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Analytical study on the vibration frequencies of tapered beams

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Abstract

A vast amount of published work can be found in the field of beam vibrations dealing with analytical and numerical techniques. This paper deals with analysis of the nonlinear free vibrations of beams. The problem considered represents the governing equation of the nonlinear, large amplitude free vibrations of tapered beams. A new implementation of the ancient Chinese method called the Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) are presented to obtain natural frequency and corresponding displacement of tapered beams. The effect of vibration amplitude on the non-linear frequency is discussed. In the end to illustrate the effectiveness and convenience of the MMA and HPM, the obtained results are compared with the exact ones and shown in graphs and in tables. Those approaches are very effective and simple and with only one iteration leads to high accuracy of the solutions. It is predicted that those methods can be found wide application in engineering problems, as indicated in this paper.

Keywords

large amplitude free vibrations, analytical solution, tapered beam, Homotopy Perturbation Method (HPM), Max-Min Approach (MMA).

1 INTRODUCTION

Analyzing the nonlinear vibration of beams is one of the important issues in structural engineering. Applications such as high-rise buildings, long-span bridges, aerospace vehicles have necessitated the study of their dynamic behavior at large amplitudes. Many researchers have studied tapered beams, which are very important for the design of many engineering structures. The non-linear vibration of beams is governed by a non-linear partial-differential equation in space and time. Generally, finding an exact or close-form solution for nonlinear problems is very difficult. Goorman [7] is given the governing differential equation corresponding to fundamental vibration mode of a tapered beam. Evensen [6] studied on the nonlinear vibrations of beams with various boundary conditions by using the perturbation method. Pillai & Rao [24] considered the different types of solutions to the nonlinear equation of motion such as

Galerkin, harmonic balance method and simple harmonic oscillations. Singh *et al* [32] later studied the large-amplitude vibration problem of unsymmetrically laminated beams based on classical, first-order and higher-order formulations by using the numerical integration technique introduced earlier by Singh *et al* [31]. Qaisi [25] used an analytical method for determining the vibration modes of geometrically nonlinear beams under various edge conditions. Rehfield [27] proposed an approximate method for nonlinear vibration problems with material nonlinear effects for various boundary conditions. Sathyamoorthy [28] developed the work on finite element method for nonlinear beams under static and dynamic loads and classical methods for the analysis of beams with material, geometric and other types of nonlinearities. Raju *et al* [26] studied the large amplitude vibration problem of beams and plates using Rayleigh-Ritz method by incorporating the inplane deformation as well as inertia, which were absent in the earlier studies, and also by retaining the equivalent linearization function. Klein [13] used finite element approach and Rayleigh-Ritz for analyzing the vibration of the tapered beams. A dynamic discretization technique was applied to calculate the natural frequencies of a non-rotating double tapered beam based on both the Euler-Bernoulli and Timoshenko Beam Theories by Downs [5]. Sato [29] improved the Ritz method to study a linearly tapered beam with ends restrained elastically against rotation and subjected to an axial force. Lau [14] used the exact method for studying on the free vibration of tapered beam with end mass. The Green's function method in Laplace transform domain was used to study the vibration of general elastically restrained tapered beams by Lee *et al* [15] for obtaining the approximate fundamental solution by using a number of stepped beams to represent the tapered beam. Junior *et al* [21] proposed Galerkin Method and the Askey-Wiener scheme as solutions of the stochastic beam bending problem. Oni and Awodola considered [3] the Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. Boukhalfa and Hadjoui [4] analyzed the free vibration of an embarked rotating composite shaft using the hp- version of the FEM.

The main objectives of this study are to use analytical methods for analyzing the free vibration of the tapered beams. Finding an exact analytical solution for nonlinear equations is extremely difficult. Therefore, many analytical and numerical approaches have been investigated. The most useful methods for solving nonlinear equations are perturbation methods. They are not valid for strongly nonlinear equations and they have many shortcomings. Many new techniques have appeared in the open literature to overcome the shortcomings of traditional analytical methods such as Variational Iteration [10, 18], Parameter-Expansion [19], Energy Balance [2, 12, 16, 22], Variational Approach [33], Iteration Perturbation [23], the Improved Amplitude-frequency Formulation [35] and differential transformation [20], etc. In this study, which is an extension of the authors' previous work [1], Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) were developed by J.H. He [8, 9] and investigated in different works [17, 30, 34], which have the following advantages, over above-mentioned methods:

1. MMA and HPM lead us to a very rapid convergence of the solution and they can be easily extended to other nonlinear oscillations.

