dt^2} + \gamma_{mf}^2\overline{V}(m,t) = 0 \tag{31}$$

Hence, the entire equation (17) takes the form

$$\begin{aligned}\frac{d^2\overline{V}(m,t)}{dt^2} + \gamma_{mf}^2\overline{V}(m,t) = P_{mf}^0\Big[&\sin\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) + A_m\cos\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\\ &+B_m\sinh\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) + C_m\cosh\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\Big]\end{aligned} \tag{32}$$

where

$$P_{mf}^{0} = \frac{Mg}{\mu\left[1 - \eta_0 L H_a(m,m)\right]} \tag{33}$$

Solving equation (32) in conjunction with the initial conditions gives expression for $\bar{V}(m,t)$which on inversion yields

$$\begin{aligned}
V(x,t) = & \frac{1}{\tau^*(x)} \sum_{m=1}^{\infty} \left(\frac{P_{mf}^{0} \sin\gamma_{mf} t}{\gamma_{mf}} \left(D_{13} S\left[q_{12} + q_{10}t\right] - D_{14} C\left[q_{12} + q_{10}t\right] - D_{11} S\left[q_{11} + q_{10}t\right] + D_{12} C\left[q_{11} + q_{10}t\right] \right.\right.\\
& + D_{23} C\left[q_{11} + q_{10}t\right] + D_{24} S\left[q_{11} + q_{10}t\right] - D_{21} C\left[q_{12} + q_{10}t\right] - D_{22} S\left[q_{12} + q_{10}t\right] + E_{14} erfi\left[q_{22} + q_{20}t\right]\\
& - E_{13} erf\left[q_{22} + q_{20}t\right] + E_{12} erfi\left[q_{21} + q_{20}t\right] - E_{11} erf\left[q_{21} + q_{20}t\right] + E_{24} erf\left[q_{22} + q_{20}t\right] + E_{23} erfi\left[q_{22} + q_{20}t\right]\\
& \left. + E_{22} erf\left[q_{21} + q_{20}t\right] + - E_{21} erfi\left[q_{21} + q_{20}t\right] - C_2^0 \right) - \frac{P_{mf}^{0} \cos\gamma_{mf} t}{\gamma_{mf}} \left(D_{11} C\left[q_{11} + q_{10}t\right] + D_{12} S\left[q_{11} + q_{10}t\right] \right.\\
& - D_{13} C\left[q_{12} + q_{10}t\right] - D_{14} S\left[q_{12} + q_{10}t\right] + D_{21} S\left[q_{12} + q_{10}t\right] - D_{22} C\left[q_{12} + q_{10}t\right] - D_{23} S\left[q_{11} + q_{10}t\right]\\
& - D_{24} S\left[q_{11} + q_{10}t\right] + iE_{11} erf\left[q_{21} + q_{20}t\right] + iE_{12} erfi\left[q_{21} + q_{20}t\right] + iE_{13} erf\left[q_{22} + q_{20}t\right]\\
& + iE_{14} erfi\left[q_{22} + q_{20}t\right] + iE_{21} erfi\left[q_{21} + q_{20}t\right] - iE_{22} erf\left[q_{21} + q_{20}t\right] - iE_{23} erfi\left[q_{22} + q_{20}t\right]\\
& \left.\left. + iE_{24} erf\left[q_{22} + q_{20}t\right] + C_1^0 \right)\right) \left(\sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L} \right)
\end{aligned} \tag{34}$$

where $C(x)$ and $S(x)$ are the well known time-dependent Fresnel integrals defined by

$$C(x) = \int_0^x \cos\frac{\pi t^2}{2} dt \quad \text{and} \quad S(x) = \int_0^x \sin\frac{\pi t^2}{2} dt$$

and

$$D_{11} = \frac{1}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Cos\left(\frac{b_1^2}{4a_0} - c_0\right), D_{12} = \frac{1}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Sin\left(\frac{b_1^2}{4a_0} - c_0\right)$$

$$D_{13} = \frac{1}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Cos\left(\frac{b_2^2}{4a_0} - c_0\right), D_{14} = \frac{1}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Sin\left(\frac{b_2^2}{4a_0} - c_0\right)$$

$$D_{21} = \frac{A_m}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Cos\left(\frac{b_2^2}{4a_0} - c_0\right), D_{22} = \frac{A_m}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Sin\left(\frac{b_2^2}{4a_0} - c_0\right)$$

$$D_{23} = \frac{A_m}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Cos\left(\frac{b_1^2}{4a_0} - c_0\right), D_{24} = \frac{A_m}{2\sqrt{a_0}}\sqrt{\frac{\pi}{2}} Sin\left(\frac{b_1^2}{4a_0} - c_0\right)$$

