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Eigenvalue based inverse model of beam for structural modification and diagnostics. Part I: Theoretical formulation

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Abstract

In the work, the problems of the beam structural modification through coupling the additional mass or elastic support, as well as the problem of diagnostics of the beam cracks, are discussed. The common feature for both problems is that the material parameters in each of the discussed cases change only in one point (additional mass, the support in one point, the crack described by the elastic joint). These systems, after determination of the value of additional element and its localization, should have a given natural vibration frequency. In order to solve the inverse problem, i.e. the problem of finding values of the additional quantities (mass, elasticity), the beam inverse model was proposed. Analysis of this model allows finding such a value of additional mass (elasticity) as a function of its localization so that the system has the free vibration frequency, which is desired in the modification problem or measured on the object in the diagnostics.

Keywords

inverse model, structural modification, diagnostics, generalized functions.

1 INTRODUCTION

The dynamic characteristics are one of the most important factors which should be taken into consideration during design and using of the mechanical systems. It is essential for the designer to ensure that none of the system natural frequency corresponds to the excitation frequency. The structural modification can be used to change (to shift) the natural vibration frequencies away from the excitation frequency or to change the position of the vibration node. This part of the paper concerns searching for such an additional mass, added to the main system, and its position or for such a position of the elastic support and its coefficient of elasticity that the system after modification achieves the required eigenfrequencies or (and) eigenmodes.

Additionally, the dynamic characteristics are the source of the information about the technical condition of the object. The measured natural vibration frequencies can be used for diagnostic of the damages in the examined object [7, 10]. This part of the paper concerns identification of the crack position and its depth. The crack is modeled as an elastic joint.

The common feature for both problems is that the material parameters in each of the discussed cases change only in one point (the point mass, the support in one point, the crack described by the joint). These systems, after determination of the value of additional element and its localization, should have a given natural vibration frequency. Thus, the both problems can be treated as the *Structural Modification* of the system aiming at achieving the given value of the natural vibration frequency.

The typical approach to the optimization problem for the system vibration is carrying out a series of modification of the numerical or analytical model to obtain the required eigenfrequencies. Such an approach, known as the *forward variation* [5] approach or *design load analysis cycle* [1] is expensive, time-consuming and rarely leads to the optimum solution. The simple modification, such as adding a mass or an elastic support can be easily carrying out by determination of the proper receptance in the point where the additional element is added [11, 12].

The structural modification can be also defined as the inverse problem [2, 5]. The inverse engineering refers to the problem where the desired response (for example eigenvalue) of the system in known (diagnostics) or decided (modification) but the physical systems is unknown [6]. These problems are difficult because a unique solution is rarely possible.

The early works concerning the inverse eigenvalue problem [4, 16] are based on Rayleigh's work and their authors utilize the first order terms of Taylor's series expansion. Such an approach was developed in the work [1], where the second order approximation in Taylor series expansion was used.

Another approach is presented in the works [15, 17], where the response of the system due the forced vibration is used to change the natural frequency of the system. The method is based on modification of either the mass or stiffness matrix by treating the modification of the system matrices as an external forced response. This external forced response is formulated in terms of the modification parameters.

In work [3] authors presented detailed review of structural modification methods and classified them into categories of the techniques based on small modification, these based on localized modification, and these based on modal approximation. In the class of small modifications, they described model-updating techniques developed from Rayleigh's principle, eigenvalue derivatives and modal perturbation. In local modification class, they presented the methods that had been developed to solve the structural eigenvalue problem of the perturbed mass or (and) stiffness matrix of system. In this case the localization an additional mass or stiffness is known. The method is based on minimization of the residual matrix [14].

Thanks to such an approach to the inverse problem, we avoid the problem of the ambiguous solution.

Generally speaking, all methods of the inverse problem solution, described above, are based on the single- or multiple-analysis of the direct problem. In this paper, a different novel beam model, called the inverse model of a beam, is proposed. Thanks to such an approach, we avoid the problems related to the measurement noise, which is inevitable in direct problem analysis.

