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On the Free Vibration Analysis of Laminated Composite and Sandwich Plates: A Layerwise Finite Element Formulation

Abstract

In this paper, a new higher-order layerwise finite element model, developed earlier by the present authors for the static analysis of laminated composite and sandwich plates, is extended to study the free vibration behavior of multilayer sandwich plates. In the present layerwise model, a first-order displacement field is assumed for the face sheets, whereas a higher-order displacement field is assumed for the core. Thanks for enforcing the continuity of the interlaminar displacement, the number of variables is independent of the number of layers. In order to reduce the computation effort, a simply four-noded C^0 continuous isoparametric element is developed based on the proposed model. In order to study the free vibration, a consistent mass matrix is adopted in the present formulation. Several examples of laminated composite and sandwich plate with different material combinations, aspect ratios, boundary conditions, number of layers, geometry and ply orientations are considered for the analysis. The performance and reliability of the proposed formulation are demonstrated by comparing the author's results with those obtained using the three-dimensional elasticity theory, analytical solutions and other advanced finite element models. From the obtained results, it can be concluded that the proposed finite element model is simple and accurate in solving the free vibration problems of laminated composite and sandwich plates.

Keywords

Layerwise, Finite element, Laminated composite, Sandwich plates, Static, Free vibration.

Mohamed-Ouejdi Belarbi [a, *]
Abdelouahab Tati [a]
Houdayfa Ounis [a]
Abdelhak Khechai [b]

[a] Laboratoire de Génie Energétique et Matériaux, LGEM. Université de Biskra, B.P. 145, R.P. 07000, Biskra, Algeria. E-mail: mo.belarbi@univ-biskra.dz, abdeltati@yahoo.fr, houdafa.ounis@gmail.com
[b] Laboratoire de Recherche en Génie Civil, LRGC. Université de Biskra, B.P. 145, R.P. 07000, Biskra, Algeria. E-mail: abdelhak_khechai@hotmail.fr

* Corresponding author

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1 INTRODUCTION

Due to their low weight, high stiffness and high strength properties, the composites sandwich structures are widely used in various industrial areas e.g. civil constructions, marine industry, automobile and aerospace applications. A sandwich is a three layered construction, where a low weight thick core layer (e.g., rigid polyurethane foam) of adequate transverse shear rigidity, is sandwiched between two thin laminated composite face layers of higher rigidity (Pal and Niyogi 2009). Despite the many advantages of sandwich structures, their behavior becomes very complex due to the large variation of rigidity and material properties between the core and the face sheets. Therefore, the accuracy of the results for sandwich structures largely depends on the computational model adopted (Pandey and Pradyumna 2015).

In the literature, several two-dimensional theories and approaches have been used to study the behavior of composite sandwich structures. Starting by the simple classical laminated plate theory (CLPT), based on the Kirchhoff's assumptions, which does not includes the effect of the transverse shear deformation, the first-order shear deformation theory (FSDT), where the effect of the transverse shear deformation is considered (Reissner 1975, Whitney and Pagano 1970, Mindlin 1951, Yang et al. 1966), but this theory gives a state of constant shear stresses through the plate thickness, and the higher-order shear deformation theories (HSDT) where a better representation of transverse shear effect can be obtained (Lo et al. 1977, Manjunatha and Kant 1993, Reddy 1984, Lee and Kim 2013). Regarding the approaches used to model the behavior of composite structures, we distinguish the equivalent single layers (ESL) approach where all the laminate layers are referred to the same degrees-of-freedom (DOFs). The main advantages of ESL models are their inherent simplicity and their low computational cost, due to the small number of dependent variables. However, ESL approach fails to capture precisely the local behavior of sandwich structures. This drawback in ESL was circumvented by the Zig-Zag (ZZ) and layerwise (LW) approaches in which the variables are linked to specific layers (Belarbi and Tati 2015, Belarbi et al. 2016, Ćetković and Vuksanović 2009, Chakrabarti and Sheikh 2005, Kapuria and Nath 2013, Khalili et al. 2014, Khandelwal et al. 2013, Marjanović and Vuksanović 2014, Maturi et al. 2014, Sahoo and Singh 2014, Singh et al. 2011, Thai et al. 2016). For more details, the reader may refer to (Carrera 2002, Ha 1990, Khandan et al. 2012, Sayyad and Ghugal 2015).

In the last decades, the finite element method (FEM) has become established as a powerful method and as the most widely used method to analyze the complex behavior of composite sandwich structures (e.g., bending, vibration, buckling). This is due to the limitations in the analytical methods which are applicable only for certain geometry and boundary conditions (Kant and Swaminathan 2001, Mantari and Ore 2015, Maturi et al. 2014, Noor 1973, Plagianakos and Papadopoulos 2015, Srinivas and Rao 1970). Khatua and Cheung (1973) were one of the first to use the FEM in the analysis of this type of structures. They developed two triangular and rectangular elements to study the bending and free vibration of sandwich plates.