$$E_{11} = \frac{B_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_3^2}{4a_0} - c_0} e^{\frac{b_3^2}{2a_0}}, E_{12} = \frac{B_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_3^2}{4a_0} - c_0} e^{2c_0}, E_{13} = \frac{B_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_4^2}{4a_0} - c_0} e^{\frac{b_4^2}{2a_0}}$$

$$E_{14} = \frac{B_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_4^2}{4a_0} - c_0} e^{2c_0}, E_{21} = \frac{C_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_3^2}{4a_0} - c_0} e^{2c_0}, E_{22} = \frac{C_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_3^2}{4a_0} - c_0} e^{\frac{b_3^2}{2a_0}}$$

$$E_{23} = \frac{C_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_4^2}{4a_0} - c_0} e^{2c_0}, E_{24} = \frac{C_m\sqrt{\pi}}{8\sqrt{a_0}} e^{-\frac{b_4^2}{4a_0} - c_0} e^{\frac{b_4^2}{2a_0}}$$

$$q_{10} = \frac{2a_0}{\sqrt{2\pi a_0}}, q_{11} = \frac{b_1}{\sqrt{2\pi a_0}}, q_{12} = \frac{b_2}{\sqrt{2\pi a_0}}, q_{20} = \frac{2a_0}{2\sqrt{a_0}}, q_{21} = \frac{b_3}{2\sqrt{a_0}}, q_{22} = \frac{b_4}{2\sqrt{a_0}}$$

$$a_0 = \frac{\lambda_m a}{2L}, b_1 = \frac{c\lambda_m}{L} - \gamma_{mf}, b_2 = \frac{c\lambda_m}{L} + \gamma_{mf}, b_3 = \frac{c\lambda_m}{L} - i\gamma_{mf}, b_3 = \frac{c\lambda_m}{L} + i\gamma_{mf}$$

$$\begin{aligned} C_1^0 = & D_{11}C(q_{11}) + D_{12}S(q_{11}) - D_{13}C(q_{12}) - D_{14}S(q_{12}) + D_{21}S(q_{12}) - D_{22}C(q_{12}) - D_{23}S(q_{11}) - D_{24}C(q_{11}) \\ & + iE_{11}erf(q_{12}) + iE_{12}erfi(q_{21}) + iE_{13}erf(q_{22}) + iE_{14}erfi(q_{22}) + iE_{21}erfi(q_{21}) - iE_{22}erf(q_{21}) - iE_{23}erfi(q_{22}) \\ & + iE_{24}erf(q_{22}) \end{aligned}$$

$$\begin{aligned} C_2^0 = & D_{13}S(q_{12}) - D_{14}C(q_{12}) - D_{11}S(q_{11}) + D_{12}C(q_{11}) + D_{23}C(q_{11}) + D_{24}S(q_{11}) - D_{21}C(q_{12}) - D_{22}S(q_{12}) \\ & + E_{14}erfi(q_{22}) - E_{13}erf(q_{22}) + E_{12}erfi(q_{22}) - E_{11}erf(q_{21}) + E_{24}erf(q_{22}) + E_{23}erfi(q_{22}) + E_{22}erf(q_{21}) \\ & + E_{21}erfi(q_{21}) \end{aligned}$$

$$i = \sqrt{-1}$$

and

$$\tau^*(x) = \int_0^L U_m^2(x)dx \tag{35}$$

Equation (34) represents the transverse displacement response to forces moving with non-uniform velocities of prestressed uniform Rayleigh beam resting on elastic foundation and having arbitrary end support conditions.

Case II: The Moving Mass Problem. In the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus, in this case, $\varepsilon^* \neq 0$, and the solution of the entire equation (14) is required. This is termed the moving mass problem. Evidently, a closed form solution of equation (14) is not possible. Again, an approximate analytical method due to Struble is resorted to. It is remarked at this juncture that neglecting the terms representing the inertia term of the moving mass, we obtain equation (17). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency γ_{mf} due to the presence of the effect of rotatory inertia. Thus, equation (14) can be written in the form

$$\begin{aligned}
&\overline{V}_{tt}(m,t)+\gamma_{mf}^2\overline{V}(m,t)+\varepsilon^*\left\{\sum_{k=1}^{\infty}\overline{V}_{tt}(k,t)H_b(k,m)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}_{tt}(k,t)H_c(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right.\\
&+2(c+at)\sum_{k=1}^{\infty}\overline{V}_t(k,t)H_d(k,m)+4(c+at)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}_t(k,t)H_e(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\\
&+(c+at)^2\sum_{k=1}^{\infty}\overline{V}(k,t)H_f(k,m)+2(c+at)^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}(k,t)H_g(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\\
&\left.+a\sum_{k=1}^{\infty}\overline{V}(k,t)H_i(k,m)+2a\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}(k,t)H_j(k,m,n)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right\}=\frac{Mg}{\mu}\left[\sin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right.\\
&\left.+A_m\cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+B_m\sinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+C_m\cosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right]
\end{aligned}\tag{36}$$

