www.lajss.org
Latin American Journal of Solids and Structures 1 (2003) 119–140

A fully nonlinear multi-parameter rod model incorporating general cross-sectional in-plane changes and out-of-plane warping

P. M. Pimenta* and E. M. B. Campello

Polytechnic School at University of São Paulo, P.O. Box 61548, 05424-970 São Paulo, Brazil

Abstract

This work presents a fully nonlinear geometrically-exact multi-parameter rod model that incorporates general in-plane cross-sectional changes as well as general out-of-plane cross-sectional warping. The formulation constitutes an extension of the earlier works, in the sense that the restrictions to a rigid cross-section and to a Saint-Venant-like elastic warping are now removed from the theory. The definition of energetically conjugated cross-sectional resultants in terms of generalized stresses and strains, based on the concept of a cross-section director, simplifies the derivation of equilibrium equations and the enforcement of boundary conditions, in either weak or strong senses. In addition, the corresponding tangent bilinear weak form is obtained in a more expedient way, rendering always symmetric for hyper-elastic materials and conservative loadings. The definition of a cross-section director allows also the introduction of independent degrees-of-freedom to describe both the in-plane cross-sectional changes and the out-of-plane warping. Fully three-dimensional finite strain constitutive equations can therefore be employed with no spurious stiffening. Finite rotations are treated consistently by the Euler-Rodrigues formula in a pure Lagrangean framework. Altogether, the present assumptions allow a consistent basis for the proper representation of profile (distortional) deformations, which are typical of cold-formed thin-walled rod structures. This is one of the main features of the formulation, as the use of more complex shell models in order to capture such phenomena can be needless.

1 Introduction

Three-dimensional beam-like structures undergoing large displacements and large rotations are increasingly common in engineering practice. The development of geometrically-exact models for rod assemblages has consequently attracted much attention over the past decades, and numerous papers have been published up to now on the subject (see e.g. [3, 5–7, 9, 13–17], and references therein to name just a few).

The main purpose of this work is to present a fully nonlinear geometrically-exact multi-parameter rod model that incorporates general in-plane cross-sectional changes as well as general out-of-plane cross-sectional warping. The formulation constitutes an extension of the earlier

* Corresponding author Email: ppimenta@usp.br Received 13 Oct 2003; In revised form 28 Oct 2003

works presented in [3, 9, 13–17], in the sense that the restrictions to a rigid cross-section and to a Saint-Venant-like elastic warping are now removed from the theory.

Our approach defines energetically conjugated cross-sectional resultants in terms of generalized stresses and strains, based on the concept of a cross-section director. Besides their practical importance, the use of cross-sectional resultants simplifies the derivation of equilibrium equations and the enforcement of boundary conditions, in either weak or strong senses. In addition, the corresponding tangent bilinear weak form is obtained in a more expedient way, rendering always symmetric for hyper-elastic materials and conservative loadings (even far from equilibrium states).

Definition of a cross-section director plays a central role in the present model. Accordingly, it allows the introduction of independent degrees-of-freedom to describe both the in-plane cross-sectional changes and the out-of-plane warping. Fully three-dimensional finite strain constitutive equations can therefore be employed with no spurious stiffening. The ideas are general and extension to inelastic rods, in particular to those of elastic-plastic materials, is straightforward once a stress integration scheme within a time step is at hand.

Finite rotations are treated here by the Euler-Rodrigues formula in a pure Lagrangean framework [8]. We assume a straight reference configuration for the rod axis, but initially curved rods can also be considered if regarded as a stress-free deformed state from the straight position (see [10]). The use of convective non-Cartesian coordinate systems is this way avoided and only components on orthogonal frames are employed. Moreover, initial curvatures that are completely independent of the isoparametric concept are possible to be attained, which can be used even in (for example) straight finite elements.

Altogether, the present assumptions allow a consistent basis for the proper representation of profile (distortional) deformations, which are typical of cold-formed thin-walled rod structures. We believe this is one of the main features of our formulation, as the use of more complex shell models in order to capture such phenomena becomes unnecessary.

Throughout the text, italic Latin or Greek lowercase letters $(a, b, \ldots, \alpha, \beta, \ldots)$ denote scalar quantities, bold italic Latin or Greek lowercase letters ($\boldsymbol{a}$, $\boldsymbol{b}$, ... $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, ...) denote vectors, bold italic Latin or Greek capital letters ($\boldsymbol{A}$, $\boldsymbol{B}$, ...) denote second-order tensors, bold calligraphic Latin capital letters ($\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, ...) denote third-order tensors and bold blackboard italic Latin capital letters ($\mathbb{A}$, $\mathbb{B}$, ...) denote fourth-order tensors in a three-dimensional Euclidean space. Vectors and matrices built of tensor components on orthogonal frames (e.g. for computational purposes) are expressed by boldface upright Latin letters $(\mathbf{A}, \mathbf{B}, \ldots \mathbf{a}, \mathbf{b}, \ldots)$. Summation convention over repeated indices is adopted in the entire text, with Greek indices ranging from 1 to 2 and Latin indices from 1 to 3.

2 A multi-parameter rod theory incorporating general cross-sectional in-plane changes and out-of-plane warping

2.1 Kinematical assumptions

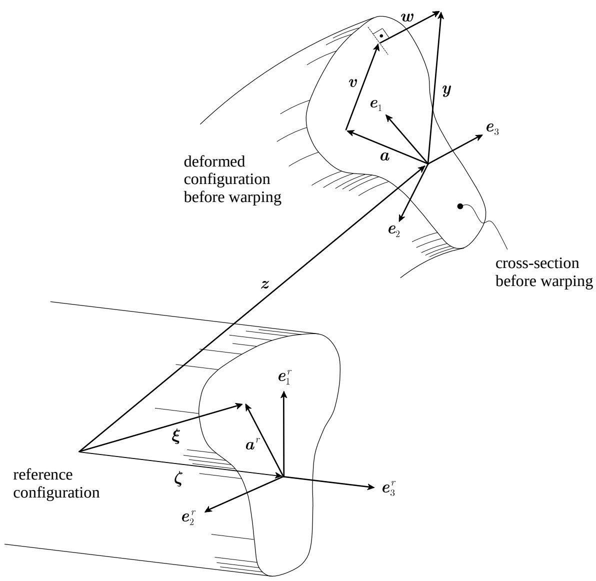

Figure 1: Rod description and basic kinematical quantities

It is assumed at the outset that the rod is straight at the initial configuration, which is used as reference. Initially curved rods can be mapped by standard isoparametric means, or can be regarded as a stress-free deformed state from the straight reference position (see [10]).

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a unit orthogonal system in the reference configuration, with e_3^r placed along the rod axis as depicted in Fig. 1. Cross-sectional planes in this configuration are uniquely defined by the vectors e_α^r. The position of the rod material points in the reference configuration can be described by

$$\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{a}^r, \tag{1}$$

where

$$\boldsymbol{\zeta} = \zeta \boldsymbol{e}_3^r, \ \zeta \in \Omega = [0, \ell] \tag{2}$$

defines a point on the rod axis and

$$\boldsymbol{a}^r = \xi_\alpha \boldsymbol{e}_\alpha^r \tag{3}$$

is the cross-section director at this point. The axis-coordinate ζ defines the rod length ℓ in the reference configuration (observe that $\{\xi_1, \xi_2, \zeta\}$ sets a three-dimensional Cartesian frame).

Let now $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a unit orthogonal system on the current configuration, with e_α attached to the cross-sectional plane before its warping. The rotation of the cross-section in the 3-D space is described by a rotation tensor $\boldsymbol{Q} = \hat{\boldsymbol{Q}}(\zeta)$, such that $\boldsymbol{e}_i = \boldsymbol{Q}\boldsymbol{e}_i^r$. In the current configuration the position $\boldsymbol{x}$ of the material points (see Fig. 1) is given by the vector field

$$\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{y} \ , \tag{4}$$

where $\boldsymbol{z} = \hat{\boldsymbol{z}}(\zeta)$ represents the current position of a point on the rod axis and $\boldsymbol{y}$ the position of the remaining points on the cross-section relative to the axis. We suppose that the cross-sections are first rigidly rotated from the reference configuration, then undergo an in-plane deformation and then are warped in the out-of-plane direction, so that the vector $\boldsymbol{y}$ can be decomposed as follows

$$\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{v} + \boldsymbol{w}. \tag{5}$$

Here

$$\boldsymbol{a} = \boldsymbol{Q}\boldsymbol{a}^r \tag{6}$$

is the current cross-section director, representing the rotational part of the deformation,

$$\boldsymbol{v} = v_\beta \boldsymbol{e}_\beta \tag{7}$$

is the vector of in-plane (or transversal) displacements, describing the in-plane changes of the cross-section, and

$$\boldsymbol{w} = w\boldsymbol{e}_3 \tag{8}$$

is the vector of out-of-plane displacements, embodying the cross-sectional warping. Notice that first-order shear deformations are accounted for since $\boldsymbol{e}_3$ is not necessarily coincident with the deformed rod axis.

Remark 1

As in (6), throughout the text we use the notation $(\cdot) = \boldsymbol{Q}(\cdot)^r \Leftrightarrow (\cdot)^r = \boldsymbol{Q}^T(\cdot)$ for any vectors $(\cdot), (\cdot)^r$. Vector $(\cdot)^r$ is said to be the back-rotated counterpart of $(\cdot)$ and is not affected by superimposed rigid body motions. It is noteworthy that vector $(\cdot)$ has the same components on the local system $\{\boldsymbol{e}_i,\ i = 1,2,3\}$ as $(\cdot)^r$ has on $\{\boldsymbol{e}_i^r,\ i = 1,2,3\}$.

Cross-sectional in-plane changes

Several kinematical assumptions are possible for the transversal displacement $\boldsymbol{v}$. Let $\boldsymbol{r} = \hat{\boldsymbol{r}}(\zeta)$ be a vector that collects the n_v transversal degrees-of-freedom, necessary to describe the cross-sectional in-plane changes. We assume here that $\boldsymbol{v}$ is a linear function of $\boldsymbol{r}$ such that

$$\boldsymbol{v} = \left(\boldsymbol{e}_\beta \otimes \boldsymbol{\phi}_\beta\right)\boldsymbol{r}, \tag{9}$$

where $\boldsymbol{\phi}_\beta = \hat{\boldsymbol{\phi}}_\beta(\xi_\alpha)$ are two vectors of shape functions describing the transversal distribution of the components of $\boldsymbol{v}$ on the cross-section. We remark that nonlinear relations may be necessary for the modeling of local buckling in cold-formed thin-walled metallic profiles; this will be addressed in a forthcoming paper [2]. From (7), these components are after that given by

$$v_\beta = \boldsymbol{\phi}_\beta \cdot \boldsymbol{r}. \tag{10}$$

We discuss next some possible choices for $\boldsymbol{\phi}_\beta$.