2 DESCRIPTION OF VIBRATION OF A BEAM WITH POINT CHANGE IN MATERIAL PROPERTIES

As it was mentioned in the introduction, the material parameters of the system, such as mass (additional mass) or elasticity (elastic support and crack modeled by elastic joint), in each of the problems discussed in the work change only in one point. Generalized functions (some basic properties see appendix A) are employed to describe the vibrations of such discrete-continuous systems.

2.1 Model of a beam with an additional mass

The model investigated in the work is the beam shown in Fig. 1.

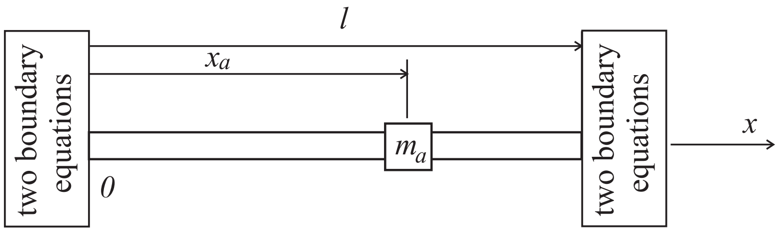

Figure 1 Beam with an addition mass

The differential equation for the free vibrations of a beam with an additional mass has the form:

$$EI \cdot \frac{\partial^4 y(x,t)}{\partial x^4} + \left(\rho F + m_a \cdot \delta(x,x_a)\right) \cdot \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{1}$$

where: EI - bending stiffness, ρ - material density , F - beam cross-section, m_a - addition mass applied to a beam in the point $x = x_a$, $\delta(x,x_a)$ - Dirac delta function in the point $x = x_a$.

The equation (1) will be solved using the Fourier method of separation of variables. The basis of the method is the assumption that the solution $y(x,t)$ can be treated as product of the function depending only on position and the function depending only on time.

$$y(x,t) = X(x) \cdot T(t)$$

With this assumption, the vibration amplitudes equation can be written in the form:

$$X(x)^{(4)} - \lambda^4 \cdot X(x) = \lambda^4 \cdot \frac{m_a}{\rho F} \cdot X(x) \cdot \delta(x,x_a) \tag{2}$$

where: $\lambda^4 = \omega^2 \cdot \rho F/EI$, ω - natural frequency of the beam.

The solution of the equation (2) is given by a function in the form:

$$\begin{aligned} X(x) = {} & P\cosh\lambda x + Q\sinh\lambda x + R\cos\lambda x + S\sin\lambda x + \\ & + \frac{\lambda m_a}{2\rho F} \cdot X(x_a) \cdot \left[\sinh\lambda(x-x_a) - \sin\lambda(x-x_a)\right] \cdot H(x,x_a) \end{aligned} \tag{3}$$

where:

- $H(x, x_a)$ - Heaviside (step) function in point $x = x_a$.

and its derivatives are described by a functions:

$$X'(x) = \lambda \cdot \Big(P \sinh \lambda x + Q \cosh \lambda x - R \sin \lambda x + S \cos \lambda x\Big) + \\ + \frac{\lambda^2 m_a}{2\rho F} \cdot X(x_a) \cdot \Big[\cosh \lambda(x - x_a) - \cos \lambda(x - x_a)\Big] \cdot H(x, x_a)$$

$$X''(x) = \lambda^2 \cdot \Big(P \cosh \lambda x + Q \sinh \lambda x - R \cos \lambda x - S \sin \lambda x\Big) + \\ + \frac{\lambda^3 m_a}{2\rho F} \cdot X(x_a) \cdot \Big[\sinh \lambda(x - x_a) + \sin \lambda(x - x_a)\Big] \cdot H(x, x_a)$$

$$X'''(x) = \lambda^3 \cdot \Big(P \sinh \lambda x + Q \cosh \lambda x + R \sin \lambda x - S \cos \lambda x\Big) + \\ + \frac{\lambda^4 m_a}{2\rho F} \cdot X(x_a) \cdot \Big[\cosh \lambda(x - x_a) + \cos \lambda(x - x_a)\Big] \cdot H(x, x_a)$$

Integration constants P, Q, R, S depends on boundary conditions related to the initial-boundary problem under consideration.