As in the previous case, an exact analytical solution to the above equation is not possible. The same technique used in case I is employed to obtain the modified frequency due to the presence of the moving mass, namely

$$\gamma_{mm}=\gamma_{mf}\left\{1-\frac{\eta^*}{2}\left[H_b(m,m)-\frac{\left(c^2H_f(m,m)+aH_i(m,m)\right)}{\gamma_{mf}^2}\right]\right\}\tag{37}$$

where

$$\begin{aligned}
&\eta^*=\frac{\varepsilon^*}{1+\varepsilon^*}\ \text{ and }\ \varepsilon^*=\frac{M}{\mu L},\quad H_b(m,m)=H_b(k,m)\big|_{k=m},\\
&H_f(m,m)=H_f(k,m)\big|_{k=m}\quad H_i(m,m)=H_i(k,m)\big|_{k=m}
\end{aligned}\tag{38}$$

retaining $O(\lambda)$ only.

Thus, equation (36) takes the form

$$\begin{aligned}
\frac{d^2\overline{V}(m,t)}{dt^2}+\gamma_{mm}^2\overline{V}(m,t)=\eta^*Lg&\left[\sin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+A_m\cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right.\\
&\left.+B_m\sinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+C_m\cosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right]
\end{aligned}\tag{39}$$

This is analogous to equation (32). Thus, using similar argument as in case I, $\overline{V}(m,t)$ can be obtained and which on inversion yields

$$\begin{aligned}
V(x,t)=&\frac{1}{\tau^*(x)}\sum_{m=1}^{\infty}\frac{\eta^*Lg}{\gamma_{mm}}(\sin\gamma_{mm}t\,(D_{13}S[q_{12}+q_{10}t]-D_{14}C[q_{12}+q_{10}t]-D_{11}S[q_{11}+q_{10}t]+D_{12}C[q_{11}+q_{10}t]\\
&+D_{23}C[q_{11}+q_{10}t]+D_{24}S[q_{11}+q_{10}t]-D_{21}C[q_{12}+q_{10}t]-D_{22}S[q_{12}+q_{10}t]+E_{14}erfi[q_{22}+q_{20}t]\\
&-E_{13}erf[q_{22}+q_{20}t]+E_{12}erfi[q_{21}+q_{20}t]-E_{11}erf[q_{21}+q_{20}t]+E_{24}erf[q_{22}+q_{20}t]+E_{23}erfi[q_{22}+q_{20}t]\\
&+E_{22}erf[q_{21}+q_{20}t]+-E_{21}erfi[q_{21}+q_{20}t]-C_2^0)-\cos\gamma_{mm}t\,(D_{11}C[q_{11}+q_{10}t]+D_{12}S[q_{11}+q_{10}t]\\
&-D_{13}C[q_{12}+q_{10}t]-D_{14}S[q_{12}+q_{10}t]+D_{21}S[q_{12}+q_{10}t]-D_{22}C[q_{12}+q_{10}t]-D_{23}S[q_{11}+q_{10}t]\\
&-D_{24}S[q_{11}+q_{10}t]+iE_{11}erf[q_{21}+q_{20}t]+iE_{12}erfi[q_{21}+q_{20}t]+iE_{13}erf[q_{22}+q_{20}t]\\
&+iE_{14}erfi[q_{22}+q_{20}t]+iE_{21}erfi[q_{21}+q_{20}t]-iE_{22}erf[q_{21}+q_{20}t]-iE_{23}erfi[q_{22}+q_{20}t]\\
&+iE_{24}erf[q_{22}+q_{20}t]+C_1^0)\ )\left(\sin\frac{\lambda_mx}{L}+A_m\cos\frac{\lambda_mx}{L}+B_m\sinh\frac{\lambda_mx}{L}+C_m\cosh\frac{\lambda_mx}{L}\right)
\end{aligned}\tag{40}$$

where all parameters are as previously defined, but γ_{mm} has replaced γ_{mf}

Equation (40) represents the transverse displacement response to concentrated masses, moving with non-uniform velocity of highly prestressed uniform Rayleigh beam resting on elastic foundation. Equation (40) is valid for all variants of classical boundary conditions.

4 APPLICATIONS

In this section, the forgoing analyses are illustrated by various practical examples. Specifically, classical boundary conditions such as simply supported boundary conditions, clamped-clamped end conditions and clamped-free end conditions are considered.