1. The simplest hypothesis for the transversal displacements is the traditional assumption of rigid cross-sections, what means

$$\boldsymbol{v} = \boldsymbol{o}. \tag{11}$$

where $\boldsymbol{o}$ is the zero vector. This restriction is adopted in nearly all rod formulations, such as [3, 9, 13–17] to name just a few. The cross-sectional transversal deformation is taken as zero and so a spurious stiffening is generated when 3-D constitutive models are utilized in the regular way.

2. A more general but yet simple assumption that partially corrects this drawback is to adopt a homothetic cross-sectional change, i.e.

$$\boldsymbol{v} = r\boldsymbol{a}, \tag{12}$$

where $r = \hat{r}(\zeta)$ is the only transversal degree-of-freedom. This assumption is adequate for isotropic materials (or for transversely isotropic materials with the material axis of symmetry parallel to $\boldsymbol{e}_3^r$) subjected to stretching-dominated deformations, as those observed in pure compression or pure tension. If (12) is adopted, we have $n_v = 1$, with

$$\boldsymbol{\phi}_\beta = [\xi_\beta] \quad \text{and} \quad \boldsymbol{r} = [r] \;. \tag{13}$$

3. For bending-dominated deformations, the artificial stiffening is not completely circumvented by (12). A better assumption is

$$\boldsymbol{\phi}_1 = \begin{bmatrix} \xi_1 \\ 1/2\,\xi_1^2 \\ \xi_1\xi_2 \end{bmatrix}, \quad \boldsymbol{\phi}_2 = \begin{bmatrix} \xi_2 \\ \xi_1\xi_2 \\ 1/2\,\xi_2^2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{14}$$

wherein 3 cross-sectional degrees-of-freedom are employed to describe the in-plane changes. In this case, r_1 corresponds to r in (12) and is important for stretching-dominated deformations, while r_2 and r_3 are imperative for the bending-dominated situations. Statement (14) is the simplest kinematical assumption that does not imply artificial stiffening in bending. It may be more adequate for isotropic materials, but can be employed whenever simplicity is aimed. We observe at this point that assumption (14) is equivalent to that one adopted in [12] for variable-thickness shells.

4. A complete quadratic assumption for $\boldsymbol{v}$ can be more appropriated, in particular for solid sections of anisotropic materials (or for plasticity-induced anisotropy). It is given by (9) with

$$\boldsymbol{\phi}_1 = \begin{bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{o}_5 \end{bmatrix}, \quad \boldsymbol{\phi}_2 = \begin{bmatrix} \boldsymbol{o}_5 \\ \boldsymbol{\varphi} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varphi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ 1/2\,\xi_1^2 \\ \xi_1\xi_2 \\ 1/2\,\xi_2^2 \end{bmatrix} \tag{15}$$

   Here the vector $\boldsymbol{r}$ collects $n_v = 10$ transversal degrees-of-freedom, that entirely describe the quadratic in-plane cross-sectional changes.

5. More general functions must be assumed if one attempts to capture profile (distortional) deformations, typical of cold-formed thin-walled sections. The degrees-of-freedom r_v, $v = 1, \ldots, n_v$ will then correspond to additional transversal displacements and axial rotations of points along the mid-line of the thin-walled section. The corresponding shape functions describe the profile deformation as transversal frames between these points, as done for example in [18]. We will return to this issue in a forthcoming paper ([2]).

Out-of-plane displacements

There is also a number of possible kinematical assumptions for the out-of-plane displacements $\boldsymbol{w}$ due to warping. Let us write these displacements as

$$\boldsymbol{w} = (\boldsymbol{e}_3 \otimes \boldsymbol{\psi})\, \boldsymbol{p}, \tag{16}$$

where $\boldsymbol{p} = \hat{\boldsymbol{p}}(\zeta)$ is a vector that collects the n_w degrees-of-freedom describing the cross-sectional warping, and $\boldsymbol{\psi} = \hat{\boldsymbol{\psi}}(\xi_\alpha)$ is the vector of corresponding warping shape functions ψ_w. According to (16) the component w of (8) on the current local system is given by

$$w = \boldsymbol{\psi} \cdot \boldsymbol{p}. \tag{17}$$

The consideration of $\boldsymbol{w}$ is of central importance in torsion- and other shear-dominated deformations, and we discuss some possibilities in what follows.

1. The simplest assumption, which was employed in [3,9,13,15,16] among many others, is to neglect all warping effects, what means

$$\boldsymbol{w} = \boldsymbol{o}. \tag{18}$$

   Accordingly, the cross-sections are forced to remain plane after the deformation, and therefore artificial stiffening is produced in torsion-dominated deformations as 3-D constitutive models are directly employed.

2. A simple choice that corrects this shortcoming is to adopt the classical warping functions $\psi = \hat{\psi}(\xi_\alpha)$ from the Saint-Venant's torsion theory, which can be found in any text-book on Theory of Elasticity, as follows

$$w = \psi p \ . \tag{19}$$

Here $p = \hat{p}(\zeta)$ is the only cross-sectional degree-of-freedom, describing the warping magnitude. Therefore $n_w = 1$ and

$$\boldsymbol{\psi} = [\psi] \quad \text{and} \quad \boldsymbol{p} = [p] \ . \tag{20}$$

This attractive kinematical assumption can be successfully used for either elastic (see [17]) or inelastic applications, whenever simplicity is desired.

3. For thin-walled cross-sections, the function ψ in (19) can be approximated by Vlasov's sectorial area, see [18]. In this case the warping function is assumed to be constant across the wall thickness, and piecewise linear along the section mid-line as in [1, 11]. Further improvements are still possible if we add to the Vlasov's warping assumption a secondary warping function, taken as linearly-varying across the wall thickness.

4. More general functions must be assumed in order to capture the real warping in the fully nonlinear regime. For solid functions the cross-section domain can be discretized by simple triangular and/or quadrilateral finite elements. For thin-walled rods the functions ψ_w can be assumed to be constant across the wall thickness and linear between arbitrarily chosen nodal points on the wall mid-lines. This assumption can be further improved if we add secondary warping functions, which can be assumed to be linear across the wall thickness. We will return to this issue in a forthcoming paper [2].

Remark 2

It is sometimes convenient that the warping functions $\psi_w,\ w = 1, \ldots, n_w$, satisfy the following orthogonality conditions

$$\int_A \psi_w dA = \int_A \xi_2 \psi_w dA = \int_A \xi_1 \psi_w dA = 0 \ , \tag{21}$$

in which A is the cross-section area. If the warping functions $\left\{\bar{\psi}_w,\ w = 1, \ldots n_w\right\}$ are initially chosen that do not satisfy (21), they can be made orthogonal by

$$\psi_w = \bar{\psi}_w - a_w - b_w \xi_2 + c_w \xi_1 \ , \tag{22}$$

where the coefficients $a_w,\ b_w$ and c_w are computed through

$$\begin{bmatrix} a_w \\ b_w \\ c_w \end{bmatrix} = \begin{bmatrix} A & S_1 & S_2 \\ S_1 & I_{11} & I_{12} \\ S_2 & I_{21} & I_{22} \end{bmatrix}^{-1} \begin{bmatrix} \int_A \psi_w dA \\ \int_A \xi_2 \psi_w dA \\ -\int_A \xi_1 \psi_w dA \end{bmatrix} . \tag{23}$$

Here, the following cross-sectional geometric constants have been introduced

$$\begin{aligned} &A = \int_A dA\ , \quad S_1 = \int_A \xi_2 dA\ , \quad S_2 = -\int_A \xi_1 dA\ , \\ &I_{11} = \int_A \xi_2^2 dA\ , \quad I_{22} = \int_A \xi_1^2 dA \quad \text{and} \quad I_{12} = I_{21} = -\int_A \xi_1 \xi_2 dA\ . \end{aligned} \tag{24}$$

2.2 Kinematics

The displacements of the points on the rod axis can be computed by

$$\boldsymbol{u} = \boldsymbol{z} - \boldsymbol{\zeta}. \tag{25}$$

The rotation tensor $\boldsymbol{Q}$, describing the rotation of the cross-sections, may be expressed in terms of the Euler rotation vector $\boldsymbol{\theta}$, by means of the well-known Euler-Rodrigues formula

$$\boldsymbol{Q} = \boldsymbol{I} + h_1(\theta)\,\boldsymbol{\Theta} + h_2(\theta)\,\boldsymbol{\Theta}^2\ . \tag{26}$$

In this case $\theta = \|\boldsymbol{\theta}\|$ is the true rotation angle and

$$h_1(\theta) = \frac{\sin\theta}{\theta} \quad \text{and} \quad h_2(\theta) = \frac{1}{2}\left(\frac{\sin\theta/2}{\theta/2}\right)^2 \tag{27}$$

are two trigonometric functions, with $\boldsymbol{\Theta} = \text{Skew}(\boldsymbol{\theta})$ as the skew-symmetric tensor whose axial vector is $\boldsymbol{\theta}$. Altogether, the components of $\boldsymbol{u}$, $\boldsymbol{\theta}$, $\boldsymbol{r}$ and $\boldsymbol{p}$ on a global Cartesian system constitute the $3 + 3 + n_v + n_w$ parameters (or cross-sectional degrees-of-freedom) of this rod model.