In the recent years, many researchers have investigated the dynamic response of laminated composite and sandwich plates using finite element models based on Zig-Zag theory. Chakrabarti and Sheikh (2004) developed a C^1 continuous six-noded triangular plate element with 48 DOFs for dynamic analyses of laminate-faced sandwich plate using higher-order zig-zag theory (HZZT). Afterwards, Kulkarni and Kapuria (2008) extended the application of a newly improved discrete Kirchoff quadrilateral element, based on third order zigzag theory to vibration analysis of composite and sandwich plates. Zhen and Wanji (2006, 2010) carried out free vibration analyses of laminated composite and sandwich plates using an eight-noded quadrilateral element based on a global-local higher order shear deformation theories (GLHSDT).

Chalak et al. (2013) developed a nine node finite element by taking out the nodal field variables in such a manner to overcome the problem of continuity requirement of the derivatives of transverse displacements for the free vibration analysis of laminated sandwich plate. An efficient nine-noded quadratic element with 99 DOFs is developed by Khandelwal et al. (2013) for accurately predicting natural frequencies of soft core sandwich plate. The formulation of this element is based on combined theory, HZZT and least square error (LSE) method. Therefore, the zig-zag plate theory presents good performance but it has a problem in its finite element implementation as it requires C^1 continuity of transverse displacement at the nodes which involves finite element implantation difficulties. Also, it requires high-order derivatives for displacement when obtaining transverse shear stresses from equilibrium equations (Pandey and Pradyumna 2015).

Recently, various authors have adopted the layerwise approach to assume separate displacement field expansions within each material layer. Lee and Fan (1996) described a new layerwise model using the FSDT for the face sheets whereas the displacement field at the core is expressed in terms of the two face sheets displacements. They used a nine-nodded isoparametric finite element for bending and free vibration of sandwich plates. On the other hand, a new three-dimensional (3D) layerwise finite element model with 240 DOFs has been developed by Nabarrete et al. (2003) for dynamic analyses of sandwich plates with laminated face sheets. They used the FSDT for the face sheets, whereas for the core a cubic and quadratic function for the in-plane and transverse, displacements, was adopted. In the same year, Desai et al. (2003) developed an eighteen-node layerwise mixed brick element with 108 DOFs for the free vibration analysis of multi-layered thick composite plates. Later, an eight nodes quadrilateral element having 136 DOFs was developed by Araújo et al. (2010) for the analysis of sandwich laminated plates with a viscoelastic core and laminated anisotropic face layers. The construction of this element is based on layerwise approach where the HSDT is used to model the core layer and the face sheets are modeled according to a FSDT. Elmalich and Rabinovitch (2012) have undertaken an analysis on the dynamics of sandwich plates, using a C0 four-node rectangular element. The formulation of this element is based on the use of a new layerwise model, where the FSDT is used for the face sheets and the HSDT is used for the core. In 2015, Pandey and Pradyumna (2015) presented a new higher-order layerwise plate formulation for static and free vibration analyses of laminated composite plates. A high-order displacement field is used for the middle layer and a first-order displacement field for top and bottom layers. The authors used an eight-noded isoparametric element containing 104 DOFs to model the plate. The performance of these layerwise models is good but it requires high computational effort as the number of variables dramatically increases with the number of layers.

According to the presented literature review on the sandwich models, we found that many authors used finite element models having large number of nodes and/or DOFs, especially those based on the layerwise approach. Therefore, the present work aims to propose a new C^0 layerwise model competitor to the majority of aforementioned finite element models, having a reduced number of nodes and DOFs. This new model is used for the calculation of natural frequencies of laminated composite and sandwich plates. Thanks for enforcing the continuity of the interlaminar displacement, the number of variables is independent of the number of layers. The numerical results obtained by developed model are compared favorably with those obtained via analytical solution and numerical results obtained by other models. The results obtained from this investigation will be useful for a more understanding of the bending and free vibration behavior of sandwich laminates plates.

2 MATHEMATICAL MODEL

Sandwich plate is a structure composed of three principal layers as shown in Figure.1, two face sheets (top-bottom) of thicknesses (h_t), (h_b) respectively, and a central layer named core of thickness (h_c) which is thicker than the previous ones. Total thickness (h) of the plate is the sum of these thicknesses. The plane (x, y) coordinate system coincides with mid-plane plate.