The equations describing the boundary conditions constitute the system of 4 algebraic homogeneous equations, to which one can add, as the fifth equation, the equation connecting the beam vibration amplitude in the point $x = x_a$ (i.e. the place where the mass is added) to the constants of integration in the form:

$$P \cosh \lambda x_a + Q \sinh \lambda x_a + R \cos \lambda x_a + S \sin \lambda x_a - X(x_a) = 0$$

In this way, the homogeneous system of 5 algebraic equations is obtained, where the unknowns are: the constants of integration P, Q, R, S and the beam vibration amplitude in the point $X(x_a)$ where the mass is added.

This system can be written in the matrix form $\mathbf{M} \cdot \mathbf{C} = \mathbf{0}$:

$$\begin{bmatrix} \multicolumn{4}{c}{\text{four equations which}} & 0 \\ \multicolumn{4}{c}{\text{describes the boundary conditions}} & 0 \\ \multicolumn{4}{c}{\text{of the beam without}} & a_{35} \\ \multicolumn{4}{c}{\text{an additional element}} & a_{45} \\ \cosh \lambda x_a & \sinh \lambda x_a & \cos \lambda x_a & \sin \lambda x_a & -1 \end{bmatrix} \begin{bmatrix} P \\ Q \\ S \\ R \\ X(x_a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

The coefficients a_{35}, a_{45} depends on the forms of the equations describing the boundary conditions at the right end of the beam ($x = l$). They are the coefficients at the constant $X(x_a)$ in the equation (3) or its derivatives are listed in Tab. 1.

Table 1 The coefficients a_{35} and a_{45}

Boundary condition	a_{35} or a_{45}
$X(l) = 0$	$\frac{\lambda m_a}{2\rho F} \cdot \left[\sinh \lambda(l - x_a) - \sin \lambda(l - x_a) \right]$
$X'(l) = 0$	$\frac{\lambda^2 m_a}{2\rho F} \cdot \left[\cosh \lambda(l - x_a) - \cos \lambda(l - x_a) \right]$
$X''(l) = 0$	$\frac{\lambda^3 m_a}{2\rho F} \cdot \left[\sinh \lambda(l - x_a) + \sin \lambda(l - x_a) \right]$
$X'''(l) = 0$	$\frac{\lambda^4 m_a}{2\rho F} \cdot \left[\cosh \lambda(l - x_a) + \cos \lambda(l - x_a) \right]$

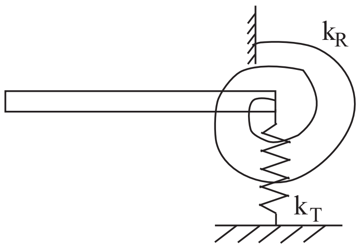

Figure 2 Right end of the beam supported in general way

In the case of the generalized boundary conditions (elastic support of the beam), the equations for the boundary conditions are the proper linear combination of the equations presented in Tab. 1 and for the support as in Fig. 2.

The equation which connect the shear force and vibration amplitude in the point $x = l$ has the form:

$$EI \cdot X'''(l) - k_T \cdot X(l) = 0$$

In this case coefficient $a_{35}(a_{45})$ has form (it is result from the rows 1 and 4 of the Tab. 1):

$$\begin{aligned} a_{35}(a_{45}) &= EI \cdot \frac{\lambda^4 m_a}{2\rho F} \cdot \left[\cosh \lambda(l - x_a) + \cos \lambda(l - x_a) \right] + \\ &- k_T \cdot \frac{\lambda m_a}{2\rho F} \cdot \left[\sinh \lambda(l - x_a) - \sin \lambda(l - x_a) \right] \end{aligned}$$