From differentiation of (4) with respect to $\boldsymbol{\xi}$ one can evaluate the deformation gradient $\boldsymbol{F}$. After some algebra one gets

$$\boldsymbol{F} = \boldsymbol{x}_{,\alpha} \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{x}' \otimes \boldsymbol{e}_3^r\ , \tag{28}$$

wherein we have used the notation $(\cdot)_{,\alpha} = \partial(\cdot)/\partial\xi_\alpha$ and $(\cdot)' = \partial(\cdot)/\partial\zeta$ for derivatives. With the aid of (4) through (8), the derivatives in (28) are

$$\boldsymbol{x}_{,\alpha} = \boldsymbol{a}_{,\alpha} + \boldsymbol{v}_{,\alpha} + \boldsymbol{w}_{,\alpha} \quad \text{and} \quad \boldsymbol{x}' = \boldsymbol{z}' + \boldsymbol{a}' + \boldsymbol{v}' + \boldsymbol{w}', \tag{29}$$

in which

$$\begin{aligned} &\boldsymbol{a}_{,\alpha} = \boldsymbol{Q}\boldsymbol{e}_\alpha^r\ , \quad \boldsymbol{v}_{,\alpha} = \left(\boldsymbol{\phi}_{\beta,\alpha} \cdot \boldsymbol{r}\right)\boldsymbol{Q}\boldsymbol{e}_\beta^r\ , \quad \boldsymbol{w}_{,\alpha} = \left(\boldsymbol{\psi}_{,\alpha} \cdot \boldsymbol{p}\right)\boldsymbol{Q}\boldsymbol{e}_3^r\ , \\ &\boldsymbol{a}' = \boldsymbol{Q}\left(\boldsymbol{\kappa}^r \times \boldsymbol{a}^r\right)\ , \quad \boldsymbol{v}' = \boldsymbol{Q}\left[\left(\boldsymbol{\phi}_\beta \cdot \boldsymbol{r}'\right)\boldsymbol{e}_\beta^r + \boldsymbol{\kappa}^r \times \boldsymbol{v}^r\right] \quad \text{and} \\ &\boldsymbol{w}' = \boldsymbol{Q}\left[\left(\boldsymbol{\psi} \cdot \boldsymbol{p}'\right)\boldsymbol{e}_3^r + \boldsymbol{\kappa}^r \times \boldsymbol{w}^r\right]\ , \end{aligned} \tag{30}$$

with

$$\boldsymbol{v}^r = v_\beta \boldsymbol{e}_\beta^r = \boldsymbol{Q}^T\boldsymbol{v} \quad \text{and} \quad \boldsymbol{w}^r = w\boldsymbol{e}_3^r = \boldsymbol{Q}^T\boldsymbol{w}\ . \tag{31}$$

Still in (30), the following vector has been introduced

$$\boldsymbol{\kappa}^r = \boldsymbol{\Gamma}^T\boldsymbol{\theta}', \tag{32}$$

in which

$$\boldsymbol{\Gamma} = \boldsymbol{I} + h_2(\theta)\,\boldsymbol{\Theta} + h_3(\theta)\,\boldsymbol{\Theta}^2 \tag{33}$$

and

$$h_3(\theta) = \frac{1 - h_1(\theta)}{\theta^2}\ . \tag{34}$$

Vector $\boldsymbol{\kappa}^r$ in (32) can be regarded as the back-rotated counterpart of $\boldsymbol{\kappa} = \text{axial}(\boldsymbol{K}) = \boldsymbol{\Gamma}\boldsymbol{\theta}'$, where $\boldsymbol{K} = \boldsymbol{Q}'\boldsymbol{Q}^T$ is a skew-symmetric tensor that shows up in deriving expressions (30). One can understand $\boldsymbol{K}$ as the tensor describing the specific rotations of the cross-sections.

Thus, the deformation gradient may be rewritten as

$$\boldsymbol{F} = \boldsymbol{Q}\left(\boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{\gamma}_3^r \otimes \boldsymbol{e}_3^r\right) = \boldsymbol{Q}\boldsymbol{F}^r\ , \tag{35}$$

where $\boldsymbol{F}^r = \boldsymbol{I} + \gamma_\alpha^r \otimes e_\alpha^r + \gamma_3^r \otimes e_3^r$ is called the back-rotated deformation gradient and

$$\begin{aligned} \boldsymbol{\gamma}_\alpha^r &= \left(\boldsymbol{\phi}_{\beta,\alpha} \cdot \boldsymbol{r}\right)\boldsymbol{e}_\beta^r + \left(\boldsymbol{\psi}_{,\alpha} \cdot \boldsymbol{p}\right)\boldsymbol{e}_3^r \quad \text{and} \\ \boldsymbol{\gamma}_3^r &= \boldsymbol{\eta}^r + \boldsymbol{\kappa}^r \times \boldsymbol{y}^r + \left(\boldsymbol{\phi}_\beta \cdot \boldsymbol{r}'\right)\boldsymbol{e}_\beta^r + \left(\boldsymbol{\psi} \cdot \boldsymbol{p}'\right)\boldsymbol{e}_3^r\ . \end{aligned} \tag{36}$$

Here

$$\boldsymbol{\eta}^r = \boldsymbol{Q}^T\boldsymbol{z}' - \boldsymbol{e}_3^r\ , \tag{37}$$

and $\boldsymbol{y}^r$ is the back-rotated counterpart of $\boldsymbol{y}$, i.e.

$$\boldsymbol{y}^r = \boldsymbol{a}^r + \boldsymbol{v}^r + \boldsymbol{w}^r = \boldsymbol{Q}^T\boldsymbol{y}\ . \tag{38}$$

It will be clear on the next items that vectors $\boldsymbol{\eta}^r$ of (37) and $\boldsymbol{\kappa}^r$ of (32) can be understood as generalized cross-sectional strains.

The material velocity gradient is given by time differentiation of (35) (denoted by a superposed dot) as follows

$$\dot{\boldsymbol{F}} = \boldsymbol{\Omega}\boldsymbol{F} + \boldsymbol{Q}\left(\dot{\boldsymbol{\gamma}}_\alpha^r \otimes \boldsymbol{e}_\alpha^r + \dot{\boldsymbol{\gamma}}_3^r \otimes \boldsymbol{e}_3^r\right)\ , \tag{39}$$

where $\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T$ represents the cross-section spin. The spin axial vector $\boldsymbol{\omega}$ is obtained in a similar way as to obtain the axial vector of $\boldsymbol{K}$, i.e. $\omega = \text{axial}(\boldsymbol{\Omega}) = \boldsymbol{\Gamma}\dot{\boldsymbol{\theta}}$. Derivatives $\dot{\boldsymbol{\gamma}}_i^r$ of (39) are computed directly from (36), what yields

$$\begin{aligned} \dot{\boldsymbol{\gamma}}_\alpha^r &= \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_{\beta,\alpha}\right)\dot{\boldsymbol{q}} + \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}_{,\alpha}\right)\dot{\boldsymbol{p}} \quad \text{and} \\ \dot{\boldsymbol{\gamma}}_3^r &= \dot{\boldsymbol{\eta}}^r + \dot{\boldsymbol{\kappa}}^r \times \boldsymbol{y}^r + \left[\left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r\right) \otimes \boldsymbol{\psi}\right]\dot{\boldsymbol{p}} + \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}\right)\dot{\boldsymbol{p}}' + \\ &\quad \left[\left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r\right) \otimes \boldsymbol{\phi}_\beta\right]\dot{\boldsymbol{r}} + \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_\beta\right)\dot{\boldsymbol{r}}' \end{aligned} \tag{40}$$

In order to fully evaluate expressions (40), the time derivatives $\dot{\boldsymbol{\eta}}^r$ and $\dot{\boldsymbol{\kappa}}^r$ are needed. From (37) and (32), after some algebra it is possible to arrive at

$$\dot{\boldsymbol{\eta}}^r = \boldsymbol{Q}^T\left(\dot{\boldsymbol{u}}' + \boldsymbol{Z}'\boldsymbol{\Gamma}\dot{\boldsymbol{\theta}}\right) \quad \text{and} \quad \dot{\boldsymbol{\kappa}}^r = \boldsymbol{Q}^T\left(\boldsymbol{\Gamma}'\dot{\boldsymbol{\theta}} + \boldsymbol{\Gamma}\dot{\boldsymbol{\theta}}'\right)\ , \tag{41}$$

where $\boldsymbol{Z}' = \mathrm{Skew}\,(\boldsymbol{z}')$ and

$$\boldsymbol{\Gamma}' = h_2(\theta)\,\boldsymbol{\Theta}' + h_3(\theta)\,(\boldsymbol{\Theta}\boldsymbol{\Theta}' + \boldsymbol{\Theta}'\boldsymbol{\Theta}) + h_4(\theta)\left(\boldsymbol{\theta}\cdot\boldsymbol{\theta}'\right)\boldsymbol{\Theta} + h_5(\theta)\left(\boldsymbol{\theta}\cdot\boldsymbol{\theta}'\right)\boldsymbol{\Theta}^2\,. \tag{42}$$

Notice that in (42) $\boldsymbol{\Theta}' = \mathrm{Skew}\left(\boldsymbol{\theta}'\right)$ and

$$h_4(\theta) = \frac{h_1(\theta) - 2h_2(\theta)}{\theta^2} \quad \text{and} \quad h_5(\theta) = \frac{h_2(\theta) - 3h_3(\theta)}{\theta^2} \tag{43}$$

are two additional trigonometric functions.

2.3 Statics

Let the 1^{st} Piola-Kirchhoff stress tensor be written as

$$\boldsymbol{P} = \boldsymbol{Q}\left(\boldsymbol{\tau}\alpha^r \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{\tau}_3^r \otimes \boldsymbol{e}_3^r\right)\,. \tag{44}$$

The quantities $\boldsymbol{\tau}_i^r$ are back-rotated stress vectors and act on cross-sectional planes whose normals on the reference configuration are $\boldsymbol{e}_i^r$. Expression (44) motivates the definition of a back-rotated 1^{st} Piola-Kirchhoff stress tensor $\boldsymbol{P}^r$, such that

$$\boldsymbol{P}^r = \boldsymbol{Q}^T P = \boldsymbol{\tau}_\alpha^r \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{\tau}_3^r \otimes \boldsymbol{e}_3^r\,. \tag{45}$$

With the expressions for $\boldsymbol{P}$ and $\dot{\boldsymbol{F}}$, it is not difficult to show that the rod internal power per unit reference volume may be written as

$$\boldsymbol{P} : \dot{\boldsymbol{F}} = \boldsymbol{\tau}_\alpha^r \cdot \dot{\boldsymbol{\gamma}}_\alpha^r + \boldsymbol{\tau}_3^r \cdot \dot{\boldsymbol{\gamma}}_3^r\,, \tag{46}$$

where the property $\boldsymbol{P}\boldsymbol{F}^T : \boldsymbol{\Omega} = 0$, arising from the local moment balance, was utilized. Introducing (40) into (46) and performing some manipulation with the cross products, one gets

$$\begin{aligned}\boldsymbol{P} : \dot{\boldsymbol{F}} = {} & \boldsymbol{\tau}_3^r \cdot \dot{\boldsymbol{\eta}}^r + (\boldsymbol{y}^r \times \boldsymbol{\tau}_3^r) \cdot \dot{\boldsymbol{\kappa}}^r + \\ & + \left[(\boldsymbol{\tau}_\alpha^r \cdot \boldsymbol{e}_3^r)\,\boldsymbol{\psi}_{,\alpha} + (\boldsymbol{\tau}_3^r \cdot \boldsymbol{e}]\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r)\,\boldsymbol{\psi}\right] \cdot \dot{\boldsymbol{p}} + \left[(\boldsymbol{\tau}_3^r \cdot \boldsymbol{e}_3^r)\,\boldsymbol{\psi}\right] \cdot \dot{\boldsymbol{p}}' + \\ & + \left[\left(\boldsymbol{\tau}_\alpha^r \cdot \boldsymbol{e}_\beta^r\right)\boldsymbol{\phi}_{\beta,\alpha} + \left(\boldsymbol{\tau}_3^r \cdot \boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r\right)\boldsymbol{\phi}_\beta\right] \cdot \dot{\boldsymbol{r}} + \left[\left(\boldsymbol{\tau}_3^r \cdot \boldsymbol{e}_\beta^r\right)\boldsymbol{\phi}_\beta\right] \cdot \dot{\boldsymbol{r}}'\end{aligned} \tag{47}$$