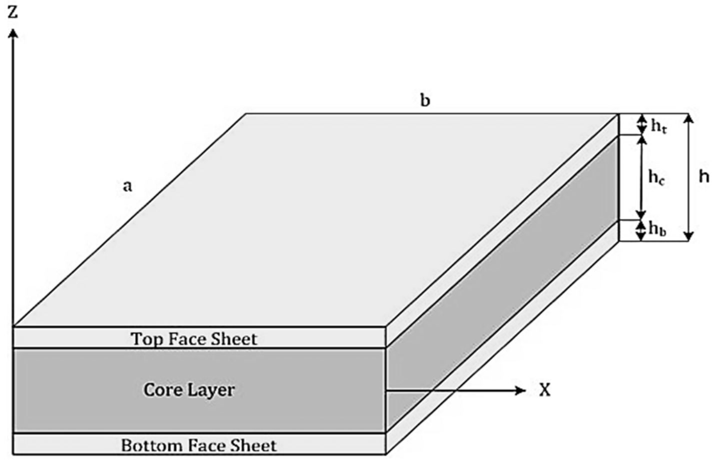

Figure 1: Geometry and notations of a sandwich plate.

2.1 Displacement Field

In the present model, the core layer is modeled using the HSDT. Hence, the displacement field is written as a third-order Taylor series expansion of the in-plane displacements in the thickness coordinate, and as a constant one for the transverse displacement:

$$\begin{aligned} u_c &= u_0 + Z\psi_x^c + Z^2\eta_x^c + Z^3\zeta_x^c \\ v_c &= v_0 + Z\psi_y^c + Z^2\eta_y^c + Z^3\zeta_y^c \\ w_c &= w_0 \end{aligned} \tag{1}$$

where u_0, v_0 and w_0 are respectively, in-plane and transverse displacement components at the mid-plane of the sandwich plate. ψ_x^c, ψ_y^c represent normal rotations about the x and y axis respectively. The parameters η_x^c, η_y^c, ζ_x^c and ζ_y^c are higher order terms.

For the two face sheets, the FSDT is adopted. The compatibility conditions as well as the displacement continuity at the interface (top face sheet-core- bottom face sheet), leads to the following improved displacement fields (Fig. 2):

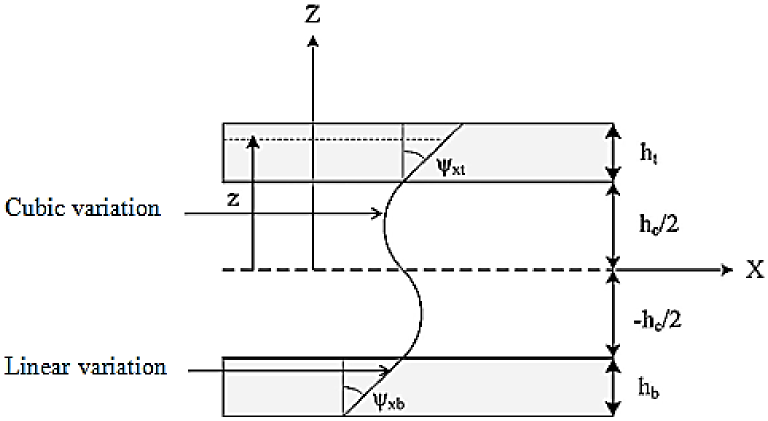

Figure 2: Representation of layerwise kinematics and coordinate system.

a) Top face sheet

$$\begin{aligned}
u_t &= u_c\left(\frac{h_c}{2}\right)+\left(Z-\frac{h_c}{2}\right)\psi_x^t\\
v_t &= v_c\left(\frac{h_c}{2}\right)+\left(Z-\frac{h_c}{2}\right)\psi_y^t\\
w_t &= w_0
\end{aligned} \tag{2}$$

where ψ_x^t and ψ_y^t are the rotations of the top face-sheet cross section about the y and x axis, respectively, and the displacement of the core for ($z=h_c/2$) is given by:

$$\begin{aligned}
u_c\left(\frac{h_c}{2}\right) &= u_0+\left(\frac{h_c}{2}\right)\psi_x^c+\left(\frac{h_c^2}{4}\right)\eta_x^c+\left(\frac{h_c^3}{8}\right)\zeta_x^c\\
v_c\left(\frac{h_c}{2}\right) &= v_0+\left(\frac{h_c}{2}\right)\psi_y^c+\left(\frac{h_c^2}{4}\right)\eta_y^c+\left(\frac{h_c^3}{8}\right)\zeta_y^c
\end{aligned} \tag{3}$$