4.1 Simply Supported Boundary Conditions

In this case, the displacement and the bending moment vanish. Thus

$$V(0,t) = 0 = V(L,t), \qquad \frac{\partial^2 V(0,t)}{\partial x^2} = 0 = \frac{\partial^2 V(L,t)}{\partial x^2} \tag{41}$$

Hence for normal modes

$$U_m(0) = 0 = U_m(L), \qquad \frac{\partial^2 U_m(0)}{\partial x^2} = 0 = \frac{\partial^2 U_m(L)}{\partial x^2} \tag{42}$$

which implies that

$$U_k(0) = 0 = U_k(L), \qquad \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2} \tag{43}$$

Applying (41) and (42), one obtains

$$\begin{gathered} A_m = A_k = 0; \quad B_m = B_k = 0; \quad C_m = C_k = 0 \\ \lambda_m = m\pi \qquad and \qquad \lambda_k = k\pi \end{gathered} \tag{44}$$

Thus, the moving force problem is reduced to a non-homogeneous second order ordinary differential equation

$$\frac{d^2\overline{V}(m,t)}{dt^2} + \alpha_{mf}^2 \overline{V}(m,t) = \frac{P_{mf}}{\mu} \sin\left(x_0 + ct + \frac{1}{2}at^2\right) \tag{45}$$

where

$$\alpha_{mf}^2 = \frac{EI\left(\frac{m\pi}{L}\right)^4 + \frac{K}{\mu} + \frac{N}{\mu}\left(\frac{m\pi}{L}\right)^2}{1 + R_0\left(\frac{m\pi}{L}\right)^2} \text{ and } P_{mf} = \frac{Mg}{1 + R_0\left(\frac{m\pi}{L}\right)^2} \tag{46}$$

Equation (45) when solved in conjunction with the initial conditions, one obtains an expression for $\overline{V}(m,t)$ which on inversion yields

$$
\begin{aligned}
V(x,t) = \frac{2}{L}\sum_{m=1}^{\infty}\frac{P_{mf}\sqrt{\pi}}{2\mu\alpha_{mf}\sqrt{2a_0}}\Bigg(&\sin\alpha_{mf}t\Bigg[\cos\left(\frac{b_2^2}{4a_0}-c_0\right)S\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)-C\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)\\
&+\cos\left(\frac{b_1^2}{4a_0}-c_0\right)S\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)-C\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)-\cos\left(\frac{b_2^2}{4a_0}-c_0\right)S\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\\
&+C\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)-\cos\left(\frac{b_1^2}{4a_0}-c_0\right)S\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)+C\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)\Bigg]\\
&-\cos\alpha_{mf}t\Bigg[\cos\left(\frac{b_1^2}{4a_0}-c_0\right)C\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)+S\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)-\\
&\cos\left(\frac{b_2^2}{4a_0}-c_0\right)C\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)-S\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)-\cos\left(\frac{b_1^2}{4a_0}-c_0\right)C\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)-\\
&S\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)\\
&+\cos\left(\frac{b_2^2}{4a_0}-c_0\right)C\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)+\\
&S\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)\Bigg]\Bigg)\left(\sin\frac{m\pi}{L}x\right)
\end{aligned}
\tag{47}
$$

which represents the transverse displacement response to forces moving with non-uniform velocity of a simply supported uniform Rayleigh beam resting on elastic foundation.

Following arguments similar to those in the last sections, use is made of the modified asymptotic method due to Struble to obtain the modified natural frequency due to the presence of inertia terms for the simply supported beam given as

$$
\alpha_{mm} = \alpha_{mf}\left\{1-\frac{\eta}{2}\left[2+\frac{\left(2c^2R(m,m)-R_2(m,m,n)\right)}{\alpha_{mf}^2}\right]\right\}
\tag{48}
$$

where

$$
R(m,m) = \left(\frac{m\pi}{L}\right)^2 \quad\text{and}\quad R_2(m,m,n) = \sum_{n=1}^{\infty}\frac{8am^2}{L\left(4m^2-n^2\right)}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)
\tag{49}
$$

neglecting higher order terms of λ. Thus, the simply supported moving mass problem reduces to

$$
\frac{d^2\overline{V}(m,t)}{dt^2}+\alpha_{mm}^2\overline{V}(m,t) = \eta Lg\sin\frac{m\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)
\tag{50}
$$

which when solved in conjunction with the initial conditions gives expression for $\overline{V}(m,t)$ and on inversion gives

$$
\begin{aligned}
V(x,t) =& \frac{2}{L}\sum_{m=1}^{\infty}\frac{\eta L g\sqrt{\pi}}{2\mu\alpha_{mm}\sqrt{2a_0}}\left(\sin\alpha_{mm}t\left[Cos\left(\frac{b_2^2}{4a_0}-c_0\right)S\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)-C\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)\right.\right.\\
&+\cos\left(\frac{b_1^2}{4a_0}-c_0\right)S\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)-C\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)-\cos\left(\frac{b_2^2}{4a_0}-c_0\right)S\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\\
&\left.+C\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)-\cos\left(\frac{b_1^2}{4a_0}-c_0\right)S\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)+C\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)\right]\\
&-\cos\alpha_{mm}t\left[\cos\left(\frac{b_1^2}{4a_0}-c_0\right)C\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)+S\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)-\right.\\
&\cos\left(\frac{b_2^2}{4a_0}-c_0\right)C\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)-S\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)-\cos\left(\frac{b_1^2}{4a_0}-c_0\right)C\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)-\\
&\left.\left.S\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_1^2}{4a_0}-c_0\right)+\cos\left(\frac{b_2^2}{4a_0}-c_0\right)C\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)+S\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\sin\left(\frac{b_2^2}{4a_0}-c_0\right)\right]\right)\left(\sin\frac{m\pi}{L}x\right)
\end{aligned}
\tag{51}
$$