The equation which connect the bending moment and angle of cross-section rotation in the point $x = l$ has the form:

$$EI \cdot X''(l) + k_R \cdot X'(l) = 0$$

In this case coefficient $a_{35}(a_{45})$ has form (rows 2 and 3 of the Tab. 1):

$$\begin{aligned} a_{35}(a_{45}) &= EI \cdot \frac{\lambda^3 m_a}{2\rho F} \cdot \big[\sinh\lambda(l-x_a) + \sin\lambda(l-x_a)\big] + \\ &+ k_R \cdot \frac{\lambda^2 m_a}{2\rho F} \cdot \big[\cosh\lambda(l-x_a) - \cos\lambda(l-x_a)\big] \end{aligned}$$

2.2 Model of a beam with an elastic support

Model of a beam with additional internal elastic support is shown in Fig. 3.

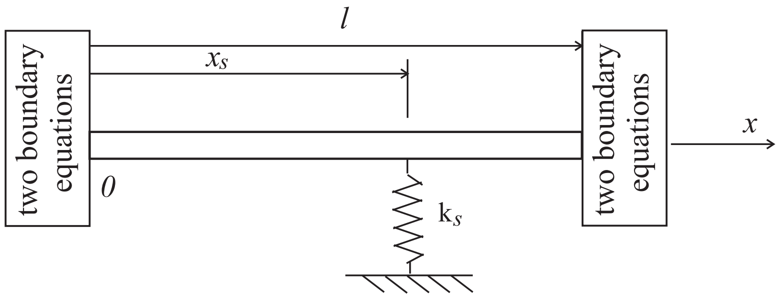

Figure 3 Beam with an additional internal elastic support

The differential equation for free vibrations can be written in the form:

$$EI \cdot \frac{\partial^4 y(x,t)}{\partial x^4} + \rho F \cdot \frac{\partial^2 y(x,t)}{\partial t^2} - k_s \cdot y(x,t) \cdot \delta(x, x_s) = 0 \tag{5}$$

After separation of the variables, the vibration amplitude equation, takes the form:

$$X(x)^{(4)} - \lambda^4 \cdot X(x) = \frac{k_s}{EI} \cdot X(x) \cdot \delta(x, x_a) \tag{6}$$

the solution of which is the function:

$$\begin{aligned} X(x) &= P\cosh\lambda x + Q\sinh\lambda x + R\cos\lambda x + S\sin\lambda x + \\ &+ \frac{k_s}{2 \cdot EI \cdot \lambda^3} \cdot X(x_s) \cdot \Big[\sinh\lambda(x-x_s) - \sin\lambda(x-x_s)\Big] \cdot H(x, x_s) \end{aligned} \tag{7}$$

The integration constants P, Q, R, S depend on the boundary conditions corresponding to the initial-boundary problem under consideration. If the fifth equation, relating the beam vibration amplitude in the point $x = x_s$ to the constants of integration, is added to four equations describing the boundary conditions then such the system of equations written in the matrix form will have a form identical to that in the case described in the previous section, i.e. as in the equation (4), although now the coefficients $a_{35}(a_{45})$ have the form presented in the Tab. 2.