The substitution of Eq. (3) in Eq. (2) led finally to the following expressions:

$$\begin{aligned}
u_t &= u_0+\left(\frac{h_c}{2}\right)\psi_x^c+\left(\frac{h_c^2}{4}\right)\eta_x^c+\left(\frac{h_c^3}{8}\right)\zeta_x^c+\left(Z-\frac{h_c}{2}\right)\psi_x^t\\
v_t &= v_0+\left(\frac{h_c}{2}\right)\psi_y^c+\left(\frac{h_c^2}{4}\right)\eta_y^c+\left(\frac{h_c^3}{8}\right)\zeta_y^c+\left(Z-\frac{h_c}{2}\right)\psi_y^t\\
w_t &= w_0
\end{aligned} \tag{4}$$

b) Bottom face sheet

According to Figure 2, the displacement components of the bottom face-sheet can be written as:

$$
\begin{aligned}
u_b &= u_c\left(-\frac{h_c}{2}\right)+\left(Z+\frac{h_c}{2}\right)\psi_x^b \\
v_b &= v_c\left(-\frac{h_c}{2}\right)+\left(Z+\frac{h_c}{2}\right)\psi_y^b \\
w_b &= w_0
\end{aligned}
\tag{5}
$$

where ψ_x^b and ψ_y^b are the rotations of the bottom face-sheet cross section about the y and x axis respectively, and the displacement of the core for ($z=-h_c/2$) is given by:

$$
\begin{aligned}
u_c\left(-\frac{h_c}{2}\right) &= u_0-\left(\frac{h_c}{2}\right)\psi_x^c+\left(\frac{h_c^2}{4}\right)\eta_x^c-\left(\frac{h_c^3}{8}\right)\zeta_x^c \\
v_c\left(-\frac{h_c}{2}\right) &= v_0-\left(\frac{h_c}{2}\right)\psi_y^c+\left(\frac{h_c^2}{4}\right)\eta_y^c-\left(\frac{h_c^3}{8}\right)\zeta_y^c
\end{aligned}
\tag{6}
$$

Substituting equation (6) into equation (5), leads to the following expression:

$$
\begin{aligned}
u_b &= u_0-\left(\frac{h_c}{2}\right)\psi_x^c+\left(\frac{h_c^2}{4}\right)\eta_x^c-\left(\frac{h_c^3}{8}\right)\zeta_x^c+\left(Z+\frac{h_c}{2}\right)\psi_x^b \\
v_b &= v_0-\left(\frac{h_c}{2}\right)\psi_y^c+\left(\frac{h_c^2}{4}\right)\eta_y^c-\left(\frac{h_c^3}{8}\right)\zeta_y^c+\left(Z+\frac{h_c}{2}\right)\psi_y^b \\
w_b &= w_0
\end{aligned}
\tag{7}
$$

2.2 Strain–Displacement Relations

The strain-displacement relations derived from the displacement model of Eqs. (1), (4) and (7) are given as follows:

For the core layer,

$$
\begin{aligned}
\varepsilon_{xx}^c &= \frac{\partial u_0}{\partial x}+Z\frac{\partial \psi_x^c}{\partial x}+Z^2\frac{\partial \eta_x^c}{\partial x}+Z^3\frac{\partial \zeta_x^c}{\partial x} \\
\varepsilon_{yy}^c &= \frac{\partial v_0}{\partial y}+Z\frac{\partial \psi_y^c}{\partial y}+Z^2\frac{\partial \eta_y^c}{\partial y}+Z^3\frac{\partial \zeta_y^c}{\partial y} \\
\gamma_{xy}^c &= \left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+Z\left(\frac{\partial \psi_x^c}{\partial y}+\frac{\partial \psi_y^c}{\partial x}\right)+Z^2\left(\frac{\partial \eta_x^c}{\partial y}+\frac{\partial \eta_y^c}{\partial x}\right)+Z^3\left(\frac{\partial \zeta_x^c}{\partial y}+\frac{\partial \zeta_y^c}{\partial x}\right) \\
\gamma_{yz}^c &= \psi_y^c+\frac{\partial w_0}{\partial y}+Z2\eta_y^c+Z^2 3\zeta_y^c \\
\gamma_{xz}^c &= \psi_x^c+\frac{\partial w_0}{\partial x}+Z2\eta_x^c+Z^2 3\zeta_x^c
\end{aligned}
\tag{8}
$$