2. In MMA and HPM, just one iteration leads us to high accuracy of solutions which are valid for a wide range of vibration amplitudes.

The Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) are used to find analytical solutions for this problem with the nonlinear governing differential equation. It is shown that the solutions are quickly convergent and their components can be simply calculated. The results of the MMA and HPM are compared with the exact one, it can be observed that MMA and HPM are accurate and require smaller computational effort. An excellent accuracy of the MMA and HPM results indicates that those methods can be used for problems in which the strong nonlinearities are taken into account.

2 TAPERED BEAM FORMULATION

In dimensionless form, the governing differential equation corresponding to fundamental vibration mode of a tapered beam is given by Goorman [7] and the schematic of a tapered beam represented by Fig. 1:

$$\left(\frac{d^2u}{dt^2}\right) + \varepsilon_1\left(u^2\left(\frac{d^2u}{dt^2}\right) + u\left(\frac{du}{dt}\right)^2\right) + u + \varepsilon_2 u^3 = 0 \tag{1}$$

Where u is displacement and ε_1 and ε_2 are arbitrary constants. Subject to the following initial conditions:

$$u(0) = A, \qquad \frac{du(0)}{dt} = 0 \tag{2}$$

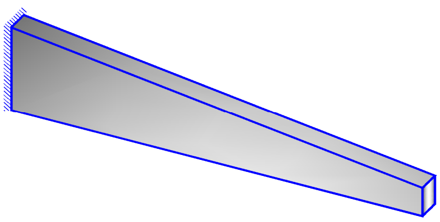

Figure 1 Schematic representation of a tapered beam.

3 OVERVIEW OF HE'S MAX-MIN APPROACH METHOD

We consider a generalized nonlinear oscillator in the form;

$$u'' + u\,f(u) = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{3}$$

Where $f(u)$ is a non-negative function of u. According to the idea of the max-min method, we choose a trial-function in the form;

$$u(t) = A\cos(\omega t), \tag{4}$$

Where ω the unknown frequency to be further is determined.
Observe that the square of frequency, ω^2, is never less than that in the solution

$$u_1(t) = A\cos(\sqrt{f_{\min}}t), \tag{5}$$

Of the following linear oscillator

$$u'' + u f_{\min} = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{6}$$

Where $f_{\min}$ is the minimum value of the function $f(u)$.
In addition, ω^2 never exceeds the square of frequency of the solution

$$u_1(t) = A\cos(\sqrt{f_{\min}}t), \tag{7}$$

Of the following oscillator

$$u'' + u f_{\min} = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{8}$$

Where $f_{\max}$ is the maximum value of the function $f(u)$.
Hence, it follows that

$$\frac{f_{\min}}{1} < \omega^2 < \frac{f_{\max}}{1}. \tag{9}$$

According to the Chentian interpolation [8, 11], we obtain

$$\omega^2 = \frac{m f_{\min} + n f_{\max}}{m + n}, \tag{10}$$

Or

$$\omega^2 = \frac{f_{\min} + k f_{\max}}{1 + k}, \tag{11}$$

Where m and n are weighting factors, $k = n/m$. So the solution of Eq. (3) can be expressed as

$$u(t) = A\cos\sqrt{\frac{f_{\min} + k f_{\max}}{1 + k}}t, \tag{12}$$

The value of k can be approximately determined by various approximate methods [22, 23, 33]. Among others, hereby we use the residual method [11]. Substituting Eq. (12) into Eq. (3) results in the following residual:

$$R(t; k) = -\omega^2 A\cos(\omega t) + (A\cos(\omega t)) \cdot f\,(A\cos(\omega t)) \tag{13}$$

Where $\omega = \sqrt{\frac{f_{\min}+kf_{\max}}{1+k}}$

If, by chance, Eq. (12) is the exact solution, then $R(t;k)$ is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^T R(t;k)\cos\sqrt{\frac{f_{\min}+kf_{\max}}{1+k}}tdt = 0, \tag{14}$$

Where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{f_{\max} - f_{\min}}{1 - \sqrt{\frac{A}{\pi}\int_0^\pi \cos^2 x.f\left(A\cos x\right)dx}}. \tag{15}$$

Substituting the above equation into Eq. (12), we obtain the approximate solution of Eq. (3).