Integration of (47) over the cross-section provides

$$\int_A \left(\boldsymbol{P} : \dot{\boldsymbol{F}}\right) dA = \boldsymbol{n}^r \cdot \dot{\boldsymbol{\eta}}^r + \boldsymbol{m}^r \cdot \dot{\boldsymbol{\kappa}}^r + \boldsymbol{\pi} \cdot \dot{\boldsymbol{p}} + \boldsymbol{\alpha} \cdot \dot{\boldsymbol{p}}' + \boldsymbol{\rho} \cdot \dot{\boldsymbol{r}} + \boldsymbol{\beta} \cdot \dot{\boldsymbol{r}}', \tag{48}$$

in which

$$\begin{aligned}&\boldsymbol{n}^r = \textstyle\int_A \boldsymbol{\tau}_3^r dA\,,\\ &\boldsymbol{m}^r = \textstyle\int_A (\boldsymbol{y}^r \times \boldsymbol{\tau}_3^r)\,dA\,,\\ &\boldsymbol{\pi} = \textstyle\int_A \left[(\boldsymbol{\tau}_\alpha^r \cdot \boldsymbol{e}_3^r)\,\boldsymbol{\psi}_{,\alpha} + (\boldsymbol{\tau}_3^r \cdot \boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r)\,\boldsymbol{\psi}\right] dA\,,\\ &\boldsymbol{\alpha} = \textstyle\int_A \left[(\boldsymbol{\tau}_3^r \cdot \boldsymbol{e}_3^r)\,\boldsymbol{\psi}\right] dA\,,\\ &\boldsymbol{\rho} = \textstyle\int_A \left[\left(\boldsymbol{\tau}_\alpha^r \cdot \boldsymbol{e}_\beta^r\right)\boldsymbol{\phi}_{\beta,\alpha} + \left(\boldsymbol{\tau}_3^r \cdot \boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r\right)\boldsymbol{\phi}_\beta\right] dA \quad \text{and}\\ &\boldsymbol{\beta} = \textstyle\int_A \left[\left(\boldsymbol{\tau}_3^r \cdot \boldsymbol{e}_\beta^r\right)\boldsymbol{\phi}_\beta\right] dA\end{aligned} \tag{49}$$

are generalized cross-sectional stresses energetically conjugated with the cross-sectional strains $\boldsymbol{\eta}^r$, $\boldsymbol{\kappa}^r$, $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{r}$ and $\boldsymbol{r}'$. In this case $\boldsymbol{n}^r$ is said to be the back-rotated cross-sectional forces and $\boldsymbol{m}^r$ the back-rotated cross-sectional moments (notice the effect of the in-plane-changes and out-of-plane warping on the definition of $\boldsymbol{m}^r$). Vector $\boldsymbol{\pi}$ represents the axial bi-shears, $\boldsymbol{\alpha}$ the axial bi-moments, $\boldsymbol{\rho}$ the transversal bi-shears and $\boldsymbol{\beta}$ the transversal bi-moments.

It is important to remark that $\boldsymbol{\tau}_i^r$, γ_i^r, $\boldsymbol{n}^r$, $\boldsymbol{m}^r$, $\boldsymbol{\eta}^r$, $\boldsymbol{\kappa}^r$, $\boldsymbol{p}$, $\boldsymbol{r}$, $\boldsymbol{\pi}$, $\boldsymbol{\alpha}$, $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ are not affected by superimposed rigid body motions and in this sense fulfill the objectivity requirements. We now collect these cross-sectional quantities into three vectors, as displayed below

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{n}^r \\ \boldsymbol{m}^r \\ \boldsymbol{\pi} \\ \boldsymbol{\alpha} \\ \boldsymbol{\rho} \\ \boldsymbol{\beta} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\eta}^r \\ \boldsymbol{\kappa}^r \\ \boldsymbol{p} \\ \boldsymbol{p}' \\ \boldsymbol{r} \\ \boldsymbol{r}' \end{bmatrix} \quad \text{and} \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\theta} \\ \boldsymbol{p} \\ \boldsymbol{r} \end{bmatrix}. \tag{50}$$

Note that both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have $6 + 2\,(n_v + n_w)$ elements, whilst $\boldsymbol{d}$ encompasses the $6 + n_v + n_w$ cross-sectional degrees-of-freedom. Definitions in (50) allows us to write (48) as follows

$$\int_A \left(\boldsymbol{P} : \dot{\boldsymbol{F}} \right) dA = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} \, . \tag{51}$$

Here, the time derivative $\dot{\boldsymbol{\varepsilon}}$ may be written in a very compact manner as

$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\Psi} \boldsymbol{\Delta} \dot{\boldsymbol{d}}, \tag{52}$$

where

$$\boldsymbol{\Psi} = \begin{bmatrix} \bar{\boldsymbol{\Psi}} & \boldsymbol{O}_{6\times 2(n_v+n_w)} \\ \boldsymbol{O}_{2(n_v+n_w)\times 9} & \boldsymbol{I}_{2(n_v+n_w)} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Delta} = \begin{bmatrix} \bar{\boldsymbol{\Delta}} & \boldsymbol{O}_{9\times n_w} & \boldsymbol{O}_{9\times n_v} \\ \boldsymbol{O}_{n_w\times 9} & \boldsymbol{I}_{n_w} & \boldsymbol{O}_{n_w\times n_v} \\ \boldsymbol{O}_{n_w\times 9} & \boldsymbol{I}_{n_w}\frac{\partial}{\partial\zeta} & \boldsymbol{O}_{n_w\times n_v} \\ \boldsymbol{O}_{n_v\times 9} & \boldsymbol{O}_{n_v\times n_w} & \boldsymbol{I}_{n_v} \\ \boldsymbol{O}_{n_v\times 9} & \boldsymbol{O}_{n_v\times n_w} & \boldsymbol{I}_{n_v}\frac{\partial}{\partial\zeta} \end{bmatrix} \tag{53}$$

are respectively a $[6 + 2\,(n_v + n_w)] \times [9 + 2\,(n_v + n_w)]$ linear operator and a $[9 + 2\,(n_v + n_w)] \times (6 + n_v + n_w)$ differential operator. In (53) one has

$$\bar{\boldsymbol{\Psi}} = \begin{bmatrix} \boldsymbol{Q}^T & \boldsymbol{O} & \boldsymbol{Q}^T \boldsymbol{Z}' \boldsymbol{\Gamma} \\ \boldsymbol{O} & \boldsymbol{Q}^T \boldsymbol{\Gamma} & \boldsymbol{Q}^T \boldsymbol{\Gamma}' \end{bmatrix}_{6\times 9} \quad \text{and} \quad \bar{\boldsymbol{\Delta}} = \begin{bmatrix} \boldsymbol{I}\frac{\partial}{\partial\zeta} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}\frac{\partial}{\partial\zeta} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}_{9\times 6}, \tag{54}$$

which correspond exactly to $\boldsymbol{\Psi}$ and $\boldsymbol{\Delta}$ of [11].

With (51) at hand, the rod internal power on a domain $\Omega = [0, \ell]$ is then given by

$$P_{int} = \int_\Omega (\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}) \, d\zeta \; . \tag{55}$$

The external power on the same domain $\Omega = [0, \ell]$ can be expressed by

$$\boldsymbol{P}_{\text{ext}} = \int_\Omega \left(\int_\Gamma \boldsymbol{t} \cdot \dot{\boldsymbol{x}} d\Gamma + \int_A \boldsymbol{b} \cdot \dot{\boldsymbol{x}} dA \right) d\zeta \; , \tag{56}$$

where Γ is the contour of a cross-section, $\boldsymbol{t}$ is the external surface traction per unit reference area and $\boldsymbol{b}$ is the vector of body forces per unit reference volume. By time differentiation of (4) one has

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{u}} + \boldsymbol{\omega} \times \boldsymbol{y} + \left(\boldsymbol{e}_\beta \otimes \boldsymbol{\varphi}_\beta \right) \dot{\boldsymbol{r}} \; + (\boldsymbol{e}_3 \otimes \boldsymbol{\psi}) \, \dot{\boldsymbol{p}} \; . \tag{57}$$

which can be introduced into (56) so that

$$\boldsymbol{P}_{\text{ext}} = \int_\Omega \left(\bar{\boldsymbol{q}} \cdot \dot{\boldsymbol{d}} \right) d\zeta \; , \tag{58}$$

where

$$\bar{\boldsymbol{q}} = \begin{bmatrix} \bar{\boldsymbol{n}} \\ \bar{\boldsymbol{\mu}} \\ \bar{\boldsymbol{\alpha}} \\ \bar{\boldsymbol{\beta}} \end{bmatrix} \; . \tag{59}$$

In this expression the following generalized external forces have been introduced

$$\begin{aligned} &\bar{\boldsymbol{n}} = \textstyle\int_\Gamma \boldsymbol{t} d\Gamma + \int_A \boldsymbol{b} dA \; , \\ &\bar{\boldsymbol{\mu}} = \boldsymbol{\Gamma}^T \bar{\boldsymbol{m}} \; , \quad \text{with} \quad \bar{\boldsymbol{m}} = \textstyle\int_\Gamma \boldsymbol{y} \times \boldsymbol{t} d\Gamma + \int_A \boldsymbol{y} \times \boldsymbol{b} dA \; , \\ &\bar{\boldsymbol{\alpha}} = \textstyle\int_\Gamma (\boldsymbol{e}_3 \cdot \boldsymbol{t}) \, \boldsymbol{\psi} d\Gamma + \int_A (\boldsymbol{e}_3 \cdot \boldsymbol{b}) \, \boldsymbol{\psi} dA \quad \text{and} \\ &\bar{\boldsymbol{\beta}} = \textstyle\int_\Gamma (\boldsymbol{e}_\beta \cdot \boldsymbol{t}) \, \boldsymbol{\phi}_\beta d\Gamma + \int_A (\boldsymbol{e}_\beta \cdot \boldsymbol{b}) \, \boldsymbol{\phi}_\beta dA \end{aligned} \tag{60}$$

wherein $\bar{\boldsymbol{n}}$ is the applied external force, $\bar{\boldsymbol{m}}$ the applied external moment, $\bar{\boldsymbol{\alpha}}$ the applied external axial bi-moments and $\bar{\boldsymbol{\beta}}$ the applied external transversal bi-moments, all per unit length of the rod axis in the reference configuration.

