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Latin American Journal of Solids and Structures 1 (2003) 49–73

Treatment of hypersingularities in boundary element anisotropic plate bending problems

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Abstract

This paper presents an approach for anisotropic thin-plate bending problems using the boundary element formulation when the source points are located on the boundary and resulting hypersingularities are analytically treated. When the integration is carried out with the source and field points belonging to the same element the radius between them goes to zero, leading to the singular integration. The anisotropic fundamental solution for the plate bending has a singularity of r^{-2} order. Thus, under these conditions, hypersingularities treatment can not be avoided. The used boundary element formulation includes two boundary integral equations where regular, weak singular, strong singular and hypersingular integrals are found. This work provides a procedure for the treatment of strong and hypersingular integrals. All terms of the analytical integrations are given for constant elements. Numerical examples for laminate composite materials under transversely uniform distributed load are presented. The accuracy of the proposed approach is assured by comparison with analytical and finite element results available in the literature.

Keywords: Anisotropic plate bending, Boundary element method, Hypersingular integrals

1 Introduction

The boundary element method have been widely applied to various engineering problems, among others to plate bending problems. The development of boundary element formulation applied to the analysis of bending problems in anisotropic plates is motivated by the increasing use of composite materials, due to their excellent mechanical properties. In most cases, these materials present anisotropic behaviour resulting complexity in mathematical treatment of composite structures.

The analytical solutions for problems that involves anisotropic materials are restricted to a small number of problems of simple domains. In the case of usual structures, the domain analysis becomes more dificult. However, its handling is possible through numerical or experimental methods.

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Nomenclature

d_i, e_i	real and imaginary part of complex roots of characteristic equation
D_{ij}	plate flexural rigidities
E_1, E_2	Modulus of elasticity in tension and compression
G_{12}	Modulus of elasticity in shear
M_n, M_n^*	bending moment perpendicular no n direction and its fundamental solution
n, n_x, n_y	outward unit normal vector on field point and its components
n_0, n_{0_x}, n_{0_y}	outward unit normal vector on source point and its components
P	field point
q	transverse load intensity
Q	source point
r, θ	polar coordinates
R	curvature radius at a smooth point of the boundary Γ
R_{c_i}, w_{c_i}	reaction forces and deflections at i^{th} plate corner
V_n, V_n^*	Kirchhoff's equivalent shear force and its fundamental solution
w, w^*	out-of-plane deflection of a plate and its fundamental solution
x, y	rectangular coordinates of field point
x_0, y_0	rectangular coordinates of source point
Γ	boundary of solid
μ_1, μ_2	complex roots of characteristic equation
ν_1	Poisson's ratio
δ	Dirac delta function

Due to hardware and software evolution, numerical methods have been used for solve a wider range of problems. Among the methods that have been outstanding in the treatment of structural problems there are the finite element and the boundary element methods.

In the last ten years, the boundary element method has been successfully applied to the analysis of a large number of anisotropic material problems. Plane elasticity problems were analised by Sollero and Aliabadi [24], Deb [8] and Albuquerque et al. [1, 2, 4, 5]; out-of-plane elasticity problems were shown by Zhang [29, 30] and tri-dimensional problems were analised by Kögl and Gaul [13–15].

Studies of plate bending problems using the boundary element method have been carried out by many researchers. Bending problems of isotropic plates, for statics as well as dynamics, have been widely studied [6, 9, 11, 12, 19, 20, 23, 25, 27]. On the other hand, it can be noted that the number of references in which boundary element method is applied to anisotropic structures is significantly smaller than those treating isotropic ones. Boundary element formulation has been applied to plate bending anisotropic problems using Kirchhoff's theory. Shi and Bezine [22] presented a boundary element analysis of plate bending problems, based on Kirchhoff's plate bending assumptions, using fundamental solutions proposed by Wu and Altiero [28]. Similar procedure was used by Rajamohan and Raamachandran [21] who presented a formulation for anisotropic plate bending in which the singularities were avoided by placing source points outside the domain. An analysis of the fundamental solution for anisotropic thin plates was presented by Portilho de Paiva et. al. [7] who compared it with isotropic fundamental solution using quasi-isotropic material properties. An analysis of symmetric laminate composites under bending using the boundary element method was presented by Albuquerque et. al. [3] who carried out analysis of symmetric cross-ply and angle-ply laminate composites under several boundary conditions.

This paper presents detailed procedures for the treatment of singularities inherent to boundary element formulation for anisotropic plate bending. Similar procedure was developed by Rashed et. al. [10] for isotropic thick plates. In order to be self-contained, all terms of the analytical integration for constant element are presented here. Numerical examples for laminate composite materials under transversely uniform distributed load are presented. The accuracy of the proposed approach is assured by comparison with analytical and finite element results available in the literature.

2 Theory of anisotropic thin plate bending

In this work, a plate is understood as a structural element defined by two parallel plane surfaces (Figure 1). The distance between these two surfaces defines the thickness of the plate, which is small when compared with other plate dimensions. In the theory of anisotropic plate bending, loads are always transversely applied in the surface of the plate.

Depending on its material properties, a plate can be considered either anisotropic, with different properties in different directions, or isotropic, with same properties in all directions. In this work the Kirchhoff theory will be applied to anisotropic thin plates.

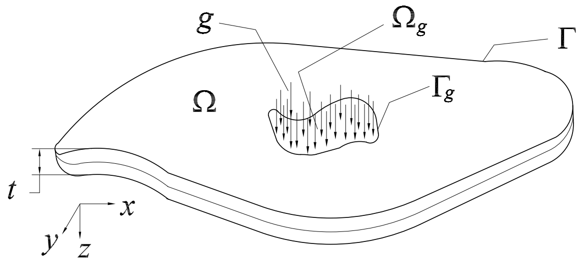

Figure 1: Definition of thin plate.

According to Timoshenko and Woinowsky-Krieger [26] the thin plate bending theory is based on the following assumptions:

1. The middle plane of the plate does not undergo deformations;

2. Straight sections, which are normal to middle surface when the plate is in the undeformed state, remain straight and normal to the deformed middle surface, after loading;
3. The normal stress perpendicular to the middle plane can be disregarded.

Consider a plate following these assumptions. The lateral midsurface deflection w satisfies the differential equation (Lekhnitskii [17]):

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = q \tag{1}$$

where D_{ij} are the flexural rigidities of the anisotropic plate, q is the transverse load intensity.
General solution to w in Equation (1) depends on μ_k, the roots of characteristic equation given by:

$$D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12} + 2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0. \tag{2}$$

Roots of this equation are always complex for homogeneous materials. The complex roots $\mu_k = d_k + e_k i$, where $k = 1, 2$ and does not imply summation, are known as deflection complex parameters. In general, these roots are different complex numbers.