4 OVERVIEW OF HOMOTOPY PERTURBATION METHOD (HPM)

To explain the basic idea of the HPM for solving nonlinear differential equations, one may consider the following nonlinear differential equation:

$$A\left(u\right) - f\left(r\right) = 0 \quad r \in \Omega \tag{16}$$

That is subjected to the following boundary condition:

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0 \quad r \in \Gamma \tag{17}$$

Where A is a general differential operator, B a boundary operator, *f(r)* is a known analytical function, Γ is the boundary of the solution domain (Ω), and $\partial u/\partial t$ denotes differentiation along the outwards normal to Γ. Generally, the operator A may be divided into two parts: a linear part L and a nonlinear part N. Therefore, Eq. (16) may be rewritten as follows:

$$L\left(x\right) + N\left(x\right) - f\left(r\right) = 0 \quad r \in \Omega \tag{18}$$

In cases where the nonlinear Eq. (16) includes no small parameter, one may construct the following homotopy equation

$$H\left(\nu, p\right) = \left(1 - p\right)\left[L\left(\nu\right) - L\left(x_0\right)\right] + p\left[A\left(\nu\right) - f\left(r\right)\right] = 0 \tag{19}$$

Where

$$\nu\left(r, p\right) : \Omega \times \left[0, 1\right] \to R \tag{20}$$

In Eq. (19), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. One may assume that solution of Eq. (19) may be written as a power series in p, as the following:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots \tag{21}$$

The homotopy parameter p is also used to expand the square of the unknown angular frequency ω as follows:

$$\omega_0 = \omega^2 - p\omega_1 - p^2\omega_2 - \cdots \tag{22}$$

Or

$$\omega^2 = \omega_0 + p\omega_1 + p^2\omega_2 + \cdots \tag{23}$$

where ω_0 is the coefficient of $u(r)$ in Eq. (16) and should be substituted by the right hand side of Eq. (23). Besides, $\omega_i\,(i = 1, 2, \ldots)$ are arbitrary parameters that have to be determined.

The best approximations for the solution and the angular frequency ω are

$$u = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots \tag{24}$$

$$\omega^2 = \omega_0 + \omega_1 + \omega_2 + \cdots \tag{25}$$

when Eq. (19) corresponds to Eq. (16) and Eq. (24) becomes the approximate solution of Eq. (16).

5 APPLICATIONS

5.1 Solution using MMA

We can re-write Eq. (1) in the following form

$$\left(\frac{d^2u}{dt^2}\right) + \left(\frac{1 + \varepsilon_1\left(\frac{du}{dt}\right)^2 + \varepsilon_2 u^2}{1 + \varepsilon_1 u^2}\right) u = 0 \tag{26}$$

We choose a trial-function in the form

$$u = A\cos(\omega t) \tag{27}$$

Where ω the frequency to be is determined.

By using the trial-function, the maximum and minimum values of ω^2 will be:

$$\omega_{\min} = \frac{1 + \varepsilon_1 A^2\omega^2}{1}, \quad \omega_{\max} = \frac{1 + \varepsilon_2 A^2}{1 + \varepsilon_1 A^2}. \tag{28}$$

So we can write:

$$\frac{1 + \varepsilon_1 A^2\omega^2}{1} < \omega^2 < \frac{1 + \varepsilon_2 A^2}{1 + \varepsilon_1 A^2} \tag{29}$$

According to the Chengtian's inequality [8, 11], we have

$$\omega^2 = \frac{m.\left(1+\varepsilon_1 A^2\omega^2+\varepsilon_2 A^2\right)+n.\left(1+\varepsilon_1 A^2\omega^2\right)}{m+n} = 1+\varepsilon_1 A^2\omega^2+k\varepsilon_2 A^2 \tag{30}$$

Where m and n are weighting factors, $k = n/m+n$. Therefore the frequency can be approximated as:

$$\omega = \sqrt{\frac{1+k\varepsilon_2 A^2}{1-\varepsilon_1 A^2}} \tag{31}$$

Its approximate solution reads

$$u = A\cos\sqrt{\frac{1+k\varepsilon_2 A^2}{1-\varepsilon_1 A^2}}t \tag{32}$$

In view of the approximate solution, Eq. (32), we re-write Eq.(**??**) in the form

$$\frac{d^2u}{dt^2}+\left(\frac{1+k\varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)u = \left(\frac{d^2u}{dt^2}\right)+\varepsilon_1\left(u^2\left(\frac{d^2u}{dt^2}\right)+u\left(\frac{du}{dt}\right)^2\right)+u+\varepsilon_2 u^3+\Psi \tag{33}$$

$$\Psi = \left(\frac{1+k\varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)u-\varepsilon_1 u^2\left(\frac{d^2u}{dt^2}\right)-\varepsilon_1 u\left(\frac{du}{dt}\right)^2-u-\varepsilon_2 u^3 \tag{34}$$