Remark 3

The vector $\bar{\boldsymbol{\mu}} = \boldsymbol{\Gamma}^T \bar{\boldsymbol{m}}$ emerging from (60) is the distributed external moment truly power-conjugated with $\boldsymbol{\theta}$, and not purely $\bar{\boldsymbol{m}}$ as one would expect. The same holds for external concentrated moments and for the natural boundary conditions. This fact has far-reaching consequences in the nonlinear analysis of structures with rotational degrees of freedom, since a non-trivial geometric contribution of the applied moments is introduced in the tangent bilinear form (see appendix C).

2.4 Equilibrium equations

In the same way as to obtain (55), one can have the expression for the rod internal virtual work on a domain $\Omega = [0, \ell]$ as follows

$$\delta W_{int} = \int_{\Omega} (\boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon}) \, d\zeta \ , \ \text{with } \delta \boldsymbol{\varepsilon} = \boldsymbol{\Psi} \boldsymbol{\Delta} \delta \boldsymbol{d}. \tag{61}$$

The external virtual work on the same domain $\Omega = [0, \ell]$ may be evaluated similarly to (58), i.e.

$$\delta W_{\text{ext}} = \int_{\Omega} (\bar{\boldsymbol{q}} \cdot \delta \boldsymbol{d}) \, d\zeta \ , \tag{62}$$

so that the rod local equilibrium can be stated by means of the virtual work theorem in a standard way:

$$\delta W = \delta W_{\text{int}} - \delta W_{\text{ext}} = 0 \ \text{ in } \Omega, \quad \forall \delta \boldsymbol{d}. \tag{63}$$

Introducing (61) and (62) into this expression, and performing partial integration on the terms with $\delta \boldsymbol{u}'$, $(\boldsymbol{\Gamma} \delta \boldsymbol{\theta})'$, $\delta \boldsymbol{p}'$ and $\delta \boldsymbol{r}'$, the following local equilibrium equations in Ω are obtained by usual arguments of variational calculus

$$\begin{aligned} &\boldsymbol{n}' + \bar{\boldsymbol{n}} = o \ , \\ &\boldsymbol{m}' + \boldsymbol{z}' \times \boldsymbol{n} + \bar{\boldsymbol{m}} = \boldsymbol{o} \ , \\ &\boldsymbol{\alpha}' - \boldsymbol{\pi} + \bar{\boldsymbol{\alpha}} = \boldsymbol{o} \quad \text{and} \\ &\boldsymbol{\beta}' - \boldsymbol{\rho} + \bar{\boldsymbol{\beta}} = \boldsymbol{o} \ . \end{aligned} \tag{64}$$

Here

$$\boldsymbol{n} = \boldsymbol{Q} \boldsymbol{n}^r \quad \text{and} \quad \boldsymbol{m} = \boldsymbol{Q} \boldsymbol{m}^r \tag{65}$$

are the true cross-sectional stress resultants with respect to the current configuration. Equations $(64)_1$ and $(64)_2$ could be obtained by Statics as well.

Remark 4

The essential boundary conditions emanating from (63) are prescribed in terms of $\boldsymbol{d}$, i.e. $\boldsymbol{u}$, $\boldsymbol{\theta}$, $\boldsymbol{p}$ and $\boldsymbol{r}$. On the other hand, the natural boundary conditions are prescribed in terms of the static quantities $\boldsymbol{n}$, $\boldsymbol{\mu} = \boldsymbol{\Gamma}^T \boldsymbol{m}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We draw the attention of the reader to the fact that the pseudo-moment $\boldsymbol{\mu} = \boldsymbol{\Gamma}^T \boldsymbol{m}$ must be prescribed, and not purely $\boldsymbol{m}$ as one would expect.

2.5 Tangent bilinear form

The Gateaux derivative of δW in (63) with respect to $\boldsymbol{d}$, after some lengthy algebraic, leads to the tangent bilinear form

$$\delta \left(\delta W \right) = \int_{\Omega} \left[\left(\boldsymbol{\Psi} \boldsymbol{\Delta} \delta \boldsymbol{d} \right) \cdot \left(\boldsymbol{D} \Psi \boldsymbol{\Delta} \delta \boldsymbol{d} \right) + \left(\boldsymbol{\Delta} \delta \boldsymbol{d} \right) \cdot \left(\boldsymbol{G} \boldsymbol{\Delta} \delta \boldsymbol{d} \right) - \delta \boldsymbol{d} \cdot \left(\boldsymbol{L} \delta \boldsymbol{d} \right) \right] d\Omega \ , \tag{66}$$

in which

$$\boldsymbol{D}=\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}}\ ,\quad \boldsymbol{G}=\begin{bmatrix}\bar{\boldsymbol{G}} & \boldsymbol{O}_{9\times2(n_v+n_w)}\\ \boldsymbol{O}_{2(n_v+n_w)\times9} & \boldsymbol{O}_{2(n_v+n_w)\times2(n_v+n_w)}\end{bmatrix}\quad\text{and}\quad \boldsymbol{L}=\frac{\partial\bar{\boldsymbol{q}}}{\partial\boldsymbol{d}}\ . \tag{67}$$

are tangent operators. $\boldsymbol{D}$ and $\boldsymbol{G}$ represent the constitutive contribution and the geometrical effects of the internal forces on the tangent bilinear form, respectively. It is worth mentioning that operator $\bar{\boldsymbol{G}}$ in (67) is identical to $\boldsymbol{G}$ of [9] (where it was first derived, see appendix A for more details), what remarkably means that the consideration of cross-sectional in-plane changes and out-of-plane warping does not introduce any additional geometric terms in (66). Consequently, $\boldsymbol{G}$ is a function of $\boldsymbol{n}^r$, $\boldsymbol{m}^r$, $\boldsymbol{u}$ and $\boldsymbol{\theta}$ only, remaining always symmetric even far from equilibrium states. Operator $\boldsymbol{L}$, however, stands for the geometrical effects of the external forces and depends directly on the character of the external loading, as one can see in (67). More details on the operator $\boldsymbol{L}$ can be found in appendix C. The bilinear form (66) is therefore symmetric whenever $\boldsymbol{D}=\boldsymbol{D}^T$ and $\boldsymbol{L}=\boldsymbol{L}^T$, i.e. whenever the material is hyperelastic (or whenever the stress integration algorithm for inelastic materials possesses a potential) and the external loading is locally conservative.

We introduce now the following tensors of elastic (or algorithmic) tangent moduli

$$\frac{\partial\boldsymbol{\tau}_i^r}{\partial\boldsymbol{\gamma}_j^r}=\boldsymbol{C}_{ij}\ . \tag{68}$$

With the aid of (68) together with the derivatives

$$\begin{array}{ll}
\frac{\partial\boldsymbol{\gamma}_\alpha^r}{\partial\boldsymbol{\eta}^r}=\boldsymbol{O}\ , & \frac{\partial\boldsymbol{\gamma}_3^r}{\partial\boldsymbol{\eta}^r}=\boldsymbol{I}\ ,\\
\frac{\partial\boldsymbol{\gamma}_\alpha^r}{\partial\boldsymbol{\kappa}^r}=\boldsymbol{O}\ , & \frac{\partial\boldsymbol{\gamma}_3^r}{\partial\boldsymbol{\kappa}^r}=-\boldsymbol{Y}^r,\\
\frac{\partial\boldsymbol{\gamma}_\alpha^r}{\partial\boldsymbol{p}}=\boldsymbol{e}_3^r\otimes\boldsymbol{\psi}_{,\alpha}\ , & \frac{\partial\boldsymbol{\gamma}_3^r}{\partial\boldsymbol{p}}=(\boldsymbol{\kappa}^r\times\boldsymbol{e}_3^r)\otimes\boldsymbol{\psi}\ ,\\
\frac{\partial\boldsymbol{\gamma}_\alpha^r}{\partial\boldsymbol{p}'}=\boldsymbol{O}\ , & \frac{\partial\boldsymbol{\gamma}_3^r}{\partial\boldsymbol{p}'}=\boldsymbol{e}_3^r\otimes\boldsymbol{\psi}\ ,\\
\frac{\partial\boldsymbol{\gamma}_\alpha^r}{\partial\boldsymbol{r}}=\boldsymbol{e}_\beta^r\otimes\boldsymbol{\phi}_{\beta,\alpha}\ , & \frac{\partial\boldsymbol{\gamma}_3^r}{\partial\boldsymbol{r}}=\left(\boldsymbol{\kappa}^r\times\boldsymbol{e}_\beta^r\right)\otimes\boldsymbol{\phi}_\beta\quad\text{and}\\
\frac{\partial\boldsymbol{\gamma}_\alpha^r}{\partial\boldsymbol{r}'}=\boldsymbol{O}\ , & \frac{\partial\boldsymbol{\gamma}_3^r}{\partial\boldsymbol{r}'}=\boldsymbol{e}_\beta^r\otimes\boldsymbol{\phi}_\beta\ ,
\end{array} \tag{69}$$

where $\boldsymbol{Y}^r=\mathrm{Skew}\,(\boldsymbol{y}^r)$, one can obtain the elements of $\boldsymbol{D}$ (see appendix B) by the chain rule. We remark that $\boldsymbol{D}$ is symmetric if

$$\boldsymbol{C}_{ij}=\boldsymbol{C}_{ji}^T\ . \tag{70}$$

3 Elastic constitutive equations

3.1 General hyperelastic materials

We write the symmetric Green-Lagrange strain tensor as

$$\boldsymbol{E}=\frac{1}{2}\left(\boldsymbol{C}-\boldsymbol{I}\right)\ , \tag{71}$$

where

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = (\boldsymbol{F}^r)^T \boldsymbol{F}^r \tag{72}$$

is the right Cauchy-Green strain tensor. The second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ is energetically conjugated to $\boldsymbol{E}$ and is such that $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$, or equivalently

$$\boldsymbol{P}^r = \boldsymbol{F}^r \boldsymbol{S}. \tag{73}$$

A general hyperelastic material can be fully described by a specific strain energy function $\psi = \hat{\psi}(\boldsymbol{E})$, such that $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = \frac{\partial \psi}{\partial \boldsymbol{E}} \ . \tag{74}$$

As a consequence, a fourth-order tensor of elastic tangent moduli for the pair $\{\boldsymbol{S}, \boldsymbol{E}\}$ can be defined as

$$\mathbb{D} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} = \frac{\partial^2 \psi}{\partial \boldsymbol{E}^2} \ . \tag{75}$$

With the aid of the following third-order tensors

$$\boldsymbol{\mathcal{B}}_i = \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\gamma}_i^r} = (\mathrm{Sym}\,(\boldsymbol{e}_k^r \otimes \boldsymbol{e}_i^r)) \otimes \boldsymbol{f}_k^r \ , \tag{76}$$

where $\boldsymbol{f}_k^r = \boldsymbol{e}_i^r + \boldsymbol{\gamma}_k^r$, the relations

$$\boldsymbol{\tau}_i^r = \boldsymbol{\mathcal{B}}_i^T \boldsymbol{S} \tag{77}$$

can be readily derived from (73), where

$$\boldsymbol{\mathcal{B}}_i^T = \boldsymbol{f}_k^r \otimes (\mathrm{Sym}\,(\boldsymbol{e}_k^r \otimes \boldsymbol{e}_i^r)) \ . \tag{78}$$

From these last three expressions and from (68) we arrive at

$$\boldsymbol{C}_{ij} = \boldsymbol{\mathcal{B}}_i^T \mathbb{D} \boldsymbol{\mathcal{B}}_j + \left(\boldsymbol{e}_i^r \cdot \boldsymbol{S}\boldsymbol{e}_j^r\right) \boldsymbol{I} \ . \tag{79}$$

with which $\boldsymbol{D}$ can be computed.