3 Boundary integral equation

Using Rayleigh-Green identity, an integral equation for an anisotropic thin plate under transversal load $q(p)$ is obtained. In this equation one has the boundary integral equation given in terms of four basic boundary values, namely, deflection w, normal slope $\partial w/\partial n$, bending moment M_n and Kirchhoff's equivalent shear force V_n. Two of these four values should be the unknowns of the problem and other two are determined by boundary conditions.

As shown by Shi and Bezine [22] the first boundary integral equation over the boundary Γ is:

$$\begin{aligned} cw(Q)+ \quad & \int_\Gamma V_n^*(Q,P)w(P)d\Gamma(P) - \int_\Gamma M_n^*(Q,P)\frac{\partial w}{\partial n}(P)d\Gamma(P) + \\ & \sum_{i=1}^{N_c} R_{c_i}^*(Q,P)w_{c_i}(P) = \\ & \int_\Gamma w^*(Q,P)V_n(P)d\Gamma(P) - \int_\Gamma \frac{\partial w^*}{\partial n}(Q,P)M_n(P)d\Gamma(P) + \\ & \sum_{i=1}^{N_c} w_{c_i}^*(Q,P)R_{c_i}(P) \end{aligned} \tag{3}$$

where the constant c is introduced in order to consider that the Dirac delta function can be applied in the domain, in the boundary, or outside the domain. In the particular case, when the point is taken in a smooth part of boundary, $c = 1/2$. Besides, N_c is the number of corner points on the boundary. Q is the point where the load is applied, so-called source point, and P is the point where the deflection is observed, so-called field point. Stars indicate the known state fundamental solution. R_{c_i} and w_{c_i} are reaction forces and deflections at the i^{th} plate corner. In Equation (3) the body forces are neglected.

In plate bending problems there are always two unknowns to be determined at any boundary point. Thus, the problem solution requires a second boundary integral equation in order to have an equal number of equations and unknown variables. This second equation is obtained by differentiating the displacement $w(Q)$ in relation to a Cartesian coordinate system fixed in the source point, i.e., the point where the Dirac delta of the fundamental state is applied, in the direction of the outward unit normal vector n_0 on source point. It is given by:

$$
\begin{aligned}
c\frac{\partial w}{\partial n_0}(Q) \quad & \int_\Gamma \frac{\partial V_n^*}{\partial n_0}(Q,P)w(P)d\Gamma(P) - \int_\Gamma \frac{\partial M_n^*}{\partial n_0}(Q,P)\frac{\partial w}{\partial n}(P)d\Gamma(P) + \\
& \sum_{i=1}^{N_c} \frac{\partial R_{c_i}^*}{\partial n_0}(Q,P)w_{c_i}(P) = \\
& \int_\Gamma \frac{\partial w^*}{\partial n_0}(Q,P)V_n(P)d\Gamma(P) - \int_\Gamma \frac{\partial^2 w^*}{\partial n_0 \partial n}(Q,P)M_n(P)d\Gamma(P) + \\
& \sum_{i=1}^{N_c} \frac{\partial w_{c_i}^*}{\partial n_0}(Q,P)R_{c_i}(P).
\end{aligned} \tag{4}
$$

The detailed development of Equations (3) and (4) can be seen in [6,12,22]. It is important to say that it is possible to use only Equation (3) in a boundary element formulation by using the boundary nodes as source points and an equal number of points external to the domain of the problem.

4 Anisotropic fundamental solution

The fundamental solutions are the solutions of the differential Equation (1) with the non-homogeneous term equal to a concentrated force given by a Dirac delta function $\delta(Q,P)$, i.e.,

$$\Delta\Delta w^*(Q,P) = \delta(Q,P) \tag{5}$$

where $\Delta\Delta(.)$ is the differential operator given by:

$$\Delta\Delta(.) = \frac{D_{11}}{D_{22}}\frac{\partial^4(.)}{\partial x^4} + 4\frac{D_{16}}{D_{22}}\frac{\partial^4(.)}{\partial^3 \partial y} + \frac{2(D_{12}+2D_{66})}{D_{22}}\frac{\partial^4(.)}{\partial x^2 \partial y^2} + 4\frac{D_{26}}{D_{22}}\frac{\partial^4(.)}{\partial x \partial y^3} + \frac{\partial^4(.)}{\partial y^4}. \tag{6}$$

As presented by Shi and Bezine [22], the deflection fundamental solution for anisotropic plate bending is:

$$w^*(r,\theta) = \frac{1}{8\pi}\left\{C_1 R_1(r,\theta) + C_2 R_2(r,\theta) + C_3\left[S_1(r,\theta) - S_2(r,\theta)\right]\right\} \tag{7}$$

where r is the distance between the source point $P(x_0, y_0)$ and field point $Q(x, y)$,

$$\theta = \arctan\frac{y - y_o}{x - x_o}, \tag{8}$$

$$C_1 = \frac{(d_1 - d_2)^2 - (e_1^2 - e_2^2)}{GHe_1}, \tag{9}$$

$$C_2 = \frac{(d_1 - d_2)^2 + (e_1^2 - e_2^2)}{GHe_2}, \tag{10}$$

$$C_3 = \frac{4(d_1 - d_2)}{GH}, \tag{11}$$

$$G = (d_1 - d_2)^2 + (e_1 + e_2)^2, \tag{12}$$

$$H = (d_1 - d_2)^2 + (e_1 - e_2)^2, \tag{13}$$

$$\begin{aligned} R_i(r,\theta) = \quad & r^2\left[(\cos\theta + d_i\sin\theta)^2 - e_i^2\sin^2\theta\right]\times \\ & \left\{\log\left[\frac{r^2}{a^2}\left((\cos\theta + d_i\sin\theta)^2 + e_i^2\sin^2\theta\right)\right] - 3\right\} - \\ & 4r^2 e_i\sin\theta\,(\cos\theta + d_i\sin\theta)\arctan\frac{e_i\sin\theta}{\cos\theta + d_i\sin\theta} \end{aligned} \tag{14}$$

and

$$\begin{aligned} S_i(r,\theta) = \quad & r^2 e_i\sin\theta\,(\cos\theta + d_i\sin\theta)\times \\ & \left\{\log\left[\frac{r^2}{a^2}\left((\cos\theta + d_i\sin\theta)^2 + e_i^2\sin^2\theta\right)\right] - 3\right\} + \\ & r^2\left[(\cos\theta + d_i\sin\theta)^2 - e_i^2\sin^2\theta\right]\arctan\frac{e_i\sin\theta}{\cos\theta + d_i\sin\theta}. \end{aligned} \tag{15}$$

The index i of functions $R_i(r,\theta)$ and $S_i(r,\theta)$ given by Equations (14) and (15) does not imply summation and the coefficient a is an arbitrary constant. In this work it is assumed that $a = 1$.