This represents the transverse-displacement response to a concentrated mass moving with non-uniform velocity of a simply supported uniform Rayleigh beam resting on elastic foundation.

5 COMMENTS ON CLOSED FORM SOLUTIONS

It is pertinent at this juncture to establish conditions under which resonance occurs. This phenomenon in structural and highway engineering is of great concern to researchers or in particular, design engineers, because, for example, it causes cracks, permanent deformation and destruction in structures. Bridges and other structures are known to have collapsed as a result of resonance occurring between the structure and some signals traversing them.

Equation (47) clearly shows that the Simply Supported elastic beam resting on elastic foundation and traversed by moving force reaches a state of resonance whenever

$$\alpha_{mf} = \frac{m\pi c_c}{L}, \alpha_{mf} = \frac{m\pi c_c}{L} + 2a_0 t_c \tag{52}$$

while equation (51) indicates that the same beam under the action of moving mass will experience resonance effect whenever

$$\alpha_{mm} = \frac{m\pi c_c}{L}, \alpha_{mm} = \frac{m\pi c_c}{L} + 2a_0 t_c \tag{53}$$

where c_c and t_c are respectively the critical velocity and critical time at which resonance occurs.

From equation (48), we know that

$$\alpha_{mm} = \alpha_{mf}\left\{1-\frac{\eta}{2}\left[2+\frac{\left(2c^2R(m,m)-R_2(m,m,n)\right)}{\alpha_{mf}^2}\right]\right\} \tag{54}$$

which implies

$$\alpha_{mf} = \frac{\frac{m\pi c_c}{L}}{\left\{1 - \frac{\eta}{2}\left[2 + \frac{(2c^2 R(m,m) - R_2(m,m,n))}{\alpha_{mf}^2}\right]\right\}} \tag{55}$$

It is therefore evident, that for the same natural frequency, the critical velocity for the system consisting of a Simply Supported Elastic Beam resting on an elastic foundation and traversed by concentrated forces moving with a non-uniform speed is greater than that of the moving mass problem. Thus, for the same natural frequency of an elastic beam, resonance is reached earlier in the moving mass system than in the moving force system.

For the resonance conditions for other classical boundary conditions, equation (34) clearly shows that the uniform elastic beam resting on an elastic foundation and traversed by concentrated forces moving with variable velocities reaches a state of resonance whenever

$$\gamma_{mf} = \frac{\lambda_m c_c}{L} \text{ and } \gamma_{mf} = \frac{\lambda_m c_c}{L} + \frac{2a_0 t}{L} \tag{56}$$

while equation (40) shows that the same beam under the action of a moving mass experiences resonance effect whenever

$$\gamma_{mm} = \frac{\lambda_m c_c}{L} \text{ and } \gamma_{mm} = \frac{\lambda_m c_c}{L} + \frac{2a_0 t}{L} \tag{57}$$

From equation (37)

$$\gamma_{mm} = \gamma_{mf}\left\{1 - \frac{\eta^*}{2}\left[(H_b(m,m)) - \frac{c^2\left(H_f(m,m) + aH_i(m,m)\right)}{\gamma_{mf}^2}\right]\right\} \tag{58}$$

which implies

$$\gamma_{mf} = \frac{\frac{\lambda_m c_c}{L}}{1 - \frac{\eta^*}{2}\left[(H_b(m,m)) - \frac{c^2(H_f(m,m) + aH_i(m,m))}{\gamma_{mf}^2}\right]} \tag{59}$$

Evidently, from equation (58) and (59), the same results and analysis obtained in the case of a Simply Supported Bernoulli-Euler beam are obtained for all other examples of classical boundary conditions.

6 NUMERICAL RESULTS AND DISCUSSION

We shall illustrate the analysis proposed in this paper by considering a homogenous beam of modulus of elasticity $E = 3.1 \times 10^{10}\text{N/m}^2$, the moment of inertia $I = 2.87698 \times 10^{-3}\text{m}^4$,

the beam span $L = 150\text{m}$ and the mass per unit length of the beam μ=2758.291 Kg/m. The values of foundation moduli are varied between 0N/m^3 and 400000N/m^3, the values of axial force N is varied between 0 N and $2 \cdot 0 \times 10^8$N.