Remark 5

The just developed approach for hyperelastic materials is general and can be straightforwardly extended to inelastic rods, once a stress integration scheme within a time step is available.

3.2 General isotropic hyperelastic materials

For isotropic hyperelasticity, the strain energy function ψ can be written in terms of the invariants of the right Cauchy-Green strain tensor $\boldsymbol{C}$. We adopt here the following set of invariants

$$I_1 = \boldsymbol{I} : \boldsymbol{C} \ , \quad I_2 = \frac{1}{2}\boldsymbol{I} : \boldsymbol{C}^2 \quad \text{and} \quad J = \det \boldsymbol{F} \ , \tag{80}$$

with which we write $\psi = \hat{\psi}(I_1, I_2, J)$. Using (73) and (74), the back-rotated first Piola-Kirchhoff stress tensor is then obtained via

$$\boldsymbol{P}^r = 2\boldsymbol{F}^r \left(\frac{\partial \psi}{\partial \boldsymbol{C}}\right) , \tag{81}$$

what yields

$$\boldsymbol{P}^r = \frac{\partial \psi}{\partial J} J (\boldsymbol{F}^r)^{-T} + 2\boldsymbol{F}^r \left(\frac{\partial \psi}{\partial I_1}\boldsymbol{I} + \frac{\partial \psi}{\partial I_2}\boldsymbol{C}\right) \tag{82}$$

if the chain rule is applied with the derivatives

$$\frac{\partial J}{\partial \boldsymbol{C}} = \frac{1}{2} J \boldsymbol{C}^{-1} , \quad \frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I} \quad \text{and} \quad \frac{\partial I_2}{\partial \boldsymbol{C}} = \boldsymbol{C} . \tag{83}$$

Conversely, as one can readily verify, if we write the back-rotated deformation gradient as

$$\boldsymbol{F}^r = \boldsymbol{f}_i^r \otimes \boldsymbol{e}_i^r , \tag{84}$$

where $\boldsymbol{f}_i^r = \boldsymbol{e}_i^r + \boldsymbol{\gamma}_i^r$ (see expression (35)), then

$$J = (\boldsymbol{f}_1^r \times \boldsymbol{f}_2^r) \cdot \boldsymbol{f}_3^r , \quad J(\boldsymbol{F}^r)^{-T} = \boldsymbol{g}_i^r \otimes \boldsymbol{e}_i^r \quad \text{and} \quad \boldsymbol{F}^r \boldsymbol{C} = \left(\boldsymbol{f}_i^r \cdot \boldsymbol{f}_j^r\right) \boldsymbol{f}_i^r \otimes \boldsymbol{e}_j^r , \tag{85}$$

in which

$$\boldsymbol{g}_1^r = \boldsymbol{f}_2^r \times \boldsymbol{f}_3^r , \quad \boldsymbol{g}_2^r = \boldsymbol{f}_3^r \times \boldsymbol{f}_1^r \quad \text{and} \quad \boldsymbol{g}_3^r = \boldsymbol{f}_1^r \times \boldsymbol{f}_2^r . \tag{86}$$

Introducing (85) into (82), one arrives at the following expression for the vector-columns of $\boldsymbol{P}^r$:

$$\boldsymbol{\tau}_i^r = \frac{\partial \psi}{\partial J}\boldsymbol{g}_i^r + 2\frac{\partial \psi}{\partial I_1}\boldsymbol{f}_i^r + 2\frac{\partial \psi}{\partial I_2}\left(\boldsymbol{f}_j^r \otimes \boldsymbol{f}_j^r\right)\boldsymbol{f}_i^r . \tag{87}$$

3.3 A neo-Hookean hyperelastic material

A simple poly-convex neo-Hookean material as proposed in [4] is represented by the strain energy function

$$\psi(J, I_1) = \frac{1}{2}\lambda\left[\frac{1}{2}\left(J^2 - 1\right) - \ln J\right] + \frac{1}{2}\mu\left(I_1 - 3 - 2\ln J\right) , \tag{88}$$

in which λ and μ are material parameters (or Lamé coefficients). With this expression at hand, from (87) we get

$$\tau_i^r = \frac{1}{J}\left[\lambda\frac{1}{2}\left(J^2 - 1\right) - \mu\right]\boldsymbol{g}_i^r + \mu\boldsymbol{f}_i^r , \tag{89}$$

and then the tangent tensors in (68) are given by

$$\boldsymbol{C}_{ij} = \left[\tfrac{1}{2}\lambda\left(1 + \tfrac{1}{J^2}\right) + \tfrac{1}{J^2}\mu\right]\boldsymbol{g}_i^r \otimes \boldsymbol{g}_j^r + \mu\,\delta_{ij}\boldsymbol{I} - \tfrac{1}{J}\left[\lambda\tfrac{1}{2}\left(J^2 - 1\right) - \mu\right]\varepsilon_{ijk}\,\text{Skew}\left(\boldsymbol{f}_k^r\right) . \tag{90}$$

Here $\delta_{ij} = \boldsymbol{e}_i^r \cdot \boldsymbol{e}_j^r$ and $\varepsilon_{ijk} = \boldsymbol{e}_i^r \cdot \boldsymbol{e}_j^r \times \boldsymbol{e}_k^r$ are the usual Kronecker and permutation symbols, respectively. From (90) the constitutive matrix $\boldsymbol{D}$ can be computed.

4 Finite element solution

The description of the rod deformation generates a boundary value problem whose weak form (63) can be solved by several approximation techniques. We adopt here a Galerkin type of approximation, the trial functions of which are to be supplied by the finite element method. We write the finite element interpolation in a particular element e, $e = 1, \ldots N_e$, as follows

$$\boldsymbol{d} = \mathbf{N}\mathbf{p}_e \ , \tag{91}$$

where $\mathbf{N}$ is the matrix of element shape functions and $\mathbf{p}_e$ the vector of element nodal degrees-of-freedom. The vector of the residual nodal forces for a particular element is then given by

$$\mathbf{P}_e = \int_{\Omega_e} \left[\mathbf{N}^T \bar{\boldsymbol{q}} - (\boldsymbol{\Psi}\boldsymbol{\Delta}\mathbf{N})^T \boldsymbol{\sigma} \right] d\zeta \ , \tag{92}$$

in which Ω_e is the element domain. The element tangent stiffness matrix is straightforwardly obtained with the help of (66), leading to

$$\mathbf{k}_e = \int_{\Omega_e} \left[(\boldsymbol{\Psi}\boldsymbol{\Delta}\mathbf{N})^T \boldsymbol{D} (\boldsymbol{\Psi}\boldsymbol{\Delta}\mathbf{N}) + (\boldsymbol{\Delta}\mathbf{N})^T \boldsymbol{G} (\boldsymbol{\Delta}\mathbf{N}) - \mathbf{N}^T \boldsymbol{L}\mathbf{N} \right] d\zeta \ . \tag{93}$$

Here it is important to remark that the linearization stated in (93) can be performed either before or after discretization. Assemblage of the global residual forces and of the global tangent stiffness may be done as usual by

$$\mathbf{R} = \sum_{e=1}^{N_e} \mathbf{A}_e^T \mathbf{P}_e \quad \text{and} \quad \mathbf{K} = \sum_{e=1}^{N_e} \mathbf{A}_e^T \mathbf{k}_e \mathbf{A}_e, \tag{94}$$

respectively, where $\mathbf{A}_e$ is the connectivity matrix relating the element nodal degrees-of-freedom $\mathbf{p}_e$ with the whole domain nodal degrees-of-freedom $\mathbf{r}$, i.e.

$$\mathbf{p}_e = \mathbf{A}_e \mathbf{r} \ . \tag{95}$$

Equilibrium is then reached by vanishing the global residual forces,

$$\mathbf{R}(\mathbf{r}) = \boldsymbol{o}, \tag{96}$$

what can be iteratively solved by the Newton method for the free degrees-of-freedom.

5 Concluding Remarks

The geometrically-exact six-parameter rod model presented in [9, 13, 14] was extended to a multi-parameter formulation that allows for general cross-sectional in-plane changes and out-of-plane warping. Our approach defines cross-sectional stresses and strains in a consistent way, thus rendering a complete stress-resultant model. Large rotations are exactly treated in the context of finite elasticity, and very large strain problems can be realistically represented since the cross-sectional changes are incorporated within the rod kinematics. Remarkably, no additional geometric terms regarding these changes need to be included in the tangent bilinear form. Three-dimensional finite strain constitutive equations can be directly employed, with no spurious stiffening or approximations. The present assumptions allow a consistent basis for the proper representation of profile (distortional) deformations, typical of cold-formed thin-walled rod structures, and we believe this is one of the main features of our formulation.

Acknowledgments: Fellowship funding from FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) and CNPq (Conselho Nacional de Pesquisa), together with the material support and stimulating discussions in IBNM (Institut für Baumechanik und Numerische Mechanik) of the University of Hannover, are gratefully acknowledged in this work.

References

[1] E.M.B. Campello. Nonlinear analysis of cold formed metal structures (in portuguese: Análise não-linear de perfis metálicos conformados a frio). Master of Science Dissertation. Escola Politécnica da Universidade de São Paulo, São Paulo, 2000.

[2] E.M.B. Campello, P.M. Pimenta, and P. Wriggers. A fully nonlinear multi-parameter rod model incorporating profile deformation and general cross-sectional warping for the analysis of thin-walled structures. (to appear).

[3] A. Cardona and M. Gerardin. A beam finite element non-linear theory with finite rotations. Int. J. Numer. Meths. Engrg, 26:2403–2438, 1988.

[4] P.J. Ciarlet. Mathematical elasticity. vol 1, North Holland, Amsterdam, 1988.

[5] F. Gruttmann, R. Sauer, and W. Wagner. A geometrical nonlinear eccentric 3d-beam element with arbitrary cross-sections. Comp. Meth. Appl. Mech. Eng, 160:383–400, 1998.