Substituting the trial function into Eq. (34) , and using Fourier expansion series, it is obvious that:

$$\begin{aligned}\Psi = &\left(\frac{1+k\varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)(A\cos\omega t)-\left(2\omega^2\varepsilon_1 A^2 cos^2(\omega t)-\varepsilon_1 A^2\omega^2-1-\varepsilon_2 A^2 cos^2(\omega t)\right)Acos(\omega t)\\ &= \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right] = b_1\cos(\omega t)+b_3\cos(3\omega t)+\ldots\approx b_1\cos(\omega t)\end{aligned} \tag{35}$$

For avoiding secular term we set $b_1 = 0$

$$\int_0^{T/4}\left(\left(\frac{1+k\varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)-\left(2\omega^2\varepsilon_1 A^2 cos^2(\omega t)-\varepsilon_1 A^2\omega^2-1-\varepsilon_2 A^2 cos^2(\omega t)\right)\right)Acos(\omega t)dt = 0 \tag{36}$$

Where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = -\frac{\left(\varepsilon_1\omega^2-\varepsilon_1^2 A^2\omega^2+3\varepsilon_1-2\varepsilon_2+2\varepsilon_2 A^2\varepsilon_1\right)}{3\varepsilon_2} \tag{37}$$

Substituting Eq. (37) into Eq. (32), yields

$$\omega = \frac{\sqrt{(3+\varepsilon_1 A^2)\,(2\varepsilon_2 A^2+3)}}{(3+\varepsilon_1 A^2)} \tag{38}$$

According to Eqs. (38) and (38), we can obtain the following approximate solution:

$$u(t) = A\cos\left(\frac{\sqrt{(3+\varepsilon_1 A^2)\,(2\varepsilon_2 A^2+3)}}{(3+\varepsilon_1 A^2)}t\right) \tag{39}$$

5.2 Solution using HPM

Eq. (1) can be rewritten as the following form:

$$\left(\frac{d^2u}{dt^2}\right) + u + p.\left[\varepsilon_1 u^2\left(\frac{d^2u}{dt^2}\right) + \varepsilon_1 u\left(\frac{du}{dt}\right)^2 + \varepsilon_2 u^3\right] = 0, \quad p \in [0,1]. \tag{40}$$

To explain the analytical solution, the solution u and the square of the unknown angular frequency ω are expanded as follows:

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{41}$$

$$1 = \omega^2 - p\omega_1 - p^2\omega_2 - \cdots \tag{42}$$

Substituting Eqs.(41) and (42) into Eq. (40) and equating the terms with identical powers of p, the following set of linear differential equations is obtained:

$$p^0: \ \left(\frac{d^2u_0}{dt^2}\right) + \omega^2 u_0 = 0 \tag{43}$$

$$p^1: \ \ddot{u}_1 + \omega^2 u_1 = \varepsilon_1 u_0\left(\frac{du_0}{dt}\right)^2 + \varepsilon_1\left(\frac{d^2u_0}{dt^2}\right)u_0^2 - \omega_1 u_0 + \varepsilon_2 u_0^3, \tag{44}$$

Solving Eq. (43) gives: $u_0 = A\cos\omega t$. Substituting u_0 into Eq. (44), yield:

$$\begin{aligned} p^1: \ \left(\frac{d^2u_1}{dt^2}\right) + \omega^2 u_1 =& \varepsilon_1\omega^2A^3cos(\omega t)sin^2(\omega t) - \varepsilon_1\omega^2A^3cos^3(\omega t) \\ &+\omega_1 Acos(\omega t) + \varepsilon_2A^3cos^3(\omega t) \end{aligned} \tag{45}$$

For achieving the secular term, we use Fourier expansion series as follows:

$$\begin{aligned} \Phi(\omega,t) =& \varepsilon_1\omega^2A^3\cos(\omega t)\sin^2(\omega t) - \varepsilon_1\omega^2A^3\cos^3(\omega t) + \omega_1 A\cos(\omega t) + \varepsilon_2A^3\cos^3(\omega t) \\ =& \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right] \\ =& b_1\cos(\omega t) + b_3\cos(3\omega t) + \cdots \\ \approx& \left(\int_0^{\pi/2}\Phi(\omega,t)d(\omega t)\right)\cos(\omega t) \\ =& \left[(A\omega_1 + \frac{1}{3}A^3\varepsilon_1\omega^2 - \frac{2}{3}A^3\varepsilon_2)\right]\cos(\omega t) \end{aligned} \tag{46}$$