Figure 1 displays the transverse displacement response of a clamped-clamped uniform Rayleigh beam under the action of concentrated forces moving at variable velocity for various values of axial force N and for fixed values of foundation modulus K=40000 and Rotatory inertia correction factor R_o=50. The figure shows that as N increases, the dynamic deflection of the uniform beam decreases. Similar results are obtained when the fixed-fixed beam is subjected to a concentrated masses traveling at variable velocity as shown in figure 4. For various traveling time t, the deflection profile of the beam for various values of foundation modulus K and for fixed values of axial force N=200000 and Rotatory inertia correction factor R_o=50 are shown in figure 2. It is observed that higher values of foundation modulus reduce the deflection profile of the vibrating beam. The same behaviour characterizes the deflection profile of the clamped-clamped beam under the action of concentrated masses moving at variable velocity for various values of foundation modulus K as shown in figure 5. Also, figures 3 and 6 display the response amplitudes of the clamped-clamped uniform Rayleigh beam respectively to concentrated forces and masses traveling at variable velocity for various values of rotatory inertia R_o and for fixed values of axial force N=200000 and foundation modulus K=40000. These figures clearly show that as the values of rotatory inertia correction factor increases, the response amplitudes of the clamped-clamped uniform beam under the action of both concentrated forces and masses traveling at variable velocity decrease. Figure 7 depicts the comparison of the transverse displacement response of moving force and moving mass cases of a clamped-clamped uniform Rayleigh bean traversed by a moving load traveling at variable velocity for fixed values of N=200000, K=400000 and R_o=50.

values of rotatory inertia Ro and for fixed values of axial force N=200000 and foundation modulus K=40000.

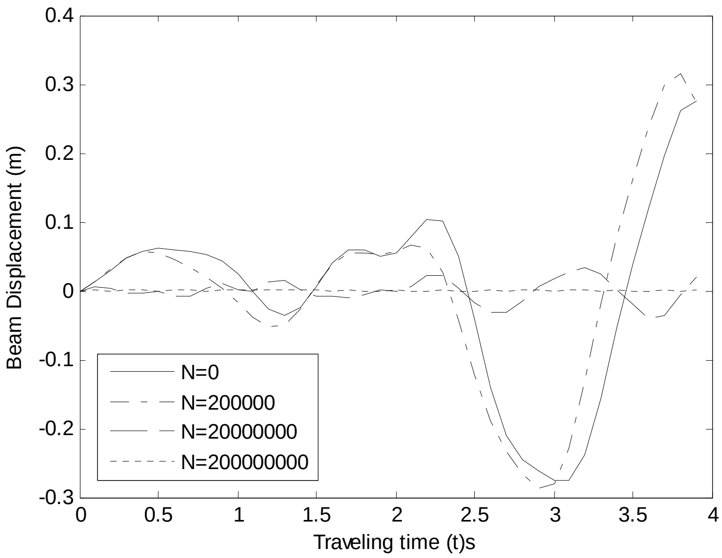

Figure 1 Transverse displacement of a clamped-clamped uniform Rayleigh beam under the actions of concentrated forces traveling at variable velocity for various values of axial force N and for fixed values of foundation modulus K=40000 and rotatory inertia R_o=50.

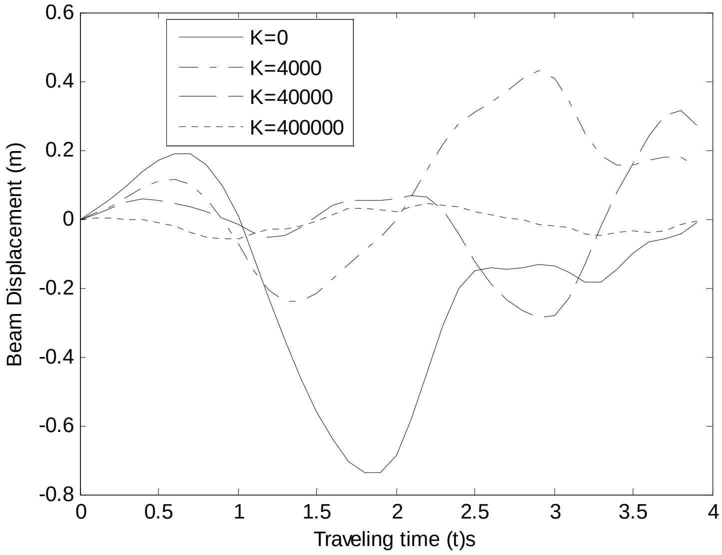

Figure 2 Deflection profile of a clamped-clamped uniform Rayleigh beam under the actions of concentrated forces traveling at variable velocity for various values of foundation modulus K and for fixed values of axial force N=200000 and rotatory inertia R_o=50.