[6] F. Gruttmann and W. Wagner. Geometrical and material nonlinear three-dimensional beams – theory and numerics. In W. Wundrlich, editor, Proceedings of the ECCM '99, pages 1–13. Munich, Germany, 276, 1999.

[7] G. Jelenic and M. Saje. A kinematically exact space finite strain beam model – finite element formulation by generalized virtual work principle. Comp. Meth. Appl. Mech. Eng, 120:131–161, 1995.

[8] P.M. Pimenta. Finite rotations (in portuguese: "rotações finitas"). Technical report, BT/PEF-8714, Department of Structural and Foundation Engineering, Polytechnic School at University of Sao Paulo, 1987.

[9] P.M. Pimenta. On a geometrically-exact finite-strain rod model. In Proceedings of the 3^{rd} Pan-American Congress on Applied Mechanics, III PACAM, São Paulo, 1993.

[10] P.M. Pimenta. Geometrically exact analysis of initially curved rods. In Advances in Computational Techniques for Structural Engineering, pages 99–108, vol.1, Edinburgh, U.K., 1996.

[11] P.M. Pimenta and E.M.B. Campello. Geometrically nonlinear analysis of thin-walled space frames. In II ECCM, Proceedings of the Second European Conference on Computational Mechanics, Cracow, Poland, 2001.

[12] P.M. Pimenta, E.M.B. Campello, and P. Wriggers. A fully nonlinear multi-parameter shell model with thickness variation and a triangular shell finite element. Computational Mechanics, 2003 (to appear).

[13] P.M. Pimenta and T. Yojo. Geometrically-exact analysis of spatial frames. Applied Mechanics Reviews, 46(11):118–128, ASME, New York, 1993.

[14] P.M. Pimenta and T. Yojo. Geometrically exact analysis of spatial frames with consideration of torsion warping. In XIV CILAMCE (Proceedings of the XIV Iberian-Latin-American congress on computational methods in engineering), pages 21–30, IPT, São Paulo, 1993.

[15] J.C. Simo. A finite strain beam formulation. The three-dimensional dynamic problem. Part I. Computer Methods in Applied Mechanics and Engineering, 49:55–70, 1985.

[16] J.C. Simo and L. Vu-Quoc. A three-dimesional finite strain rod model. Part II: Computational aspects. Computer Methods in Applied Mechanics and Engineering, 58:79–116, 1986.

[17] J.C. Simo and L. Vu-Quoc. A geometrically exact rod model incorporating shear and torsion-warping deformation. Int. J. Solid Structures, 27:371–393, 1991.

[18] B.Z. Vlassov. Pièces longues en voiles minces. Éditions Eyrolles, Paris, 1962.

Appendix

A Tangent operator $\bar{\boldsymbol{G}}$

Tangent operator $\bar{\boldsymbol{G}}$ in (67) has the following structure

$$\bar{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{G}^{u'\theta} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{G}^{\theta'\theta} \\ \boldsymbol{G}^{\theta u'} & \boldsymbol{G}^{\theta\theta'} & \boldsymbol{G}^{\theta\theta} \end{bmatrix} . \tag{A.1}$$

In order to derive the elements of (A.1), the following result is obtained by differentiation

$$\frac{\partial\left(\boldsymbol{\Gamma}^T\boldsymbol{t}\right)}{\partial\boldsymbol{\theta}} = \boldsymbol{\Gamma}^T\frac{\partial\boldsymbol{t}}{\partial\boldsymbol{\theta}} + \boldsymbol{V}\left(\boldsymbol{\theta},\boldsymbol{t}\right) , \tag{A.2}$$

where $\boldsymbol{t}$ is a generic vector and

$$\begin{aligned}\boldsymbol{V}\left(\boldsymbol{\theta},\boldsymbol{t}\right) = {} & h_2\left(\theta\right)\boldsymbol{T} + h_3\left(\theta\right)\left(\boldsymbol{T}\boldsymbol{\Theta} - 2\boldsymbol{\Theta}\boldsymbol{T}\right) + \\ & -h_4\left(\theta\right)\left(\boldsymbol{\Theta}\boldsymbol{t}\otimes\boldsymbol{\theta}\right) + h_5\left(\theta\right)\left(\boldsymbol{\Theta}^2\boldsymbol{t}\otimes\boldsymbol{\theta}\right) ,\end{aligned} \tag{A.3}$$

with $\boldsymbol{T} = \operatorname{skew}(\boldsymbol{t})$. One can show that property

$$\boldsymbol{V}\left(\boldsymbol{\theta},t\right) = \boldsymbol{V}^T\left(\boldsymbol{\theta},\boldsymbol{t}\right) + \boldsymbol{\Gamma}^T\boldsymbol{T}\boldsymbol{\Gamma} \tag{A.4}$$

holds for $V(\boldsymbol{\theta}, \boldsymbol{t})$, and this is a crucial result in proving the symmetry of (A.1). With the aid of (A.2) and (A.3) it is possible to write

$$\begin{aligned}
\boldsymbol{G}^{u'\theta} &= \left(\boldsymbol{G}^{\theta u'}\right)^T = -\boldsymbol{N}\boldsymbol{\Gamma} , \\
\boldsymbol{G}^{\theta\theta'} &= \left(\boldsymbol{G}^{\theta'\theta}\right)^T = \boldsymbol{V}(\boldsymbol{\theta},\ \boldsymbol{m}) \quad \text{and} \\
\boldsymbol{G}^{\theta\theta} &= \left(\boldsymbol{G}^{\theta\theta}\right)^T = \boldsymbol{\Gamma}^T\boldsymbol{Z}'\boldsymbol{N}\boldsymbol{\Gamma} - \boldsymbol{V}(\boldsymbol{\theta},\ \boldsymbol{z}'\times\boldsymbol{n}) + \boldsymbol{V}'\left(\boldsymbol{\theta},\ \boldsymbol{\theta}',\ \boldsymbol{m}\right) - \boldsymbol{\Gamma}'^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\Gamma} ,
\end{aligned} \tag{A.5}$$

in which $\boldsymbol{N} = \mathrm{Skew}(\boldsymbol{n})$, $\boldsymbol{M} = Skew(\boldsymbol{m})$ and

$$\begin{aligned}
\boldsymbol{V}'\left(\boldsymbol{\theta},\ \boldsymbol{\theta}',\ \boldsymbol{m}\right) = {}& h_3(\theta)\left(\boldsymbol{M}\boldsymbol{\Theta}'_2\boldsymbol{\Theta}'\boldsymbol{M}\right) - h_4(\theta)\left(\boldsymbol{\Theta}'m\otimes\theta + \boldsymbol{\Theta}m\otimes\theta_{,\alpha}\right) + \\
& + h_5(\theta)\left(\left(\boldsymbol{\Theta}'\boldsymbol{\Theta} + \boldsymbol{\Theta}\boldsymbol{\Theta}'\right)\boldsymbol{m}\otimes\boldsymbol{\theta} + \boldsymbol{\Theta}^2\boldsymbol{m}\otimes\boldsymbol{\theta}'\right) + \\
& + \left(\boldsymbol{\theta}\cdot\boldsymbol{\theta}'\right)\left[h_4(\theta)\boldsymbol{M} + h_5(\theta)\left(\boldsymbol{M}\boldsymbol{\Theta} - 2\boldsymbol{\Theta}\boldsymbol{M}\right)\right] + \\
& + \left(\boldsymbol{\theta}\cdot\boldsymbol{\theta}'\right)\left[-h_6(\theta)\left(\boldsymbol{\Theta}\boldsymbol{m}\otimes\boldsymbol{\theta}\right) + h_7(\theta)\left(\boldsymbol{\Theta}^2\boldsymbol{m}\otimes\boldsymbol{\theta}\right)\right] .
\end{aligned} \tag{A.6}$$

Here the following trigonometric functions have been introduced

$$h_6(\theta) = \frac{h_3(\theta) - h_2(\theta) - 4h_4(\theta)}{\theta^2} \quad \text{and} \quad h_7(\theta) = \frac{h_4(\theta) - 5h_5(\theta)}{\theta^2} . \tag{A.7}$$

B Tangent operator $\boldsymbol{D}$

Tangent operator $\boldsymbol{D}$ in (67) has following structure

$$\boldsymbol{D} = \begin{bmatrix}
\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{\eta}^r} & \frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{\kappa}^r} & \frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{p}} & \frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{p}'} & \frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{r}'} \\
\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{\eta}^r} & \frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{\kappa}^r} & \frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{p}} & \frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{p}'} & \frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{r}} \\
\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\eta}^r} & \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\kappa}^r} & \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{p}} & \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{p}'} & \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{r}'} \\
\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\eta}^r} & \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\kappa}^r} & \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{p}} & \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{p}'} & \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{r}'} \\
\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\eta}^r} & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\kappa}^r} & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{p}} & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{p}'} & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{r}'} \\
\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\eta}^r} & \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\kappa}^r} & \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{p}} & \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{p}'} & \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{r}'}
\end{bmatrix} . \tag{B.1}$$

The elements of (B.1) are displayed next.