Substituting Eq. (46) into Eq. (45) yields:

$$p^1: \ \ddot{u}_1 + \omega^2 u_1 = \left[\left(A\omega_1 + \frac{1}{3}A^3\varepsilon_1\omega^2 - \frac{2}{3}A^3\varepsilon_2\right)\right]\cos(\omega t) + \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right] \tag{47}$$

Avoiding secular term, gives:

$$\omega_1 = -\frac{1}{3}A^2(\varepsilon_1\omega^2 - 2\varepsilon_2) \tag{48}$$

From Eq. (42) and setting $p = 1$, we have:

$$1 = \omega^2 - \omega_1 \tag{49}$$

Comparing Eqs. (48) and (49), we can obtain:

$$\omega^2 = -\frac{1}{3}A^2(\varepsilon_1\omega^2 - 2\varepsilon_2) + 1 \tag{50}$$

Solving Eq. (50), gives:

$$\omega_{HPM} = \frac{\sqrt{(3+\varepsilon_1 A^2)\,(2\varepsilon_2 A^2 + 3)}}{(3+\varepsilon_1 A^2)} \tag{51}$$

6 RESULTS AND DISCUSSIONS

Comparisons with the analytical methods and the exact one are presented to illustrate and verify the accuracy of the Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM). The exact frequency ω_e for a dynamic system governed by Eq. (1) can be derived, as shown in Eq. (52), as follows:

$$\omega_{Exact} = 2\pi \Bigg/ 4\sqrt{2}A\int_0^{\pi/2} \frac{\sqrt{1+\varepsilon_1 A^2 cos^2 t}\, sint}{\sqrt{A^2\,(1-cos^2 t)\,(\varepsilon_2 A^2 cos^2 t + \varepsilon_2 A^2 + 2)}}dt \tag{52}$$

To demonstrate the accuracy of the MMA and HPM , the procedures explained in previous sections are applied to obtain natural frequency and corresponding displacement of tapered beams. A comparison of obtained results from the Max-Min Approach and Homotopy perturbation method and the exact one is tabulated in Table 1 for different parameters A, ε_1 and ε_2.

From Table 1, the relative error of the analytical approaches is 2.90861% for the first-order analytical approximations, for $A = 10$, $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 0.5$. To further illustrate and verify the accuracy of these approximate analytical approaches, a comparison of the time history oscillatory displacement response for tapered beams with exact solutions is presented in Figs. 2-4. From Figs. 2 and 3, the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. Fig. 4 presents the high accuracy of both

Table 1 Comparison of frequency corresponding to various parameters of system.

Constant parameters			Approximate solution	Exact solution	Relative error %
A	ε_1	ε_2	$\omega_{MMA=HPM}$	ω_{Exact}	$\left\vert\frac{\omega_{MMA=HPM}-\omega_{Ex}}{\omega_{Ex}}\right\vert$
0.5	0.1	0.5	1.03652	1.03924	0.26199
	0.5	1	1.05830	1.05727	0.09679
	5	10	1.37198	1.34555	1.96424
	10	50	2.25630	2.20503	2.32529
2	0.1	0.5	1.43486	1.44100	0.42665
	0.5	1	1.48323	1.44506	2.64192
	5	10	1.8996	1.85323	2.50516
	10	50	3.06138	3.0103	1.69512
10	0.1	0.5	2.81479	2.73523	2.90861
	0.5	1	1.95708	1.92710	1.55604
	5	10	1.99552	1.98950	0. 1842
	10	50	3.15801	3.15265	0.17001
20	0.1	0.5	3.06138	3.0103	1.69512
	0.5	1	1.98888	1.97683	0.60943
	5	10	1. 6369	1.99692	0.09776
	10	50	3.16121	3.15951	0.05356

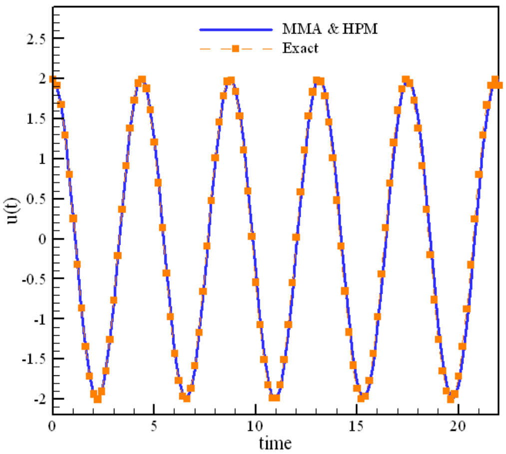

Figure 2 Comparison of analytical solutions of $u(t)$ based on t with the exact solution for $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.5$, $A = 2$.