For the top face sheet,

$$
\begin{aligned}
\varepsilon_{xx}^{t} &= \frac{\partial u_t}{\partial x} = \frac{\partial u_0}{\partial x} + \left(\frac{h_c}{2}\right)\frac{\partial \psi_x^c}{\partial x} + \left(\frac{h_c^2}{4}\right)\frac{\partial \eta_x^c}{\partial x} + \left(\frac{h_c^3}{8}\right)\frac{\partial \zeta_x^c}{\partial x} + (Z - \frac{h_c}{2})\frac{\partial \psi_x^t}{\partial x} \\
\varepsilon_{yy}^{t} &= \frac{\partial v_t}{\partial y} = \frac{\partial v_0}{\partial x} + \left(\frac{h_c}{2}\right)\frac{\partial \psi_y^c}{\partial y} + \left(\frac{h_c^2}{4}\right)\frac{\partial \eta_y^c}{\partial y} + \left(\frac{h_c^3}{8}\right)\frac{\partial \zeta_y^c}{\partial y} + (Z - \frac{h_c}{2})\frac{\partial \psi_y^t}{\partial y} \\
\gamma_{xy}^{t} &= \frac{\partial u_t}{\partial y} + \frac{\partial v_t}{\partial x} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \frac{h_c}{2}\left(\frac{\partial \psi_x^c}{\partial y} + \frac{\partial \psi_y^c}{\partial x}\right) + \frac{h_c^2}{4}\left(\frac{\partial \eta_x^c}{\partial y} + \frac{\partial \eta_y^c}{\partial x}\right) \\
&\quad + \frac{h_c^3}{8}\left(\frac{\partial \zeta_x^c}{\partial y} + \frac{\partial \zeta_y^c}{\partial x}\right) + (Z - \frac{h_c}{2})\left(\frac{\partial \psi_x^t}{\partial y} + \frac{\partial \psi_y^t}{\partial x}\right) \\
\gamma_{yz}^{t} &= \frac{\partial w_0}{\partial y} + \psi_y^t \\
\gamma_{xz}^{t} &= \frac{\partial w_0}{\partial x} + \psi_x^t
\end{aligned}
\tag{9}
$$

For the bottom face sheet,

$$
\begin{aligned}
\varepsilon_{xx}^{b} &= \frac{\partial u_b}{\partial x} = \frac{\partial u_0}{\partial x} - \left(\frac{h_c}{2}\right)\frac{\partial \psi_x^c}{\partial x} + \left(\frac{h_c^2}{4}\right) - \frac{\partial \eta_x^c}{\partial x}\left(\frac{h_c^3}{8}\right)\frac{\partial \zeta_x^c}{\partial x} + (Z + \frac{h_c}{2})\frac{\partial \psi_x^b}{\partial x} \\
\varepsilon_{yy}^{b} &= \frac{\partial v_b}{\partial y} = \frac{\partial v_0}{\partial y} - \left(\frac{h_c}{2}\right)\frac{\partial \psi_y^c}{\partial y} + \left(\frac{h_c^2}{4}\right) - \frac{\partial \eta_y^c}{\partial y}\left(\frac{h_c^3}{8}\right)\frac{\partial \zeta_y^c}{\partial y} + (Z + \frac{h_c}{2})\frac{\partial \psi_y^b}{\partial y} \\
\gamma_{xy}^{b} &= \frac{\partial u_b}{\partial y} + \frac{\partial v_b}{\partial x} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - \frac{h_c}{2}\left(\frac{\partial \psi_x^c}{\partial y} + \frac{\partial \psi_y^c}{\partial x}\right) + \frac{h_c^2}{4}\left(\frac{\partial \eta_x^c}{\partial y} + \frac{\partial \eta_y^c}{\partial x}\right) \\
&\quad - \frac{h_c^3}{8}\left(\frac{\partial \zeta_x^c}{\partial y} + \frac{\partial \zeta_y^c}{\partial x}\right) + (Z + \frac{h_c}{2})\left(\frac{\partial \psi_x^b}{\partial y} + \frac{\partial \psi_y^b}{\partial x}\right) \\
\gamma_{yz}^{b} &= \frac{\partial w_0}{\partial y} + \psi_y^b \\
\gamma_{xz}^{b} &= \frac{\partial w_0}{\partial x} + \psi_x^b
\end{aligned}
\tag{10}
$$

2.3 Constitutive Relationships

In this work, the two face sheets (top and bottom) are considered as laminated composite. Hence, the stress-strain relations for k^{th} layer in the global coordinate system are expressed as:

$$
\begin{Bmatrix} \sigma_{xx}^f \\ \sigma_{yy}^f \\ \tau_{yz}^f \\ \tau_{xz}^f \\ \sigma_{xy}^f \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & 0 & 0 & \overline{Q}_{26} \\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & \overline{Q}_{54} & \overline{Q}_{55} & 0 \\ \overline{Q}_{61} & \overline{Q}_{62} & 0 & 0 & \overline{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^f \\ \varepsilon_{yy}^f \\ \gamma_{yz}^f \\ \gamma_{xz}^f \\ \gamma_{xy}^f \end{Bmatrix}^{(k)} \qquad f = top, bottom \tag{11}
$$