Table 2 Coefficients a_{35} and a_{45} in dependence on boundary condition

Boundary condition	a_{35} or a_{45}
$X(l)=0$	$-\frac{k_s}{2\cdot EI\cdot\lambda^3}\cdot\left[\sinh\lambda(l-x_s)-\sin\lambda(l-x_s)\right]$
$X'(l)=0$	$-\frac{k_s}{2\cdot EI\cdot\lambda^2}\cdot\left[\cosh\lambda(l-x_s)-\cos\lambda(l-x_s)\right]$
$X''(l)=0$	$-\frac{k_s}{2\cdot EI\cdot\lambda}\cdot\left[\sinh\lambda(l-x_s)+\sin\lambda(l-x_s)\right]$
$X'''(l)=0$	$-\frac{k_s}{2\cdot EI}\cdot\left[\cosh\lambda(l-x_s)+\cos\lambda(l-x_s)\right]$
$EI\cdot X'''(l)-k_T\cdot X(l)=0$	$-\frac{k_s}{2}\cdot\left[\cosh\lambda(l-x_s)+\cos\lambda(l-x_s)\right]+$
	$+\frac{k_T\cdot k_s}{2\cdot EI\cdot\lambda^3}\cdot\left[\sinh\lambda(l-x_s)-\sin\lambda(l-x_s)\right]$
$EI\cdot X''(l)+k_R\cdot X'(l)=0$	$-\frac{k_s}{2\cdot\lambda}\cdot\left[\sinh\lambda(l-x_s)+\sin\lambda(l-x_s)\right]+$
	$-\frac{k_R\cdot k_s}{2\cdot EI\cdot\lambda^2}\cdot\left[\cosh\lambda(l-x_s)-\cos\lambda(l-x_s)\right]$

2.3 Model of a beam with a crack

The examined in the work problem was described by an open crack model, what is shown schematically in Fig. 4.

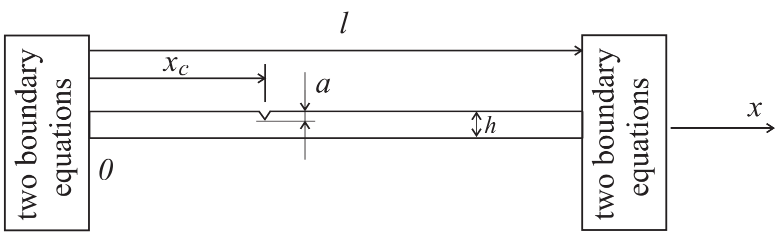

Figure 4 Beam with a crack

The crack was modeled as an elastic joint, flexibility c_b of which relates the bending moment in the intersection with the coordinate $x = x_c$ with the rotation angles at the right and left side of the intersection, at crack location:

$$y'(x_c^+) - y'(x_c^-) = c_b \cdot EI \cdot y''(x_c^-) \tag{8}$$

The equation relating flexibility of the elastic joint c_b and the crack depth, based on fracture mechanics and Castigliano's theorem, is placed in the appendix B.

The beam with the crack modeled as the elastic joint, being the subject of the subsequent analysis, is shown in Fig. 5.

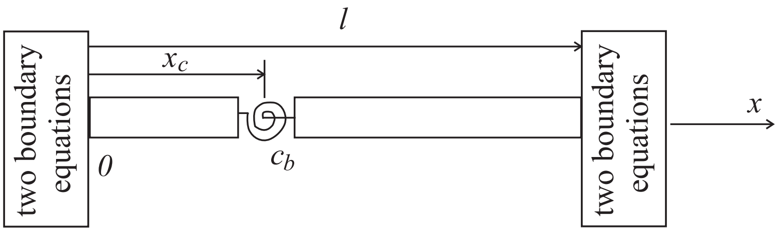

Figure 5 Cracked beam model

The free vibration equation for a beam with a crack, after separation of variables, can be written in the form:

$$X(x)^{(4)} - \lambda^4 \cdot X(x) = c_b \cdot X''(x_c) \cdot \delta''(x_c) \tag{9}$$

solution of which is the function:

$$\begin{aligned} X(x) &= P\cosh\lambda x + Q\sinh\lambda x + R\cos\lambda x + S\sin\lambda x + \\ &+ \frac{c_b}{2\lambda} \cdot X''(x_b) \cdot \Big[\sinh\lambda(x - x_c) + \sin\lambda(x - x_c) \Big] \cdot H(x, x_c) \end{aligned} \tag{10}$$

Integration constants P, Q, R, S depends on boundary conditions.