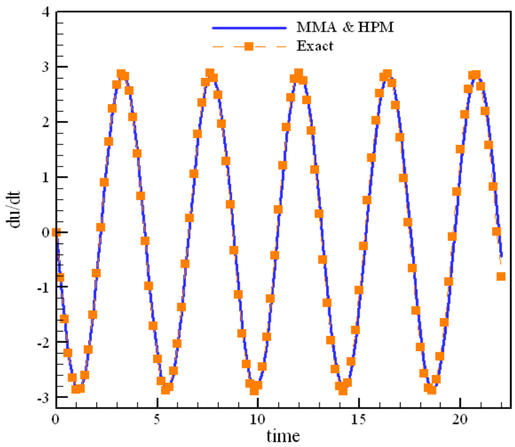

Figure 3 Comparison of analytical solutions of $u(t)$ based on t with the exact solution for $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.5$, $A = 2$.

approaches with the exact one for different values of amplitude, and shows the phase-space curves ($\dot{u}(t)$ versus $u(t)$ curve) of the Eqs. (51, 38) for amplitudes, $u(0) = 0.5, 1, 2$ and 3. It can be observed that the phase-space curve generated from MMA and HPM are close to that of the exact curve. The phase plot shows the behavior of the oscillator when $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.1$. It is periodic with a center at (0, 0). This situation also occurs in the unforced, undamped cubic Duffing oscillators.

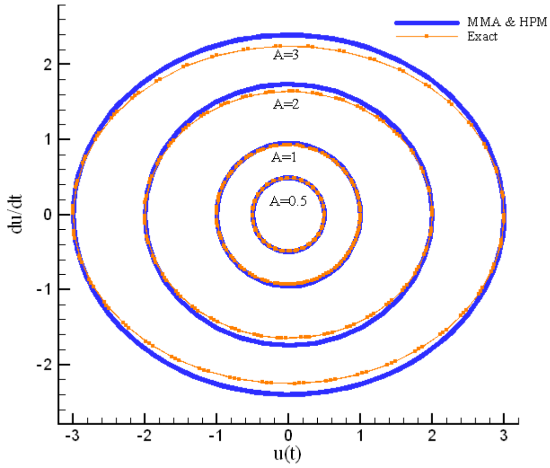

Figure 4 Comparison of analytical solutions of du/dt based on $u(t)$ with the exact solution for $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.1$.

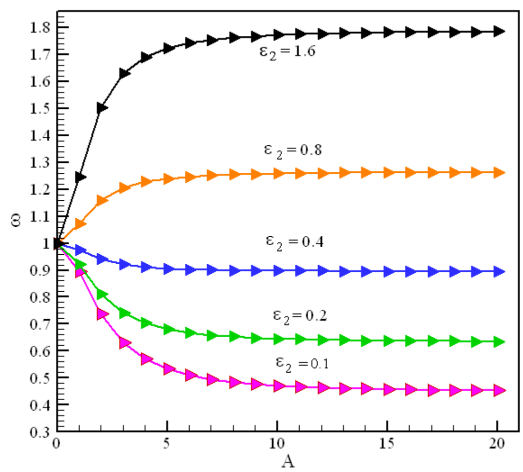

Figure 5 Comparison of frequency corresponding to various parameters of amplitude (A) and $\varepsilon_1 = 1$.

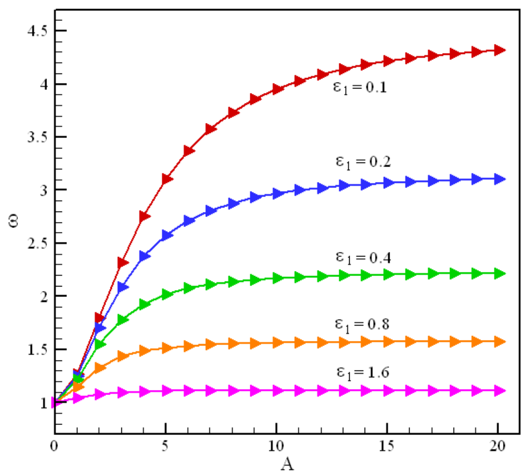

Figure 6 Comparison of frequency corresponding to various parameters of amplitude (A) and $\varepsilon_2 = 1$.