Other fundamental solutions are given by:

$$M_n^* = -\left(f_1 \frac{\partial^2 w^*}{\partial x^2} + f_2 \frac{\partial^2 w^*}{\partial x \partial y} + f_3 \frac{\partial^2 w^*}{\partial y^2}\right), \tag{16}$$

$$R_{c_i}^* = -\left(g_1 \frac{\partial^2 w^*}{\partial x^2} + g_2 \frac{\partial^2 w^*}{\partial x \partial y} + g_3 \frac{\partial^2 w^*}{\partial y^2}\right) \tag{17}$$

and

$$V_n^* = -\left(h_1 \frac{\partial^3 w^*}{\partial x^3} + h_2 \frac{\partial^3 w^*}{\partial x^2 \partial y} + h_3 \frac{\partial^3 w^*}{\partial x \partial y^2} + h_4 \frac{\partial^3 w^*}{\partial y^3}\right) - \frac{1}{R}\left(h_5 \frac{\partial^2 w^*}{\partial x^2} + h_6 \frac{\partial^2 w^*}{\partial x \partial y} + h_7 \frac{\partial^2 w^*}{\partial y^2}\right) \tag{18}$$

where R is the curvature radius at a smooth point of the boundary. Other constants of the fundamental solutions are presented in Appendix A.

The derivatives of deflection fundamental solution can be expressed by linear combination of derivatives of functions R_i and S_i. For example:

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{8\pi}\left[C_1 \frac{\partial^2 R_1}{\partial y^2} + C_2 \frac{\partial^2 R_2}{\partial y^2} + C_3\left(\frac{\partial^2 S_1}{\partial y^2} - \frac{\partial^2 S_2}{\partial y^2}\right)\right]. \tag{19}$$

All other derivative terms are obtained in a similar way. The derivatives of R_i and S_i are presented in Appendix B.

As it can be seen in equations presented in Appendix B, derivatives of R_i and S_i present weak ($\log r$), strong (r^{-1}), and hyper (r^{-2}) singularities that will need special attention during their integration in boundary element kernels.

5 Matrix equation

In order to compute the unknown boundary variables, the boundary Γ is discretized in n straight elements N_i ($i = 1, 2, ..., n$) with a node K_i defined in the middle point of each segment. In this work the boundary variables w, $\partial w/\partial n$, M_n, and V_n are supposed to be constant along each element N_i, with their values being those taken by variables at the node K_i.

Equations (3) and (4) can be written in the discretized matrix form placing the source point in a node d as:

$$\frac{1}{2}\left\{\begin{array}{c} w^{(d)} \\ \\ \frac{\partial w^{(d)}}{\partial n_0}\end{array}\right\}+\sum_{i=1}^{N_e}\left(\left[\begin{array}{cc} H_{11}^{(i,d)} & H_{12}^{(i,d)} \\ H_{21}^{(i,d)} & H_{22}^{(i,d)}\end{array}\right]\left\{\begin{array}{c} w^{(i,d)} \\ \frac{\partial w^{(i,d)}}{\partial n}\end{array}\right\}\right)+\sum_{i=1}^{N_c}\left(\left\{\begin{array}{c} K_1^{(i,d)} \\ K_2^{(i,d)}\end{array}\right\} w_c^{(i,d)}\right)=$$

$$\sum_{i=1}^{N_e}\left(\left[\begin{array}{cc} G_{11}^{(i,d)} & G_{12}^{(i,d)} \\ G_{21}^{(i,d)} & G_{22}^{(i,d)}\end{array}\right]\left\{\begin{array}{c} V_n^{(i,d)} \\ M_n^{(i,d)}\end{array}\right\}\right)+\sum_{i=1}^{N_c}\left(\left\{\begin{array}{c} F_1^{(i,d)} \\ F_2^{(i,d)}\end{array}\right\} R_c^{(i,d)}\right) \tag{20}$$

where N_e stands for the number of element, N_c stands for the number of corners. Terms of matrices and vectors are given by:

$$H_{11}^{(i,d)}=\int_{\Gamma_i} V_n^* d\Gamma, \qquad H_{12}^{(i,d)}=-\int_{\Gamma_i} M_n^* d\Gamma, \tag{21}$$

$$H_{21}^{(i,d)}=\int_{\Gamma_i} \frac{\partial V_n^*}{\partial n_0} d\Gamma, \qquad H_{22}^{(i,d)}=-\int_{\Gamma_i} \frac{\partial M_n^*}{\partial n_0} d\Gamma, \tag{22}$$

$$G_{11}^{(i,d)}=\int_{\Gamma_i} w^* d\Gamma, \qquad G_{12}^{(i,d)}=-\int_{\Gamma_i} \frac{\partial w^*}{\partial n} d\Gamma, \tag{23}$$

$$G_{21}^{(i,d)}=\int_{\Gamma_i} \frac{\partial w^*}{\partial n_0} d\Gamma, \qquad G_{22}^{(i,d)}=-\int_{\Gamma_i} \frac{\partial^2 M_n^*}{\partial n_0 \partial n} d\Gamma, \tag{24}$$

$$K_1^{(i,d)}=R_{c_i}^*, \qquad K_2=\frac{\partial R_{c_i}^*}{\partial n_0}, \tag{25}$$

$$F_1^{(i,d)}=w_{c_i}^*, \qquad F_2=\frac{\partial w_{c_i}^*}{\partial n_0}. \tag{26}$$

In matrix equation (20) we have two equations and $2N_e + N_c$ unknowns. In order to obtain a solvable linear system, the source point is placed successively in every boundary node, resulting $2N_e$ equations. Other N_c equations are obtained by writing Equation (3) to every corner node. So one obtains the matrix equation given by:

$$\left[\begin{array}{cc} \mathbf{H} & \mathbf{K} \\ \mathbf{H}' & \mathbf{K}'\end{array}\right]\left\{\begin{array}{c} \mathbf{w} \\ \mathbf{w_c}\end{array}\right\}=\left[\begin{array}{cc} \mathbf{G} & \mathbf{F} \\ \mathbf{G}' & \mathbf{F}'\end{array}\right]\left\{\begin{array}{c} \mathbf{V} \\ \mathbf{V_c}\end{array}\right\} \tag{27}$$

where $\mathbf{w}$ contains the deflection and rotation to every boundary node, $\mathbf{V}$ contains shear forces and twisting moments to every boundary node, $\mathbf{w_c}$ contains deflection to every corner and $\mathbf{V_c}$ contains the corner reactions to every corner. Terms $\mathbf{H}$, $\mathbf{K}$, $\mathbf{G}$, and $\mathbf{F}$, are matrices which contain the respective terms of Equation (20) written to every boundary node. Terms $\mathbf{H}'$, $\mathbf{K}'$, $\mathbf{G}'$, and $\mathbf{F}'$ are matrices which contain the respective first line terms of Equation (20) written to each corner.

Applying boundary conditions, equation (27) can be rearranged as

$$\mathbf{Ax} = \mathbf{b} \tag{28}$$

which can be solved by standard procedure for linear systems.