Similar to the previous sections of the work, the system of equations describing the boundary conditions will be completed with the equation:

$$\lambda^2 \cdot \Big[P\cosh\lambda x_c + Q\sinh\lambda x_c - R\cos\lambda x_c - S\sin\lambda x_c \Big] - X''(x_c) = 0$$

In such a case, the system of equations describing the boundary conditions has the form identical to this from the equation (4), although now the fifth row has the form coming from the above equation and the coefficients a_{35} and a_{45} depend on the corresponding derivatives of the equation (10) and are placed in Tab. 3.

3 INVERSE PROBLEM IN DIAGNOSTICS AND STRUCTURAL MODIFICATION

Analysis of the direct model consists in finding such values of the natural vibration frequencies ω so that the main matrix determinant of the equation (4) is equal to zero for them, i.e. $\det \mathbf{M} = 0$. Finding these values for given additional mass m_a (elasticity k_s) or crack depth (flexibility c_b) and their positions is not a problem.

Analysis of the inverse problem consists in finding such values of mass (elasticity) or flexibility and their positions so that the natural vibration frequency has the desired value (in the case of modification) or the value measured on the object (in the case of crack). So, the

Boundary condition	a_{35} or a_{45}
$X(l) = 0$	$\frac{c_b}{2\cdot EI\cdot\lambda}\cdot[\sinh\lambda(l-x_c)+\sin\lambda(l-x_c)]$
$X'(l) = 0$	$\frac{c_b}{2\cdot EI}\cdot\left[\cosh\lambda(l-x_c)+\cos\lambda(l-x_c)\right]$
$X''(l) = 0$	$\lambda\cdot\frac{c_b}{2\cdot EI}\cdot\left[\sinh\lambda(l-x_c)-\sin\lambda(l-x_c)\right]$
$X'''(l) = 0$	$\lambda^2\cdot\frac{c_b}{2\cdot EI}\cdot\left[\cosh\lambda(l-x_c)-\cos\lambda(l-x_c)\right]$
$EI\cdot X'''(l) - k_T\cdot X(l) = 0$	$\lambda^2\cdot\frac{c_b}{2}\cdot\left[\cosh\lambda(l-x_c)-\cos\lambda(l-x_c)\right]+$
	$-\frac{k_T\cdot c_b}{2\cdot EI\cdot\lambda}\cdot[\sinh\lambda(l-x_c)+\sin\lambda(l-x_c)]$
$EI\cdot X''(l) + k_R\cdot X'(l) = 0$	$\lambda\cdot\frac{c_b}{2}\cdot\left[\sinh\lambda(l-x_c)-\sin\lambda(l-x_c)\right]+$
	$+\frac{k_R\cdot c_b}{2\cdot EI}\cdot\left[\cosh\lambda(l-x_c)+\cos\lambda(l-x_c)\right]$

Table 3 Coefficients a_{35} and a_{45}

problem comes down (similarly as in the case of the direct problem) to solving the determinant equation $\det\mathbf{M} = 0$, in which the value $\lambda = \sqrt{\omega\cdot\sqrt{\frac{\rho F}{EI}}}$ in known now, while the values of mass m_a (elasticity k_s or flexibility c_b) and their positions are the searched variables. Mathematically, this can be written as searching for the solution of the two variables function in the form:

$$F_1(m_a, x_a) = 0 \qquad F_2(k_s, x_s) = 0 \qquad F_3(c_b, x_c) = 0$$

Such an equation can be solved employing sensitivity analysis, i.e. finding the impact of mass m_a (elasticity k_s or flexibility c_b) change and change of their positions on the change of free vibration frequency ω, in other words in determining:

$$\frac{\partial\det\mathbf{M}}{\partial m_a} \qquad \frac{\partial\det\mathbf{M}}{\partial x_a}$$
$$\text{or} \quad \frac{\partial\det\mathbf{M}}{\partial k_s} \qquad \frac{\partial\det\mathbf{M}}{\partial x_s}$$
$$\text{or} \quad \frac{\partial\det\mathbf{M}}{\partial c_b} \qquad \frac{\partial\det\mathbf{M}}{\partial x_c}$$