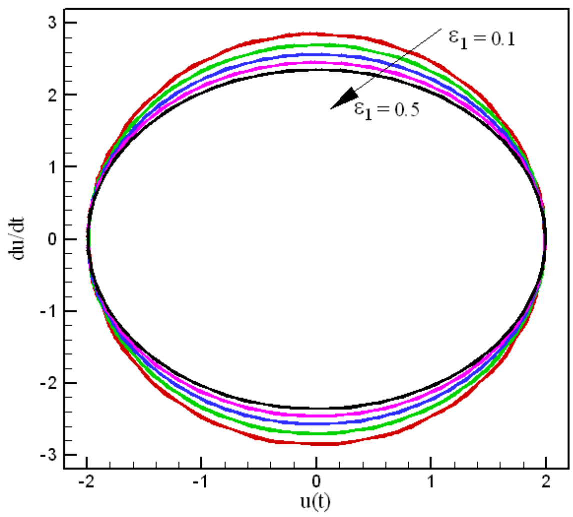

Figure 7 Phase plane, for $A = 2$, $\varepsilon_2 = 0.5$.

The effect of small parameters ε_1 and ε_2 on the frequency corresponding to various parameters of amplitude (A) has been studied in Figs. 5 and 6 for ε_1 and ε_2. Also, the phase plane for this problem obtained from MMA and HPM has been shown in Fig. 7. It is evident that MMA and HPM show excellent agreement with the numerical solution using the exact solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the MMA and HPM can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

7 CONCLUSIONS

In this paper, the MMA and HPM were employed to solve the governing equations of nonlinear oscillations of tapered beams. The analytical solutions yield a thoughtful and insightful understanding of the effect of system parameters and initial conditions. Also, Analytical solutions give a reference frame for the verification and validation of other numerical approaches. MMA and HPM are suitable not only for weak nonlinear problems, but also for strong nonlinear problems. The most significant feature of those methods is their excellent accuracy for the whole range of oscillation amplitude values. Also, it can be used to solve other conservative truly nonlinear oscillators with complex nonlinearities. The MMA and HPM solutions are quickly convergent and its components can be simply calculated. Also, compared to other analytical methods, it can be observed that the results of MMA and HPM require smaller computational effort and only the one iteration leads to accurate solutions. The successful implementation of the MMA and HPM for the large amplitude nonlinear oscillation problem considered in this paper further confirms the capability of those methods in solving nonlinear oscillation problems.

References

[1] M. Bayat, M. Shahidi, A. Barari, and G. Domairry. The approximate analysis of nonlinear behavior of structure under harmonic loading. *International Journal of the Physical Sciences*, 5(7):1074–1080, 2010.

[2] M. Bayat, M. Shahidi, A. Barari, and G. Domairry. Analytical evaluation of the nonlinear vibration of coupled oscillator systems. *Zeitschrift für Naturforschung A - A Journal of Physical Sciences*, 66(1-2):67–74, 2011.

[3] Abdelkrim Boukhalfa and Abdelhamid Hadjoui. Free vibration analysis of an embarked rotating composite shaft using the hp- version of the FEM. *Latin American Journal of Solids and Structures*, 7:105–141, 2010.

[4] Claudio Roberto Avila da Silva Junior, Andre Teolo Beck, and Edison da Rosa. Solution of the stochastic beam bending problem by Galerkin method and the Askey-Wiener scheme. *Latin American Journal of Solids and Structures*, 6:51–72, 2009.

[5] B. Downs. Transverse vibrations of cantilever beam having unequal breadth and depth tapers. *ASME J. Appl. Mech.*, 44:737–742, 1977.

[6] D.A. Evensen. Nonlinear vibrations of beams with various boundary conditions. *AIAA J.*, 6:370–372, 1968.

[7] D.J. Goorman. Free vibrations of beams and shafts. *Appl. Mech., ASME*, 18:135–139, 1975.

[8] J. H. He. Max-Min approach to nonlinear oscillators. *International Journal of Nonlinear Sciences and Numerical Simulation*, 9(2):207–210, 2008.

[9] J.H. He. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 178:257–262, 1999.

[10] J.H. He. Variational iteration method-a kind of non-linear analytical technique: Some examples. *International Journal of Non-Linear Mechanics*, 34(4):699–708, 1999.

[11] J.H. He and H. Tang. Rebuild of king fang 40 bc musical scales by He's inequality. *Appl Mech Comput*, 168(2):909–914, 2005.