6 Treatment of hypersingularities

Equations (3) and (4) present integrals of fundamental solutions where, according to Equation (7) one can see that w^*, the fundamental solution of deflection, and its derivatives $\partial w^*/\partial n$ and $\partial w^*/\partial n_0$ are regular functions i.e., they do not show singularities. Thus they can be carried out analytically or using Gauss quadrature. According to Equations (7) and (16) one can see that integrals that comprise $\partial^2 w^*/\partial n \partial n_0$ and M_n^* are improper integrals, i.e., these functions are weak singular. Their integrals can be carried out analytically or using Gauss logarithmic.

On the other hand, integrals of V_n^*, given in Equation (18), and $\partial M_n^*/\partial n_0$, that is the derivative of M_n^*, given in Equation (16), include a jump term and these functions present strong singularities, so their integrals must be computed in the Cauchy principal-value sense. Let one analyses the V_n^* fundamental solution given by Equation (18). Since constant elements have straight geometry, the second part of Equation (18) vanishes because R tends to infinite. Thus V_n^* is comprised only of third derivatives of w^*. So,

$$V_n^* = \quad -\left(h_1 \frac{\partial^3 w^*}{\partial x^3} + h_2 \frac{\partial^3 w^*}{\partial x^2 \partial y} + h_3 \frac{\partial^3 w^*}{\partial x \partial y^2} + h_4 \frac{\partial^3 w^*}{\partial y^3}\right), \tag{29}$$

and according to Equation (19) one has:

$$\frac{\partial^3 w^*}{\partial x^3} = \frac{1}{8\pi}\left[C_1 \frac{\partial^3 R_1}{\partial x^3} + C_2 \frac{\partial^3 R_2}{\partial x^3} + C_3\left(\frac{\partial^3 S_1}{\partial x^3} - \frac{\partial^3 S_2}{\partial x^3}\right)\right], \tag{30}$$

$$\frac{\partial^3 w^*}{\partial x^2 \partial y} = \frac{1}{8\pi}\left[C_1 \frac{\partial^3 R_1}{\partial x^2 \partial y} + C_2 \frac{\partial^3 R_2}{\partial x^2 \partial y} + C_3\left(\frac{\partial^3 S_1}{\partial x^2 \partial y} - \frac{\partial^3 S_2}{\partial x^2 \partial y}\right)\right], \tag{31}$$

$$\frac{\partial^3 w^*}{\partial x \partial y^2} = \frac{1}{8\pi}\left[C_1 \frac{\partial^3 R_1}{\partial x \partial y^2} + C_2 \frac{\partial^3 R_2}{\partial x \partial y^2} + C_3\left(\frac{\partial^3 S_1}{\partial x \partial y^2} - \frac{\partial^3 S_2}{\partial x \partial y^2}\right)\right], \tag{32}$$

$$\frac{\partial^3 w^*}{\partial y^3} = \frac{1}{8\pi}\left[C_1 \frac{\partial^3 R_1}{\partial y^3} + C_2 \frac{\partial^3 R_2}{\partial y^3} + C_3\left(\frac{\partial^3 S_1}{\partial y^3} - \frac{\partial^3 S_2}{\partial y^3}\right)\right]. \tag{33}$$

It can be see in Equations (B.6) to (B.9) and from Equations (B.20) to (B.23) that third derivatives of R_i and S_i reduce to

$$\frac{\partial^3 R_i}{\partial x^3} = \frac{1}{r}\,a_{1i}, \tag{34}$$

$$\frac{\partial^3 R_i}{\partial x^2 \partial y} = \frac{1}{r}\,a_{2i}, \tag{35}$$

$$\frac{\partial^3 R_i}{\partial x \partial y^2} = \frac{1}{r}\,a_{3i}, \tag{36}$$

$$\frac{\partial^3 R_i}{\partial y^3} = \frac{1}{r}\,a_{4i}, \tag{37}$$

$$\frac{\partial^3 S_i}{\partial x^3} = \frac{1}{r}\,b_{1i}, \tag{38}$$

$$\frac{\partial^3 S_i}{\partial x^2 \partial y} = \frac{1}{r}\,b_{2i}, \tag{39}$$

$$\frac{\partial^3 S_i}{\partial x \partial y^2} = \frac{1}{r}\,b_{3i}, \tag{40}$$

$$\frac{\partial^3 S_i}{\partial y^3} = \frac{1}{r}\,b_{4i}. \tag{41}$$

where a_{ji} and b_{ji} are given functions of θ. As θ is constant when straight elements are used, a_{ji} and b_{ji} are constants. The a_{ji} constants are given by:

$$a_{1i} = \frac{4\left(\cos\theta + d_i \sin\theta\right)}{\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta}, \tag{42}$$

$$a_{2i} = \frac{4\left[d_i\left(\cos\theta + d_i \sin\theta\right) + e_i^2 \sin\theta\right]}{\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta}, \tag{43}$$

$$a_{3i} = \frac{4\left[\left(d_i^2 - e_i^2\right)\cos\theta + \left(d_i^2 + e_i^2\right)d_i\sin\theta\right]}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta}, \tag{44}$$

$$a_{4i} = \frac{4\left[d_i\left(d_i^2 - 3e_i^2\right)\cos\theta + \left(d_i^4 - e_i^4\right)\sin\theta\right]}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta}, \tag{45}$$

and b_{ji} constants are given by:

$$b_{1i} = -\frac{2e_i\sin\theta}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta}, \tag{46}$$

$$b_{2i} = \frac{2e_i\cos\theta}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta}, \tag{47}$$

$$b_{3i} = \frac{2e_i\left[2d_i\left(\cos\theta + d_i\sin\theta\right) - \left(d_i^2 - e_i^2\right)\sin\theta\right)}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta}, \tag{48}$$

$$b_{4i} = \frac{2e_i\left[\left(3d_i^2 - e_i^2\right)\cos\theta + 2d_i\left(d_i^2 + e_i^2\right)\sin\theta\right]}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta}. \tag{49}$$

Substituting Equations (34) and (38) into Equation (30) results:

$$\frac{\partial^3 w^*}{\partial x^3} = \frac{1}{8\pi}\left[C_1\frac{1}{r}a_{11} + C_2\frac{1}{r}a_{12} + C_3\left(\frac{1}{r}b_{11} - \frac{1}{r}b_{12}\right)\right] \tag{50}$$

or

$$\frac{\partial^3 w^*}{\partial x^3} = \frac{1}{r}m_1. \tag{51}$$

Similarly, it can be seen that:

$$\frac{\partial^3 w^*}{\partial x^2\partial y} = \frac{1}{r}m_2, \tag{52}$$

$$\frac{\partial^3 w^*}{\partial x\partial y^2} = \frac{1}{r}m_3, \tag{53}$$

$$\frac{\partial^3 w^*}{\partial y^3} = \frac{1}{r} m_4. \tag{54}$$

where m_n are constants given by

$$m_n = \frac{1}{8\pi} \left[C_1 a_{n1} + C_2 a_{n2} + C_3 \left(b_{n1} - b_{n2} \right) \right]. \tag{55}$$