Another method for solution of the two-variable function is factor analysis that consists in so-called factorial experiments, in other words determining (from the equation $\det\mathbf{M} = 0$)

free vibration frequency for many different values of mass m_a (elasticity k_s or flexibility c_b) and their positions x_a (x_s, x_c), correspondingly. Having the results of many calculations, it is possible to determine gradient of these changes, in other words to determine "rate" of free vibration frequency ω, changes caused by the changes of the individual independent variables m_a, x_a (k_s, x_s or c_b, x_c). After determination of gradient, it is possible to minimize the difference between the determined free vibration frequency (from the condition $\det \mathbf{M} = 0$) and the desired value (the goal of modification) or the value measured (on the cracked element).

All the methods listed above consist in searching for the minimum of the function. This makes a serious problem in the analysis of the inverse problem, which is an ambiguous model - its solution is a function, for instance $m_a = g_1(x_a)$. Because of that, one should employ, in principle, the methods used for searching of a functional minimum, i.e. the methods based on calculus of variations.

Another method for solution of the inverse-model ambiguity problem is searching for the value of one of the independent variable (for instance mass m_a, elasticity k_s or flexibility c_b) for the constant fixed value of another independent variable (position x_a, x_s or x_c) - localized modification [3]. Such calculations (i.e. searching for a minimum of function of one variable) should be carried out for every possible position of the additional element from the range $(0, l)$.

In this paper, it was proposed to develop the inverse model of the beam and to carry out the analysis of this model instead of multiple analysis of the direct problem.

4 INVERSE MODEL OF A BEAM

As it was pointed out in the previous section, solution of the modification or the diagnostic problem one consists in searching for solution of the function of two variables written in form of determinant equation: $\det \mathbf{M} = 0$. To solve this equation, developing of the inverse model that allows to transform the equations of the form $F_1(m_a, x_a) = 0$ to the form $m_a = g_1(x_a)$, was proposed.

In order to determine the inverse model, the following algorithm of calculations, easy to computer implementation, was proposed:

1^o One should build a new matrix (marked as $\mathbf{A}$) that comes from the main matrix of the initial - boundary problem (eq. (4) through replacement of the quantity m_a (k_s or c_b) with the number 1

2^o One should build the second matrix (marked as $\mathbf{B}$) that comes from the matrix $\mathbf{A}$ through crossing the last row and the last column off. So, the matrix $\mathbf{B}$ is a matrix describing the eigenvalue problem for a beam without an additional element.

3^o After defining of the matrixes $\mathbf{A}$ and $\mathbf{B}$, the equation $\det \mathbf{M} = 0$ can be written in the form:

$$m_a \cdot (\det \mathbf{A} + \det \mathbf{B}) - \det \mathbf{B} = 0$$

$$\text{or} \quad k_s \cdot (\det \mathbf{A} + \det \mathbf{B}) - \det \mathbf{B} = 0$$

$$\text{or} \quad c_b \cdot (\det \mathbf{A} + \det \mathbf{B}) - \det \mathbf{B} = 0$$

4^o After some transformation, it is possible to determine the additional value for each $x_a(x_s, x_c) \in (0, l)$, with the help of the relationship:

$$m_a(k_s, c_b) = \det \mathbf{B} / (\det \mathbf{A} + \det \mathbf{B}) \tag{11}$$

The equation (11) is useful for determination of such a value of mass (elasticity or flexibility) as a function of its position so that the system has the free vibration frequency, which is desired in the modification problem or measured on the object in the diagnostics.

The choice of the "proper" value of the searched quantity and its position depends on the additional criteria, the propositions of which are given in the second part of this work.

5 SUMMARY AND CONCLUSIONS

In the work, the problems of the beam structural modification through coupling the additional mass or elastic support, as well as the problem of diagnostics of the beam cracks, are discussed.