The core is considered as an orthotropic composite material and its loads resultants are obtained by integration of the stresses through the thickness direction of laminated plate.

$$\begin{bmatrix} N_x & M_x & \overline{N}_x & \overline{M}_x \\ N_y & M_y & \overline{N}_y & \overline{M}_y \\ N_{xy} & M_{xy} & \overline{N}_{xy} & \overline{M}_{xy} \end{bmatrix} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \left(1, Z, Z^2, Z^3\right) dz$$
$$\begin{bmatrix} V_x & S_x & R_x \\ V_y & S_y & R_y \end{bmatrix} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \left(1, Z, Z^2\right) dz \tag{12}$$

where N, M denote membrane effort, bending moment, respectively, and $\overline{N}, \overline{M}$ denote higher order membrane and moment resultants, respectively. V is the shear resultant; S and R are the higher order shear resultant.

It is informative to relate the loads resultants of the core defined in Eq. (12) to the total strains in Eq. (8). From Eqs. (8) and (12), we obtain:

$$\begin{Bmatrix} N \\ M \\ \overline{N} \\ \overline{M} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [D] & [E] \\ [B] & [D] & [E] & [F] \\ [D] & [E] & [F] & [G] \\ [E] & [F] & [G] & [H] \end{bmatrix} \begin{Bmatrix} \varepsilon^{(0)} \\ \chi^{(1)} \\ \chi^{(2)} \\ \chi^{(3)} \end{Bmatrix} \tag{13a}$$

$$\begin{Bmatrix} V \\ S \\ R \end{Bmatrix} = \begin{bmatrix} \left[A^s\right] \left[B^s\right] \left[D^s\right] \\ \left[B^s\right] \left[D^s\right] \left[E^s\right] \\ \left[D^s\right] \left[E^s\right] \left[F^s\right] \end{bmatrix} \begin{Bmatrix} \gamma_s^{(0)} \\ \chi_s^{(1)} \\ \chi_s^{(2)} \end{Bmatrix} \tag{13b}$$

where $\left[\mathrm{A}_{ij}\right], \left[\mathrm{B}_{ij}\right]$, etc. are the elements of the reduced stiffness matrices of the core, defined by:

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}\right) = \int_{\frac{-h_c}{2}}^{\frac{h_c}{2}} \overline{Q}_{ij}^{c} \left(1, Z, Z^2, Z^3, Z^4, Z^5, Z^6\right) dz \qquad (i, j = 1, 2, 6)$$
$$\left(A_{ij}^s, B_{ij}^s, D_{ij}^s, E_{ij}^s, F_{ij}^s\right) = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \overline{Q}_{ij}^{c} \left(1, Z, Z^2, Z^3, Z^4\right) dz \qquad (i, j = 4, 5) \tag{14}$$

According to the FSDT, the constitutive equations for the two face sheets are given by:

$$\begin{Bmatrix} N^f \\ M^f \\ T^f \end{Bmatrix} = \begin{bmatrix} A^f & B^f & 0 \\ B^f & D^f & 0 \\ 0 & 0 & A_c^f \end{bmatrix} \begin{Bmatrix} \varepsilon_m^f \\ \varepsilon_f^f \\ \gamma_c^f \end{Bmatrix} \tag{15}$$

where the elements of reduced stiffness matrices of the face sheets are given as follows:

a. Top face sheet,

$$
\begin{aligned}
&\left(A_{ij}^{t}, B_{ij}^{t}, D_{ij}^{t}\right) = \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \overline{Q}_{ij}^{(k)} \left(1, Z, Z^2\right) dz = \sum_{k=1}^{n\ layer} \int_{h^k}^{h^{k+1}} \overline{Q}_{ij}^{(k)} \left(1, Z, Z^2\right) dz \qquad (i,j=1,2,6) \\
&\left(\overline{A}_{ij}^{t}\right) = \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \overline{Q}_{ij}^{(k)}\, dz = \sum_{k=1}^{n\ layer} \int_{h^k}^{h^{k+1}} \overline{Q}_{ij}^{(k)}\, dz \qquad (i,j=4,5)
\end{aligned}
\tag{16}
$$

b. Bottom face sheet,

$$
\begin{aligned}
&\left(A_{ij}^{b}, B_{ij}^{b}, D_{ij}^{b}\right) = \int_{-\left(\frac{h_c}{2}+h_b\right)}^{-\frac{h_c}{2}} \overline{Q}_{ij}^{(k)} \left(1, Z, Z^2\right) dz = \sum_{k=1}^{n\ layer} \int_{h^k}^{h^{k+1}} \overline{Q}_{ij}^{(k)} \left(1, Z, Z^2\right) dz \qquad (i,j=1,2,6) \\
&\left(\overline{A}_{ij}^{b}\right) = \int_{-\left(\frac{h_c}{2}+h_b\right)}^{-\frac{h_c}{2}} \overline{Q}_{ij}^{(k)}\, dz = \sum_{k=1}^{n\ layer} \int_{h^k}^{h^{k+1}} \overline{Q}_{ij}^{(k)}\, dz \qquad (i,j=4,5)
\end{aligned}
\tag{17}
$$