The substitution of Equations (51) to (54) into Equation (29) results

$$V_n^* = \quad - \left(h_1 \frac{1}{r} m_1 + h_2 \frac{1}{r} m_2 + h_3 \frac{1}{r} m_3 + h_4 \frac{1}{r} m_4 \right) \tag{56}$$

or

$$V_n^* = \frac{1}{r} M \tag{57}$$

where M is a constant given by

$$M = \quad - \left(h_1 m_1 + h_2 m_2 + h_3 m_3 + h_4 m_4 \right). \tag{58}$$

From this it can be seen that $H_{11}^{(i,d)}$ of Equation (21) can be interpreted in the Cauchy principal-value sense. It is given by:

$$\int_{\Gamma_i} V_n^* d\Gamma = M \fint_{-L}^{L} \frac{1}{r} \, dr = 0 \tag{59}$$

where L is the half of the element length.

Following the same procedure, $\partial M_n^* / \partial n_0$ can be obtained. From

$$\frac{\partial M_n^*}{\partial n_0} = \frac{\partial M_n^*}{\partial x} n_{0_x} + \frac{\partial M_n^*}{\partial y} n_{0_y} \tag{60}$$

and from Equation (16), $\partial M_n^* / \partial x$ and $\partial M_n^* / \partial y$ are obtained:

$$\frac{\partial M_n^*}{\partial x} = - \left(f_1 \frac{\partial^3 w^*}{\partial x^3} + f_2 \frac{\partial^3 w^*}{\partial x^2 \partial y} + f_3 \frac{\partial^3 w^*}{\partial x \partial y^2} \right), \tag{61}$$

$$\frac{\partial M_n^*}{\partial y} = -\left(f_1 \frac{\partial^3 w^*}{\partial x^2 \partial y} + f_2 \frac{\partial^3 w^*}{\partial x \partial y^2} + f_3 \frac{\partial^3 w^*}{\partial y^3} \right). \tag{62}$$

Then, substituting Equations (51) to (54) into (61) and (62) and after into (60), it can be rewritten as:

$$\frac{\partial M_n^*}{\partial n_0} = -\left(f_1 \frac{1}{r} b_1 + f_2 \frac{1}{r} b_2 + f_3 \frac{1}{r} b_3 \right) n_{0_x} - \left(f_1 \frac{1}{r} b_2 + f_2 \frac{1}{r} b_3 + f_3 \frac{1}{r} b_4 \right) n_{0_y} \tag{63}$$

or

$$\frac{\partial M_n^*}{\partial n_0} = \frac{1}{r} N, \tag{64}$$

where

$$N = -\left(f_1 b_1 + f_2 b_2 + f_3 b_3 \right) n_{0_x} - \left(f_1 b_2 + f_2 b_3 + f_3 b_4 \right) n_{0_y}. \tag{65}$$

Thus, $H_{22}^{(i,d)}$ of Equation (22) can be interpreted in the Cauchy principal-value sense. It results:

$$\int_{\Gamma_i} \frac{\partial M_n^*}{\partial n_0} \, d\Gamma = N \, -\!\!\!\!\!\!\int_{-L}^{L} \frac{1}{r} \, dr = 0. \tag{66}$$

Finally, from the fourth derivatives of R_i and S_i shown in the Appendix B it can be seen that the integral of $\partial V_n^* / \partial n_0$ of $H_{21}^{(i,d)}$ of Equation (22) shows an hypersingularity that must be interpreted in the Hadamard principal-value sense. From

$$\frac{\partial V_n^*}{\partial n_0} = -\left(\frac{\partial V_n^*}{\partial x} n_{0_x} + \frac{\partial V_n^*}{\partial y} n_{0_y} \right) \tag{67}$$

and from Equation (29) one has

$$\frac{\partial V_n^*}{\partial x} = -\left(h_1 \frac{\partial^4 w^*}{\partial x^4} + h_2 \frac{\partial^4 w^*}{\partial x^3 \partial y} + h_3 \frac{\partial^4 w^*}{\partial x^2 \partial y^2} + h_4 \frac{\partial^4 w^*}{\partial x \partial y^3} \right), \tag{68}$$

$$\frac{\partial V_n^*}{\partial y} = -\left(h_1 \frac{\partial^4 w^*}{\partial x^3 \partial y} + h_2 \frac{\partial^4 w^*}{\partial x^2 \partial y^2} + h_3 \frac{\partial^4 w^*}{\partial x \partial y^3} + h_4 \frac{\partial^4 w^*}{\partial y^4} \right). \tag{69}$$

Integrating $H_{21}^{(i,d)}$ of Equation (22) in the Hadamard principal-value sense results:

$$\int_{\Gamma_i} \frac{\partial V_n^*}{\partial n_0}\, d\Gamma = T \;⨎_{-L}^{L} \frac{1}{r^2}\, dr = -T\frac{2}{L}. \tag{70}$$

Where T is a function of θ.

Since all singularities are properly treated, integrals (59), (66) and (70) can be substituted into matrix equation (27) and the problem can be solved following the traditional BEM procedure.

7 Numerical results

In this section, the formulation developed in this work will be applied to the analysis of bending problem in anisotropic plates.

7.1 Orthotropic simply-supported square plate

Consider a square plate of side length $a = 1$ and thickness $h = 0.01$. The material is orthotropic and its material properties are: $E_1 = 206.8 \cdot 10^9$, $E_2 = 13.8 \cdot 10^9$, $G_{12} = 0.6055 \cdot 10^9$ and $\nu_1 = 0.3$. All values are given in SI units. This problem was analyzed by Wu and Altiero [28] under uniformly distributed load using influence load function and by Shi and Bezine [22] under concentrated and uniformly distributed load using boundary element method and domain integration to treat the distributed load. Rajamohan and Raamachandran [21] analyzed the same problem under concentrated and uniformly distributed load using charge simulation method, which is a boundary element method without singular integrals and the domain integrals were treated by particular integrals. In this work, the square plate is considered simply supported on its four edges under uniformly distributed load $q = 1$ Pa applied along its domain (Figure 2). For this case the results obtained by BEM will be compared with the solution obtained by Timoshenko and Woinowski-Krieger [26] which solve this problem using a series solution given by:

$$w = \frac{16q_o}{\pi^6} \sum_{m=1,3,\ldots}^{M} \sum_{n=1,3,\ldots}^{N} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left(\frac{m^4}{a^4}D_{11} + \frac{2m^2n^2}{a^2b^2}H + \frac{n^4}{b^4}D_{22}\right)}, \tag{71}$$

where

$$H = D_{12} + 2D_{66}. \tag{72}$$

In order to assess convergence, the problem is solved using different meshes and the results for deflections at point A and at point B are compared with series solutions using $N = 19$