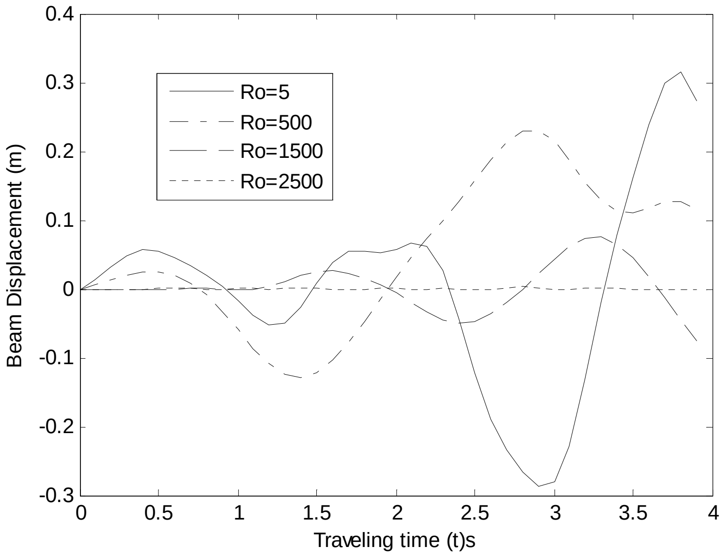

Figure 3 Response amplitude of a clamped-clamped uniform Rayleigh beam under the actions of concentrated forces traveling at variable velocity for various values of rotatory inertia Ro and for fixed values of foundation modulus K=40000 and axial force N=200000.

For other boundary conditions, namely the simply-supported and cantilever beams we obtain results similar to that of the clamped-clamped end conditions presented in this work. It is further established the results obtained in this study is in perfect agreement with existing results [5, 12, 14, 15].

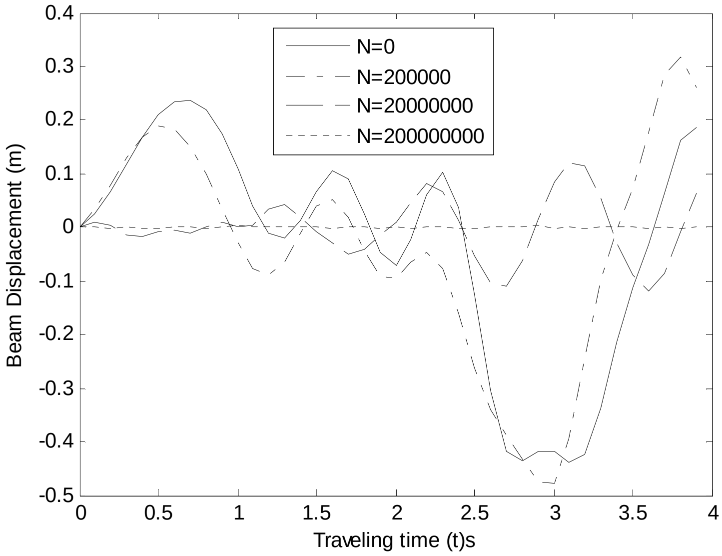

Figure 4 Transverse displacement of a clamped-clamped uniform Rayleigh beam under the actions of concentrated masses traveling at variable velocity for various values of axial force N and for fixed values of foundation modulus K=40000 and rotatory inertia R_o=50.

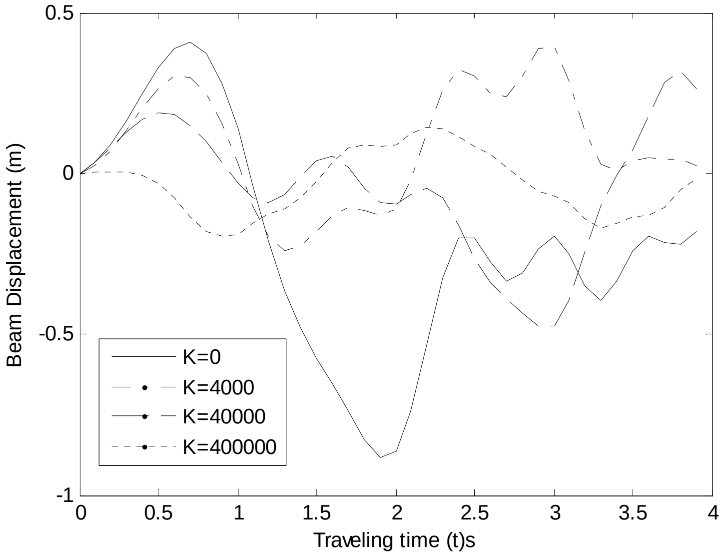

Figure 5 Deflection profile of a clamped-clamped uniform Rayleigh beam under the actions of concentrated masses traveling at variable velocity for various values of foundation modulus K and for fixed values of axial force N=200000 and rotatory inertia R_o=50.