$$
\begin{aligned}
&\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{\eta}^r} = \int_A \boldsymbol{C}_{33} dA \,, \\
&\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{\kappa}^r} = -\int_A \boldsymbol{C}_{33} \boldsymbol{Y}^r dA \,, \\
&\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{p}} = \int_A \left[\boldsymbol{C}_{3\alpha} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}_{,\alpha} \right) + \left(\boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \right) \otimes \boldsymbol{\psi} \right] dA \,, \\
&\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{p}'} = \int_A \boldsymbol{C}_{33} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi} \right) dA \,, \\
&\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{r}} = \int_A \left[\left(\boldsymbol{C}_{3\gamma} \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_{\beta,\gamma} + \left(C_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \right) \otimes \boldsymbol{\phi}_\beta \right] dA \,, \\
&\frac{\partial \boldsymbol{n}^r}{\partial \boldsymbol{r}'} = \int_A \left(\boldsymbol{C}_{33} \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta dA \,, \\
&\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{\eta}^r} = \int_A \boldsymbol{Y}^r \boldsymbol{C}_{33} dA \,, \\
&\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{\kappa}^r} = -\int_A \boldsymbol{Y}^r \boldsymbol{C}_{33} \boldsymbol{Y}^r dA \,, \\
&\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{p}} = \int_A \left[\left(\boldsymbol{Y}^r \boldsymbol{C}_{3\alpha} \boldsymbol{e}_3^r \right) \otimes \boldsymbol{\psi}_{,\alpha} + \left(\boldsymbol{Y}^r \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \right) \otimes \boldsymbol{\psi} + \left(\boldsymbol{e}_3^r \times \boldsymbol{\tau}_3^r \right) \otimes \boldsymbol{\psi} \right] dA \,, \\
&\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{p}'} = \int_A \left(\boldsymbol{Y}^r \boldsymbol{C}_{33} \boldsymbol{e}_3^r \right) \otimes \psi dA \,, \\
&\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{r}} = \int_A \left[\left(\boldsymbol{Y}^r \boldsymbol{C}_{3\gamma} \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_{\beta,\gamma} + \left(\boldsymbol{Y}^r \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \right) \otimes \boldsymbol{\phi}_\beta + \left(\boldsymbol{e}_\beta^r \times \boldsymbol{\kappa}_3^r \right) \otimes \boldsymbol{\phi}_\beta \right] dA \,, \\
&\frac{\partial \boldsymbol{m}^r}{\partial \boldsymbol{r}'} = \int_A \left(\boldsymbol{Y}^r \boldsymbol{C}_{33} \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta dA \,, \\
&\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\eta}^r} = \int_A \left[\boldsymbol{\psi}_{,\alpha} \otimes \left(\boldsymbol{C}_{\alpha 3}^T \boldsymbol{e}_3^r \right) + \boldsymbol{\psi} \otimes \left(\boldsymbol{C}_{33}^T \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \right) \right] dA \,, \\
&\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\kappa}^r} = \int_A \left[\boldsymbol{\psi}_{,\alpha} \otimes \left(\boldsymbol{Y}^r \boldsymbol{C}_{\alpha 3}^T \boldsymbol{e}_3^r \right) + \boldsymbol{\psi} \otimes \left(\boldsymbol{Y}^r \boldsymbol{C}_{33}^T \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \right) + \boldsymbol{\psi} \otimes \left(\boldsymbol{e}_3^r \times \boldsymbol{\kappa}_3^r \right) \right] dA \,, \\
&\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{p}} = \int_A \left[\left(\boldsymbol{e}_3^r \cdot \boldsymbol{C}_{\alpha\beta} \boldsymbol{e}_3^r \right) \left(\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{\psi}_{,\beta} \right) + \left(\boldsymbol{e}_3^r \cdot \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \right) \left(\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{\psi} \right) \right] dA + \\
&\qquad + \int_A \boldsymbol{\psi} \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \left[\boldsymbol{C}_{3\alpha} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}_{,\alpha} \right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \otimes \boldsymbol{\psi} \right] dA \,, \\
&\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{p}'} = \int_A \left[\left(\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{e}_3^r \right) \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi} \right) + \boldsymbol{\psi} \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \boldsymbol{C}_{33} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi} \right) \right] dA \,, \\
&\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{r}} = \int_A \left(\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{e}_3^r \right) \left[\boldsymbol{C}_{\alpha\gamma} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_{\beta,\gamma} \right) + \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta \right] dA + \\
&\qquad + \int_A \boldsymbol{\psi} \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \left[\boldsymbol{C}_{3\gamma} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_{\beta,\gamma} \right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta \right] dA \,, \\
&\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{r}'} = \int_A \left[\left(\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{e}_3^r \right) \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_\beta \right) + \boldsymbol{\psi} \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \boldsymbol{C}_{33} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_\beta \right) \right] dA \,, \\
&\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\eta}^r} = \int_A \left(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r \right) \boldsymbol{C}_{33} dA \,, \\
&\frac{\boldsymbol{\alpha}}{\partial \boldsymbol{\kappa}^r} = -\int_A \left(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r \right) \boldsymbol{C}_{33} \boldsymbol{Y}^r dA \,, \\
&\frac{\boldsymbol{\alpha}}{\partial \boldsymbol{p}} = \int_A \left(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r \right) \left[\boldsymbol{C}_{3\alpha} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}_{,\alpha} \right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \otimes \boldsymbol{\psi} \right] dA \,, \\
&\frac{\boldsymbol{\alpha}}{\partial \boldsymbol{p}'} = \int_A \left(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r \right) \boldsymbol{C}_{33} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi} \right) dA \,, \\
&\frac{\boldsymbol{\alpha}}{\partial \boldsymbol{r}} = \int_A \left(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r \right) \left[\boldsymbol{C}_{3\gamma} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_{\beta,\gamma} \right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta \right] dA \,, \\
&\frac{\boldsymbol{\alpha}}{\partial \boldsymbol{r}'} = \int_A \left(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r \right) \boldsymbol{C}_{33} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_\beta \right) dA \,, \\
&\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\eta}^r} = \int_A \left[\left(\boldsymbol{\phi}_{\delta,\alpha} \otimes \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{\alpha 3} + \boldsymbol{\phi}_\delta \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{33} \right] dA \,, \\
&\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\kappa}^r} = \int_A \left[- \left(\boldsymbol{\phi}_{\delta,\alpha} \otimes \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{\alpha 3} \boldsymbol{Y}^r - \boldsymbol{\phi}_\delta \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{33} \boldsymbol{Y}^r + \boldsymbol{\phi}_\beta \otimes \left(\boldsymbol{e}_\beta^r \times \boldsymbol{\kappa}_3^r \right) \right] dA \,, \\
&\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{p}} = \int_A \left(\boldsymbol{\phi}_{\delta,\alpha} \otimes \boldsymbol{e}_\delta^r \right) \left[\boldsymbol{C}_{\alpha\beta} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}_{,\beta} \right) + \boldsymbol{C}_{\alpha 3} \left(\left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \otimes \boldsymbol{\psi} \right) \right] dA + \\
&\qquad + \int_A \boldsymbol{\phi}_\delta \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\delta^r \right) \left[\boldsymbol{C}_{3\alpha} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi}_{,\alpha} \right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r \right) \otimes \boldsymbol{\psi} \right] dA \,, \\
&\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{p}'} = \int_A \left[\left(\boldsymbol{\phi}_{\delta,\alpha} \otimes \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi} \right) + \boldsymbol{\phi}_\delta \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{33} \left(\boldsymbol{e}_3^r \otimes \boldsymbol{\psi} \right) \right] dA \,, \\
&\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{r}} = \int_A \left(\boldsymbol{\phi}_{\delta,\alpha} \otimes \boldsymbol{e}_\delta^r \right) \left[\boldsymbol{C}_{\alpha\gamma} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_{\beta,\gamma} \right) + \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta \right] dA + \\
&\qquad + \int_A \boldsymbol{\phi}_\delta \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\delta^r \right) \left[\boldsymbol{C}_{3\gamma} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_{\beta,\gamma} \right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r \right) \otimes \boldsymbol{\phi}_\beta \right] dA \,, \\
&\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{r}'} = \int_A \left[\left(\boldsymbol{\phi}_{\delta,\alpha} \otimes \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{\alpha 3} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_\beta \right) + \boldsymbol{\phi}_\delta \otimes \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}_\delta^r \right) \boldsymbol{C}_{33} \left(\boldsymbol{e}_\beta^r \otimes \boldsymbol{\phi}_\beta \right) \right] dA \,,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
&\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{\eta}^r} = \int_A \left(\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}^r_\beta\right) \boldsymbol{C}_{33} dA \ , \\
&\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{\kappa}^r} = -\int_A \left(\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}^r_\beta\right) \boldsymbol{C}_{33}\, \boldsymbol{Y}^r dA \ , \\
&\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{p}} = \int_A \left(\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}^r_\beta\right) \left[\boldsymbol{C}_{3\alpha} \left(\boldsymbol{e}^r_3 \otimes \boldsymbol{\psi}_{,\alpha}\right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}^r_3\right) \otimes \boldsymbol{\psi}\right] dA \ , \\
&\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{p}'} = \int_A \left(\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}^r_\beta\right) \boldsymbol{C}_{33} \left(\boldsymbol{e}^r_3 \otimes \boldsymbol{\psi}\right) dA \ , \\
&\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{r}} = \int_A \left(\boldsymbol{\phi}_\alpha \otimes \boldsymbol{e}^r_\alpha\right) \left[\boldsymbol{C}_{3\gamma} \left(\boldsymbol{e}^r_\beta \otimes \boldsymbol{\phi}_{\beta,\gamma}\right) + \boldsymbol{C}_{33} \left(\boldsymbol{\kappa}^r \times \boldsymbol{e}^r_\beta\right) \otimes \boldsymbol{\phi}_\beta\right] dA \quad \text{and} \\
&\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{r}'} = \int_A \left(\boldsymbol{\phi}_\alpha \otimes \boldsymbol{e}^r_\alpha\right) \boldsymbol{C}_{33} \left(\boldsymbol{e}^r_\beta \otimes \boldsymbol{\phi}_\beta\right) dA \ .
\end{aligned}
\tag{B.3}
$$

C Tangent operator $\boldsymbol{L}$

Tangent operator $\boldsymbol{L}$ has following structure

$$
L = \begin{bmatrix}
\frac{\partial\bar{\boldsymbol{n}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{n}}}{\partial\boldsymbol{\theta}} & \frac{\partial\bar{\boldsymbol{n}}}{\partial\boldsymbol{p}} & \frac{\partial\bar{\boldsymbol{n}}}{\partial\boldsymbol{r}} \\
\frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{p}} & \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{r}} \\
\frac{\partial\bar{\boldsymbol{\alpha}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{\alpha}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{\alpha}}}{\partial\boldsymbol{p}} & \frac{\partial\bar{\boldsymbol{\alpha}}}{\partial\boldsymbol{r}} \\
\frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{u}} & \frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{p}} & \frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{r}}
\end{bmatrix} \ . \tag{C.1}
$$

For instance, semi-tangential external moments are conservative moments characterized by the following time derivative

$$
\dot{\bar{\boldsymbol{m}}} = \frac{1}{2}\boldsymbol{\omega} \times \bar{\boldsymbol{m}} \ . \tag{C.2}
$$

For this type of loading the only nonzero element of $\boldsymbol{L}$ is

$$
\frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{\theta}} = \operatorname{Sym}\left(\boldsymbol{V}\left(\boldsymbol{\theta}, \bar{\boldsymbol{m}}\right)\right) \ . \tag{C.3}
$$

In contrast, for a constant eccentric force $\bar{\boldsymbol{n}}$ whose moment is $\bar{\boldsymbol{m}} = \boldsymbol{s} \times \bar{\boldsymbol{n}}$ (with $\boldsymbol{s}$ as the eccentricity vector), this tensor is given by

$$
\frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{\theta}} = \boldsymbol{\Gamma}^T \operatorname{Sym}\left(\boldsymbol{S}\bar{\boldsymbol{N}}\right) \boldsymbol{\Gamma} + \operatorname{Sym}\left(\boldsymbol{V}\left(\boldsymbol{\theta}, \bar{\boldsymbol{m}}\right)\right) \ , \tag{C.4}
$$

where $\boldsymbol{S} = \operatorname{Skew}(\boldsymbol{s})$ and $\bar{\boldsymbol{N}} = \operatorname{Skew}(\bar{\boldsymbol{n}})$.