Thanks to using of elastic joint model to describe the crack, the common feature of both problems is that material parameters of the beam change only in one point (point mass, support in one point, crack described as joint). This allows describing vibrations of such system with the use of generalized functions, thanks to what the mathematical description of vibrations in each problem has the same form.

In order to solve the inverse problem, i.e. the problem of finding values of the additional quantities (mass, elasticity), the beam inverse model was proposed. Analysis of this model allows finding such a value of additional mass (elasticity) as a function of its localization so that the free vibration frequency changes to desirable value, when this mass is coupled to the beam.

The proposed beam inverse model can be employing to identification of the beam cracks. In such a case, however, the input quantity is free vibration frequency measured on the damaged object.

Each determined free-vibration frequency allows determining of the flexibility curve modeling the crack in the function of its position. For each pair of parameters (c_b, x_c) lying on the curve, the free vibration frequency is equal to the frequency measured. The searched parameters of the crack lay in the point that is common for two arbitrary curves.

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APPENDIX A. BASIC PROPERTIES OF THE GENERALIZED FUNCTIONS

$$\text{Dirac delta function} \quad \delta(x,x_a) = \begin{cases} 0 & \text{for } x \neq x_a \\ \infty & \text{for } x = x_a \end{cases}$$

$$\text{Heaviside function} \quad H(x,x_a) = \begin{cases} 1 & \text{for } x > x_a \\ 0 & \text{for } x < x_a \end{cases}$$

Some properties:
$\alpha(x) \cdot \delta(x,x_a) = \alpha(x_a) \cdot \delta(x,x_a)$
$\alpha(x) \cdot \delta'(x,x_a) = \alpha(x_a) \cdot \delta'(x,x_a) - \alpha'(x) \cdot \delta(x,x_a)$

$$\int_{-\infty}^{x} \delta(x,x_a) = H(x,x_a) \qquad \int_{-\infty}^{x} \delta'(x,x_a) = \delta(x,x_a)$$

$$\int_{-\infty}^{x} \delta''(x,x_a) = \delta'(x,x_a)$$

$$\int_{-\infty}^{x} f(x) \cdot H(x,x_a) = (F(x) - F(x_a)) \cdot H(x,x_a) \qquad \text{where f(x)=F'(x)}$$

simple example of using:
$\Big(\cos(x-x_a) \cdot H(x,x_a)\Big)' = -\sin(x-x_a) \cdot H(x,x_a) + \delta(x,x_a)$
$\Big(\sin(x-x_a) \cdot H(x,x_a)\Big)' = \cos(x-x_a) \cdot H(x,x_a)$

APPENDIX B. FLEXIBILITY AT THE CRACKED PLACE

The fracture mechanics studies allow to find relations between global quantity G - Energy Release Rate determining the increase in elastic strain energy for infinitesimal crack surface increase:

$$G = \frac{\partial U}{\partial A_c}$$

and local quantity K - Stress Intensity Factor (SIF), which is function of crack depth a:

$$G = \frac{1-\nu^2}{E} \cdot K_I^2$$

where: G - energy release rate represents the elastic energy per unit crack surface area, - A_c - area of the crack, - ν - Poisson ratio, - E - Young modulus, - K_I - Stress Intensity Factor (SIF) of mode I (opening the crack) for pure bending, given by equation:

$$K_I = \sigma\sqrt{\pi a} \cdot F(a/h)$$

where $F(a/h)$ - correction function also called "shape functions". Relation between the function $F(a/h)$ and crack depth for different geometries elements with crack and different ways of loading can be found in catalogs e.g. [13].

The section with the crack should be replaced with a flexible joint (rotational spring) having the same potential energy [8]. In order to determine the flexibility c_b of such joint, the Castigliano's theorem shall be used:

$$c_b = \frac{\partial^2 U}{\partial M_b^2(x_c)}$$

where $M_b(x_c)$ bending moment in coordinate where crack is situated.