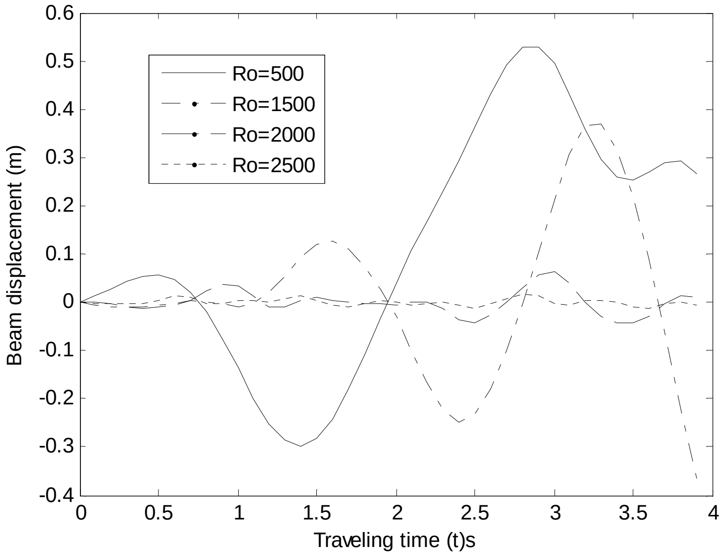

Figure 6 Response Amplitude of a clamped-clamped uniform Rayleigh beam under the actions of concentrated masses traveling at variable velocity for various values of rotatory inertia Ro and for fixed values of foundation modulus K=40000 and axial force N=200000.

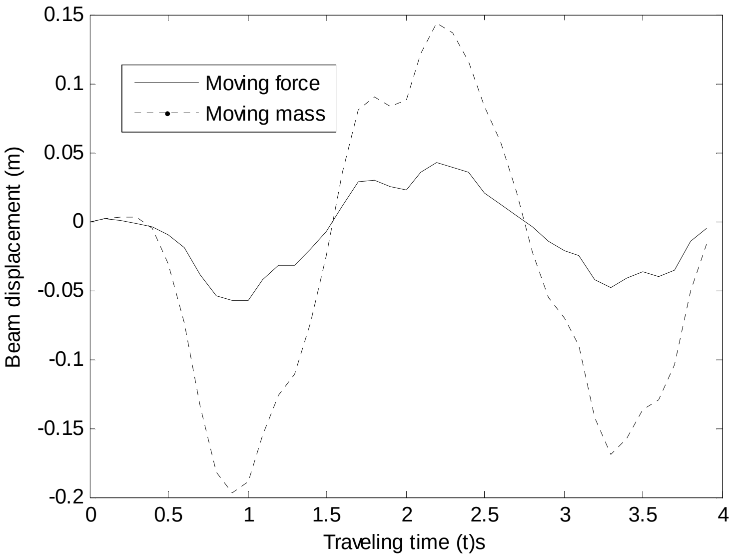

Figure 7 Comparison of the displacement response of moving force and moving mass cases of a uniform clamped-clamped Rayleigh bean for fixed values of N=200000, K=400000 and R_o=50.

7 CONCLUDING REMARKS

The problem of the flexural vibrations of a prestressed uniform Rayleigh beam resting on elastic foundation and traversed by concentrated masses traveling at variable velocity has been investigated. Closed form solutions of the governing fourth order partial differential equations with variable and singular coefficients of uniform Rayleigh beam moving mass problems are presented. For this uniform beam problem, the solution techniques is based on generalized finite integral transformation, the expansion of the Dirac delta function in series form, a modification of Struble's asymptotic method and the application of Fresnel sine and cosine integrals.

In this work, illustrative examples involving simply supported end conditions, clamped end conditions and one end clamped, one end free conditions are presented. Analytical solutions obtained are analyzed and resonance conditions for the various beam problems are established. Results show that

1. for all illustrative examples, resonance is reached earlier in a system traversed by moving mass than in that under the action of a moving force.

2. as the axial force N increases, the amplitudes of uniform Rayleigh beam under the action moving load moving at non-uniform velocity decrease.

3. when the axial force N is fixed, the displacements of a uniform Rayleigh beam resting on elastic foundation and traversed by masses traveling with variable velocity decrease as the value of foundation modulli K increases for all variants of the boundary conditions.

4. higher values of axial force N and foundation modulli K are required for a more noticeable effect in the case of other boundary condition than those of simply supported end conditions for both the moving force and moving mass problems.

5. for fixed axial force and foundation modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem for all illustrative end conditions considered.

6. it has been established that for all the illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass cases in prestressed uniform. Rayleigh beam under accelerating loads. Hence, the non-reliability of moving force solution as a safe approximation to the moving mass problem is confirmed.

7. in all the illustrative examples considered, for the same natural frequency, the critical velocity for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

Finally, this work has proposed valuable methods of analytical solution for this category of problems for all variants of classical boundary conditions.

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