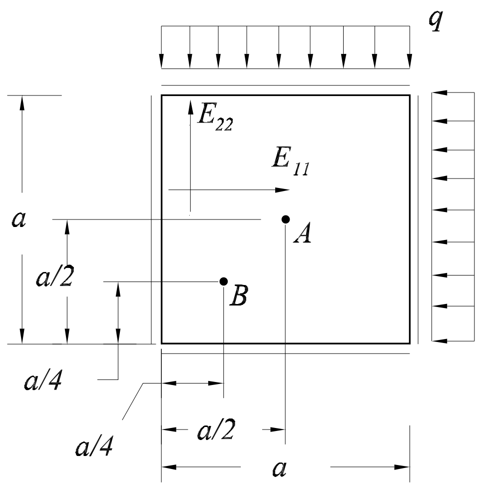

Figure 2: Square plate with simply-supported edges under uniformly distributed load.

and $M = 19$. This series solution for point A is $w_{se.} = 8.1258 \cdot 10^{-7}$ and for point B is $w_{se.} = 4.5211 \cdot 10^{-7}$. Table 1 shows deflections computed by the present BEM technique using different meshes and their respective errors compared to Timoshenko and Woinowski-Krieger [26] series solutions.

Deflections and errors				
Number of Elements	w [m] at point A	Error [%] at point A	w [m] at point B	Error [%] at point B
8	$9.2185\ 10^{-7}$	13.45	$5.3973\ 10^{-7}$	19.38
16	$8.0420\ 10^{-7}$	1.03	$4.5821\ 10^{-7}$	1.35
24	$8.0441\ 10^{-7}$	1.01	$4.4647\ 10^{-7}$	1.25
32	$8.0630\ 10^{-7}$	0.77	$4.4716\ 10^{-7}$	1.09
40	$8.0778\ 10^{-7}$	0.59	$4.5211\ 10^{-7}$	0.88

Table 1: Accuracy of deflection obtained by BEM for the orthotropic square plate with simply supported edges under uniformly distributed loads.

As it can be seen in Table 1, results are very poor when 8 elements (2 elements per side) are used. However, they converge quickly to the series solutions if the number of the element is increased. When 40 boundary elements are used (Figure 3), deflections in both points present errors below 1 % if compared with series solutions. The deformed plate is shown in Figure 4.

In order to assess the accuracy of the method with the principal axes of orthotropy not coinciding with coordinate axes, the plate was rotated 30^o around its center as shown in Figure 5. The deflection computed to a point in the center of the plate is equal to $w = 8.0645 \cdot 10^{-7}$.

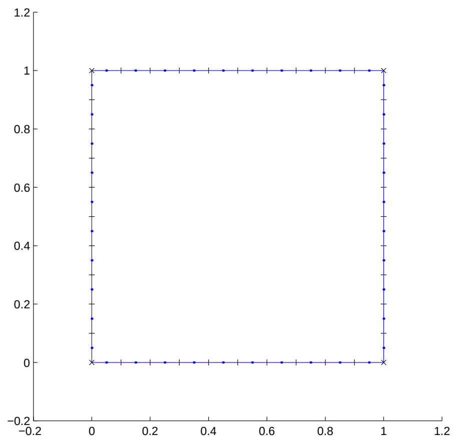

Figure 3: Boundary element mesh (40 constant boundary elements).

The error in this case is 0.75% if compared with the series solution. This shows how accurate the formulation is even for orthotropic materials with principal axes not coinciding with coordinate axes.

7.2 Cross-ply laminate graphite/epoxy composite square plate with simply supported edges

The second problem that has been analyzed is a nine-layer ply simply supported laminate [0°/90°/0°/90°/0°/90°/0°/90°/0°] of side length $a = 1$ under a uniformly distributed load $q = 6.9 \cdot 10^3$. The properties of each layer of a high modulus graphite-epoxy composite material used in this analysis are: $E_{11} = 2.07 \cdot 10^9$, $E_{22} = 5.17 \cdot 10^9$, $G_{12} = 3.10 \cdot 10^9$, and $\nu_{12} = 0.25$. All values are given in SI units. The total thickness of the laminate h is taken as $0.0254mm$. And the total thickness of the 0° and 90° laminate are the same.

This problem was analysed by Rajamohan and Raamachandran [21] using charge simulation method and by Lakshminarayana and Murthy [16] using finite element method. The center point deflection for such plate are compared in Table 2 with the finite element solution and with an analytical solution, which is derived by treating the plate as an equivalent single layer orthotropic plate. A mesh of 22 boundary elements per side (Figure 6) was used in order to obtain the same accuracy of the finite element results published in the literature (Lakshminarayana and Murthy [16]). The analytical solution for deflection in the center of the plate, presented by Noor and Mathers [18], is given by:

$$\frac{w_{an.} E_{22} h^3}{q a^4} \times 10^3 = 4.4718 \tag{73}$$

As shown in Table 2, the same accuracy obtained by FEM was obtained by BEM. While in this work it was used 88 constant boundary elements to discretize the entire plate, Lakshminarayana and Murthy [16] used symmetry considerations and 72 cubic triangular elements

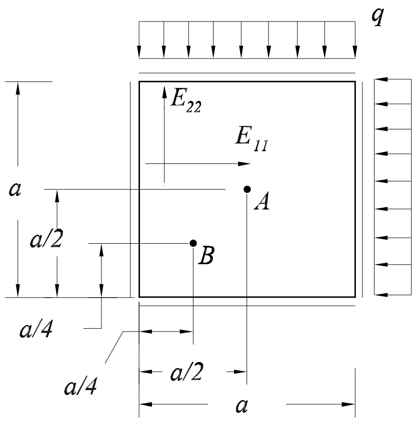

Figure 4: Deflections in a simply supported orthotropic plate (in meters).

Numerical	Deflections and errors	
Methods	$wE_{22}h^3/(qa^4) \times 10^3$	Errors [%]
BEM	4.4507	0.47
FEM	4.4508	0.47

Table 2: Accuracy of deflection obtained by BEM (88 constant boundary elements) and FEM (72 third order triangular element - discretization of one quarter of the plate) for the cross-ply laminate graphite/epoxy composite square plate with simply supported edges under uniformly distributed loads

to discretize one quarter of the plate. Of course, if the entire plate was discretized by FEM, it would be necessary larger number of elements to obtain the same accuracy. Furthermore, if we consider the number of nodes or degrees of freedom, the boundary element method has less nodes per element. On the other hand, the matrices in FEM are sparse and symmetric while in BEM are fully populated and non-symmetric.