3 FINITE ELEMENT FORMULATION

In the present study, a four-node C^0 continuous quadrilateral element, named QSFT52 (Quadrilateral Sandwich First Third with 52-DOFs), with thirteen DOFs per node $\delta = \left\{u_0 \ v_0 \ w_0 \ \psi_x^c \ \psi_y^c \ \eta_x^c \ \eta_y^c \ \zeta_x^c \ \zeta_y^c \ \psi_x^t \ \psi_y^t \ \psi_x^b \ \psi_y^b\right\}^T$ has been developed. Each node contains: two rotational DOF for each face sheet, six rotational DOF for the core, while the three translations DOF are common for sandwich layers (Figure.3).

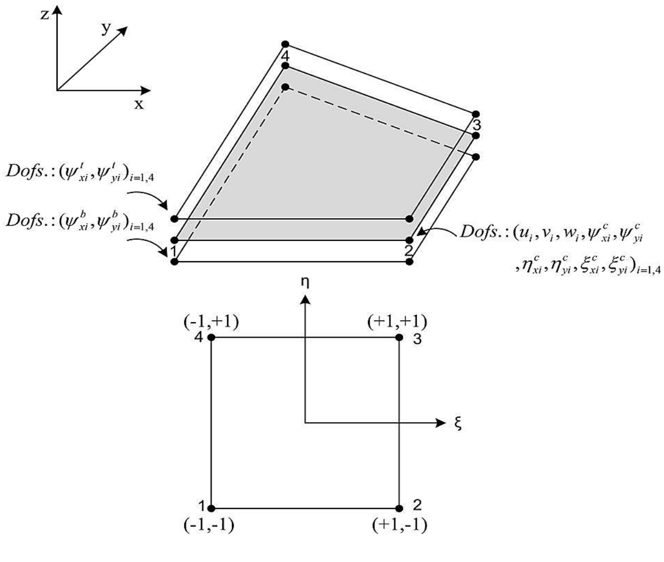

Figure 3: Geometry and corresponding DOFs of the present element.

The displacements vectors at any point of coordinates (x, y) of the plate are given by:

$$\delta(x,y)=\sum_{i=1}^{n}N_i\,(x,y)\;\delta_i \tag{18}$$

where δ_i is the nodal unknown vector corresponding to node i (i = 1, 2 3 4), N_i is the shape function associated with the node i.

The generalized strain vector for three layers can be expressed in terms of nodal displacements vector as follows:

$$\left\{\varepsilon^{(k)}\right\}=\left[B_i^{(k)}\right]\left\{\delta_i\right\} \tag{19}$$

where the matrices $\left[B_i^{(k)}\right]$ relate the strains to nodal displacements.

4 GOVERNING DIFFERENTIAL EQUATION

In this work, Hamilton's principle is applied in order to formulate governing free vibration problem, which is given as:

$$\delta\Pi=\delta\int_{t_1}^{t_2}\left(T-U\right)dt=0 \tag{20}$$

where t is the time, T is the kinetic energy of the system and U is the potential energy of the system. The first variation of kinetic energy of the three layers sandwich plate can be expressed as:

$$\begin{aligned}\delta T=&\int_{V_t}\rho_t\left(\ddot{u}_t\delta u_t+\ddot{v}_t\delta v_t+\ddot{w}_t\delta w_t\right)dV_t\\&+\int_{V_c}\rho_c\left(\ddot{u}_c\delta u_c+\ddot{v}_c\delta v_c+\ddot{w}_c\delta w_c\right)dV_c\\&+\int_{V_b}\rho_b\left(\ddot{u}_b\delta u_b+\ddot{v}_b\delta v_b+\ddot{w}_b\delta w_b\right)dV_b\end{aligned} \tag{21}$$

where u_i, v_i and w_i are the displacement in x,y and z directions, respectively, of the three-layered sandwich (i = t, c, b), ρ_i and V_i are the density of the material and volume, of each component, respectively, and ($\ddot{\ }$) is a second derivative with respect to time.