[12] N. Jamshidi and D.D. Ganji. Application of energy balance method and variational iteration method to an oscillation of a mass attached to a stretched elastic wire. *Current Applied Physics*, 10(A202):484–486, 2010.

[13] L. Klein. Transverse vibrations of non-uniform beam. *J. Sound Vibr.*, 37:491–505, 1974.

[14] J.H. Lau. Vibration frequencies of tapered bars with end mass. *ASME J. Appl. Mech.*, 51:179–181, 1984.

[15] S.Y. Lee and Y.H. Kuo. Exact solution for the analysis of general elastically restrained non-uniform beams. *ASME J. Appl. Mech.*, 59:205–212, 1992.

[16] I. Mehdipour, D.D. Ganji, and M. Mozaffari. Application of the energy balance method to nonlinear vibrating equations. *Current Applied Physics*, 10(1):104–112, 2010.

[17] Mo Miansari, Me Miansari, A. Barari, and G. Domairry. Analysis of blasius equation for flat-plate flow with infinite boundary value. *International Journal for Computational Methods in Engineering Science and Mechanics*, 11(2):79–84, 2010.

[18] S. T. Mohyud-Din. Modified variational iteration method for integro-differential equations and coupled systems. *Z. Naturforsch*, 65a:277–285, 2010.

[19] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor. Parameter-expansion techniques for strongly nonlinear oscillators. *Int. J. Nonlin. Sci. Num. Sim.*, 10(5):581–583, 2009.

[20] M. Omidvar, A. Barari, M. Momeni, and D.D. Ganji. New class of solutions for water infiltration problems in unsaturated soils. *Geomechanics and Geoengineering: An International Journal*, 5(2):127–135, 2010.

[21] S. T. Oni and T. O. Awodola. Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. *Latin American Journal of Solids and Structures*, 7:3–20, 2010.

[22] T. Ozis and A. Yildirim. Determination of the frequency-amplitude relation for a duffing-harmonic oscillator by the energy balance method. *Computers and Mathematics with Applications*, 54:1184–1187, 2007.

[23] Turgut Özis and Ahmet Yildirim. Generating the periodic solutions for forcing van der Pol oscillators by the iteration perturbation method. *Nonlinear Analysis Real world Applications*, 10:1984–1989, 2009.

[24] S.R.R. Pillai and B.N. Rao. On nonlinear free vibrations of simply supported uniform beams. *J.Sound Vib.*, 159:527–531, 1992.

[25] M.I. Qaisi. Application of the harmonic balance principle to the nonlinear free vibration of beams. *Appl. Acoust.*, 40:141–151, 1993.

[26] I.S. Raju, G.V. Rao, and K.K. Raju. Effect of longitudinal or inplane deformation and inertia on the large amplitude flexural vibrations of slender beams and thin plates. *J. Sound Vib.*, 49:415–422, 1976.

[27] L.W.A. Rehfield. Simple, approximate method for analysing nonlinear free vibrations of elastic structures. *J. Appl. Mech., ASME*, 42:509–511, 1975.

[28] M. Sathyamoorthy. Nonlinear analysis of beams, Part II: Finite-element methods. *Shock Vib. Dig.*, 14:7–18, 1982.

[29] K. Sato. Transverse vibrations of linearly tapered beams with ends restrained elastically against rotation subjected to axial force. *Int. J. Mech. Sci.*, 22:109–115, 1980.

[30] M. Shaban, D.D. Ganji, and M.M. Alipour. Nonlinear fluctuation, frequency and stability analyses in free vibration of circular sector oscillation systems. *Current Applied Physics*, 10(5):1267–1285, 2010.

[31] G. Singh, G.V. Rao, and N.G.R. Iyengar. Reinvestigation of large amplitude free vibrations of beams using finite elements. *J. Sound Vib.*, 143:351–355, 1990.

[32] G. Singh, G.V. Rao, and N.G.R. Iyengar. Analysis of the nonlinear vibrations of unsymmetrically laminated composite beams. *AIAA J.*, 29:1727–1735, 1991.

[33] L. Xu. Variational approach to solitons of nonlinear dispersive K(m,n) equations. *Chaos, Solitons & Fractals*, 37:137–143, 2008.

[34] Elçin Yusufoğlu. An improvement to homotopy perturbation method for solving system of linear equations. *Computers & Mathematics with Applications*, 58(11-12):2231–2235, 2009.

[35] Hu-Li Zhang. Application of he's frequency-amplitude formulation to an force nonlinear oscillator. *Int. J. Nonlin. Sci. Numer. Simul.*, 9(3):297–300, 2008.