From all above, comparison between BEM and FEM is not an easy task. Both of them are well-established numerical methods and both of them have advantages and disadvantages. In occasions, the decision to use one or other is due to the experience of the researcher in working with one of the formulations.

8 Conclusions

This paper presented an approach for anisotropic thin-plate bending problems using the boundary element formulation when the source points are located on the boundary. The treatment of singularities inherent of formulation was introduced and all terms of the analytical integration

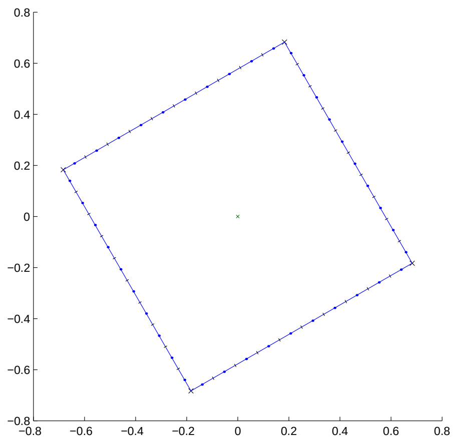

Figure 5: Rotated boundary element mesh.

for constant elements were presented. Numerical examples for laminate composite materials under transversely uniform distributed load was presented. The accuracy of the proposed approach was assured by comparison with analytical and finite element results available in the literature.

Acknowledgments: The authors are thankful to FAPESP (The State of São Paulo Research Foundation) for the financial support of this work.

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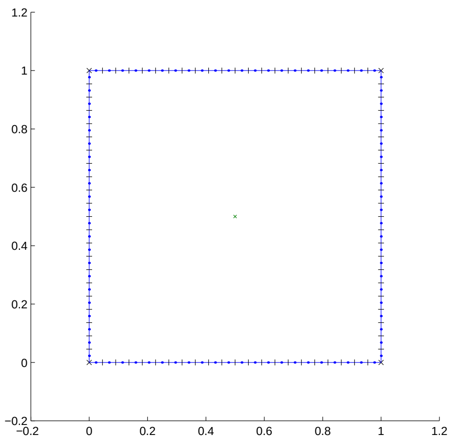

Figure 6: Boundary element mesh (22 element per side).

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Appendix

A Constants of fundamental solutions

The constants of fundamental solutions are defined as:

$$f_1 = D_{11}n_x^2 + 2D_{16}n_xn_y + D_{12}n_y^2, \tag{A.1}$$
$$f_2 = 2(D_{16}n_x^2 + 2D_{66}n_xn_y + D_{26}n_y^2), \tag{A.2}$$
$$f_3 = D_{12}n_x^2 + 2D_{26}n_xn_y + D_{22}n_y^2, \tag{A.3}$$

$$g_1 = (D_{12} - D_{11})\cos\alpha\sin\alpha + D_{16}(\cos^2\alpha - \sin^2\alpha), \tag{A.4}$$
$$g_2 = 2(D_{26} - D_{16})\cos\alpha\sin\alpha + 2D_{66}(\cos^2\alpha - \sin^2\alpha), \tag{A.5}$$
$$g_3 = (D_{22} - D_{12})\cos\alpha\sin\alpha + D_{26}(\cos^2\alpha - \sin^2\alpha), \tag{A.6}$$

$$h_1 = D_{11}n_x(1 + n_y^2) + 2D_{16}n_y^3 - D_{12}n_xn_y^2, \tag{A.7}$$
$$h_2 = 4D_{16}n_x + D_{12}n_y(1 + n_x^2) + 4D_{66}n_y^3 - D_{11}n_x^2n_y - 2D_{26}n_xn_y^2, \tag{A.8}$$
$$h_3 = 4D_{26}n_y + D_{12}n_x(1 + n_y^2) + 4D_{66}n_x^3 - D_{22}n_xn_y^2 - 2D_{16}n_x^2n_y, \tag{A.9}$$
$$h_4 = D_{22}n_y(1 + n_x^2) + 2D_{26}n_x^3 - D_{12}n_x^2n_y, \tag{A.10}$$
$$h_5 = (D_{12} - D_{11})\cos 2\alpha - 4D_{16}\sin 2\alpha, \tag{A.11}$$
$$h_6 = 2(D_{26} - D_{16})\cos 2\alpha - 4D_{66}\sin 2\alpha, \tag{A.12}$$
$$h_7 = (D_{22} - D_{12})\cos 2\alpha - 4D_{26}\sin 2\alpha. \tag{A.13}$$

B Derivatives of R_i and S_i

B.1 First derivatives of R_i

$$\frac{\partial R_i}{\partial x} = 2r\left(\cos\theta + d_i\sin\theta\right)\left\{\log\left[\frac{r^2}{a^2}\left((\cos\theta + d_i\sin\theta)^2 + e_i^2\sin^2\theta\right)\right] - 2\right\} - 4re_i\sin\theta\arctan\frac{e_i\sin\theta}{\cos\theta + d_i\sin\theta}, \tag{B.1}$$

$$\frac{\partial R_i}{\partial y} = 2r\left[d_i\left(\cos\theta + d_i\sin\theta\right) - e_i^2\sin\theta\right] \times \left\{\log\left[\frac{r^2}{a^2}\left((\cos\theta + d_i\sin\theta)^2 + e_i^2\sin^2\theta\right)\right] - 2\right\} - 4re_i\left(\cos\theta + 2d_i\sin\theta\right)\arctan\frac{e_i\sin\theta}{\cos\theta + d_i\sin\theta}. \tag{B.2}$$

B.2 Second derivatives of R_i

$$\frac{\partial^2 R_i}{\partial x^2} = 2\log\left\{\frac{r^2}{a^2}\left[(\cos\theta + d_i\sin\theta)^2 + e_i^2\sin^2\theta\right]\right\}, \tag{B.3}$$

$$\frac{\partial^2 R_i}{\partial x\partial y} = 2d_i\log\left\{\frac{r^2}{a^2}\left[(\cos\theta + d_i\sin\theta)^2 + e_i^2\sin^2\theta\right]\right\} - 4e_i\arctan\frac{e_i\sin\theta}{\cos\theta + d_i\sin\theta}, \tag{B.4}$$

$$\frac{\partial^2 R_i}{\partial y^2} = 2\left(d_i^2 - e_i^2\right)\log\left\{\frac{r^2}{a^2}\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]\right\} - 8d_ie_i\arctan\frac{e_i\sin\theta}{\cos\theta + d_i\sin\theta}. \tag{B.5}$$

B.3 Third derivatives of R_i

$$\frac{\partial^3 R_i}{\partial x^3} = \frac{4\left(\cos\theta + d_i\sin\theta\right)}{r\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]}, \tag{B.6}$$

$$\frac{\partial^3 R_i}{\partial x^2\partial y} = \frac{4\left[d_i\left(\cos\theta + d_i\sin\theta\right) + e_i^2\sin\theta\right]}{r\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]}, \tag{B.7}$$

$$\frac{\partial^3 R_i}{\partial x\partial y^2} = \frac{4\left[\left(d_i^2 - e_i^2\right)\cos\theta + \left(d_i^2 + e_i^2\right)d_i\sin\theta\right]}{r\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]}, \tag{B.8}$$

$$\frac{\partial^3 R_i}{\partial y^3} = \frac{4\left[d_i\left(d_i^2 - 3e_i^2\right)\cos\theta + \left(d_i^4 - e_i^4\right)\sin\theta\right]}{r\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]}. \tag{B.9}$$

B.4 Fourth derivatives of R_i

$$\frac{\partial^4 R_i}{\partial x^4} = -\frac{4\left[\left(\cos\theta + d_i\sin\theta\right)^2 - e_i^2\sin^2\theta\right]}{r^2\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]^2}, \tag{B.10}$$

$$\frac{\partial^4 R_i}{\partial x^3\partial y} = -\frac{4}{r^2}\left\{\frac{d_i}{\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta} + \frac{2e_i^2\sin\theta\cos\theta}{\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]^2}\right\}, \tag{B.11}$$

$$\frac{\partial^4 R_i}{\partial x^2\partial y^2} = -\frac{4}{r^2}\left\{\frac{\left(d_i^2 + e_i^2\right)}{\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]} - \frac{2e_i^2\cos^2\theta}{\left[\left(\cos\theta + d_i\sin\theta\right)^2 + e_i^2\sin^2\theta\right]^2}\right\}, \tag{B.12}$$

$$\frac{\partial^4 R_i}{\partial x \partial y^3} = -\frac{4}{r^2} \left\{ \frac{d_i\left(d_i^2+e_i^2\right)}{\left[\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right]} - \frac{2e_i^2 \cos\theta\left(2d_i\cos\theta + \left(d_i^2+e_i^2\right)\sin\theta\right)}{\left[\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right]^2} \right\}, \tag{B.13}$$

$$\frac{\partial^4 R_i}{\partial y^4} = -\frac{4}{r^2} \left\{ \frac{\left(d_i^4-e_i^4\right)}{\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta} - \frac{2e_i^2 \cos\theta\left[\left(3d_i^2-e_i^2\right)\cos\theta + 2d_i\left(d_i^2+e_i^2\right)\sin\theta\right]}{\left[\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right]^2} \right\}. \tag{B.14}$$

B.5 First derivatives of S_i

$$\frac{\partial S_i}{\partial x} = r e_i \sin\theta \left\{ \log\left[\frac{r^2}{a^2}\left(\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right)\right] - 2 \right\} + 2r\left(\cos\theta + d_i\sin\theta\right)\arctan\frac{e_i \sin\theta}{\cos\theta + d_i \sin\theta}, \tag{B.15}$$

$$\frac{\partial S_i}{\partial y} = r e_i \left(\cos\theta + 2d_i\sin\theta\right) \left\{ \log\left[\frac{r^2}{a^2}\left(\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right)\right] - 2 \right\} + 2r\left[d_i\left(\cos\theta + d_i\sin\theta\right) - e_i^2\sin\theta\right]\arctan\frac{e_i \sin\theta}{\cos\theta + d_i \sin\theta}. \tag{B.16}$$

B.6 Second derivatives of S_i

$$\frac{\partial^2 S_i}{\partial x^2} = 2\arctan\frac{e_i \sin\theta}{\cos\theta + d_i \sin\theta}, \tag{B.17}$$

$$\frac{\partial^2 S_i}{\partial x \partial y} = e_i \log\left\{\frac{r^2}{a^2}\left[\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right]\right\} + 2d_i\arctan\frac{e_i \sin\theta}{\cos\theta + d_i \sin\theta}, \tag{B.18}$$

$$\frac{\partial^2 S_i}{\partial y^2} = 2d_i e_i \log\left\{\frac{r^2}{a^2}\left[\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right]\right\} + 2\left(d_i^2 - e_i^2\right)\arctan\frac{e_i \sin\theta}{\cos\theta + d_i \sin\theta}. \tag{B.19}$$

B.7 Third derivatives of S_i

$$\frac{\partial^3 S_i}{\partial x^3} = -\frac{2e_i \sin\theta}{r\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]}, \tag{B.20}$$

$$\frac{\partial^3 S_i}{\partial x^2 \partial y} = \frac{2e_i \cos\theta}{r\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]}, \tag{B.21}$$

$$\frac{\partial^3 S_i}{\partial x \partial y^2} = \frac{2e_i\left[2d_i(\cos\theta + d_i \sin\theta) - \left(d_i^2 - e_i^2\right)\sin\theta\right)}{r\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]}, \tag{B.22}$$

$$\frac{\partial^3 S_i}{\partial y^3} = \frac{2e_i\left[\left(3d_i^2 - e_i^2\right)\cos\theta + 2d_i\left(d_i^2 + e_i^2\right)\sin\theta\right]}{r\left((\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]}. \tag{B.23}$$

B.8 Fourth derivatives of S_i

$$\frac{\partial^4 S_i}{\partial x^4} = \frac{4e_i \sin\theta(\cos\theta + d_i \sin\theta)}{r^2\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]^2}, \tag{B.24}$$

$$\frac{\partial^4 S_i}{\partial x^3 \partial y} = \frac{2e_i}{r^2}\left\{\frac{1}{(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta} - \frac{2\cos\theta(\cos\theta + d_i \sin\theta)}{\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]^2}\right\}, \tag{B.25}$$

$$\frac{\partial^4 S_i}{\partial x^2 \partial y^2} = -\frac{4e_i \cos\theta\left[d_i(\cos\theta + d_i \sin\theta) + e_i^2 \sin\ theta\right]}{r^2\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]^2}, \tag{B.26}$$

$$\frac{\partial^4 S_i}{\partial x \partial y^3} = -\frac{2e_i}{r^2}\left\{\frac{\left(d_i^2 + e_i^2\right)}{(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta} + \frac{2\left(d_i^2 + e_i^2\right)\cos\theta(\cos\theta + d_i \sin\theta) - 4e_i^2 \cos^2\theta}{\left[(\cos\theta + d_i \sin\theta)^2 + e_i^2 \sin^2\theta\right]^2}\right\}, \tag{B.27}$$

$$\frac{\partial^4 S_i}{\partial y^4} = -\frac{4e_i}{r^2} \left\{ \frac{d_i \left(d_i^2 + e_i^2\right)}{\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta} + \frac{\cos\theta \left[d_i \left(d_i^2 - 3e_i^2\right)\cos\theta + \left(d_i^4 - e_i^4\right)\sin\theta\right]}{\left[\left(\cos\theta + d_i \sin\theta\right)^2 + e_i^2 \sin^2\theta\right]^2} \right\}. \tag{B.28}$$