The first variation of the potential energy of the sandwich plate is the summation of contribution from the two face sheets and from the core as:

$$\begin{aligned}\delta U=&\int_{A_c}\int_{\frac{-h_c}{2}}^{\frac{h_c}{2}}\left(\sigma_{xx}^c\delta\varepsilon_{xx}^c+\sigma_{yy}^c\delta\varepsilon_{yy}^c+\sigma_{xy}^c\delta\varepsilon_{xy}^c+\sigma_{xz}^c\delta\varepsilon_{xz}^c+\sigma_{yz}^c\delta\varepsilon_{yz}^c\right)dV_c\\&+\int_{A_t}\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t}\left(\sigma_{xx}^t\delta\varepsilon_{xx}^t+\sigma_{yy}^t\delta\varepsilon_{yy}^t+\sigma_{xy}^t\delta\varepsilon_{xy}^t+\sigma_{xz}^t\delta\varepsilon_{xz}^t+\sigma_{yz}^t\delta\varepsilon_{yz}^t\right)dV_t\end{aligned} \tag{22}$$

$$+\int_{A_b}\int_{-\left(\frac{h_c}{2}+h_b\right)}^{-\frac{h_c}{2}}\left(\sigma_{xx}^{b}\delta\varepsilon_{xx}^{b}+\sigma_{yy}^{b}\delta\varepsilon_{yy}^{b}+\sigma_{xy}^{b}\delta\varepsilon_{xy}^{b}+\sigma_{xz}^{b}\delta\varepsilon_{xz}^{b}+\sigma_{yz}^{b}\delta\varepsilon_{yz}^{b}\right)dV_b$$

In the present analysis, the work done by external forces and the damping are neglected. Hence, Eq. (20) leads to the following dynamic equilibrium equation of a system.

$$\left[M_e\right]\left\{\ddot{\delta}\right\}+\left[K_e\right]\left\{\delta\right\}=0 \tag{23}$$

where $\left[M_e\right]$ and $\left[K_e\right]$ denote the total element mass matrix and the total element stiffness matrix respectively, which are computed using the Gauss numerical integration. The total element stiffness matrix is the summation of contribution from the two face sheets and from the core as:

$$\left[K_e\right]=\left[K_e^{(t)}\right]+\left[K_e^{(c)}\right]+\left[K_e^{(b)}\right] \tag{24}$$

where the element stiffness matrix of the core $\left[K_e^{(c)}\right]$ are given by,

$$\begin{aligned}\left[K_e^{(c)}\right]=\iint\Big(&\left[B_\varepsilon^{(0)}\right]^T[A]\left[B_\varepsilon^{(0)}\right]+\left[B_\varepsilon^{(0)}\right]^T[B]\left[B_\chi^{(1)}\right]+\left[B_\varepsilon^{(0)}\right]^T[D]\left[B_\chi^{(2)}\right]\\&+\left[B_\varepsilon^{(0)}\right]^T[E]\left[B_\chi^{(3)}\right]+\left[B_\chi^{(1)}\right]^T[B]\left[B_\varepsilon^{(0)}\right]+\left[B_\chi^{(1)}\right]^T[D]\left[B_\chi^{(1)}\right]\\&+\left[B_\chi^{(1)}\right]^T[E]\left[B_\chi^{(2)}\right]+\left[B_\chi^{(1)}\right]^T[F]\left[B_\chi^{(3)}\right]+\left[B_\chi^{(2)}\right]^T[D]\left[B_\varepsilon^{(0)}\right]\\&+\left[B_\chi^{(2)}\right]^T[E]\left[B_\chi^{(1)}\right]+\left[B_\chi^{(2)}\right]^T[F]\left[B_\chi^{(2)}\right]+\left[B_\chi^{(2)}\right]^T[L]\left[B_\chi^{(3)}\right]\\&+\left[B_\chi^{(3)}\right]^T[E]\left[B_\varepsilon^{(0)}\right]+\left[B_\chi^{(3)}\right]^T[F]\left[B_\chi^{(1)}\right]+\left[B_\chi^{(3)}\right]^T[L]\left[B_\chi^{(2)}\right]\\&+\left[B_\chi^{(3)}\right]^T[H]\left[B_\chi^{(3)}\right]+\left[B_{\gamma_s}^{(0)}\right]^T\left[A^s\right]\left[B_{\gamma_s}^{(0)}\right]+\left[B_{\gamma_s}^{(0)}\right]^T\left[B^s\right]\left[B_{\chi_s}^{(1)}\right]\\&+\left[B_{\gamma_s}^{(0)}\right]^T\left[D^s\right]\left[B_{\chi_s}^{(2)}\right]+\left[B_{\chi_s}^{(1)}\right]^T\left[B^s\right]\left[B_{\gamma_s}^{(0)}\right]+\left[B_{\chi_s}^{(1)}\right]^T\left[D^s\right]\left[B_{\chi_s}^{(1)}\right]\\&+\left[B_{\chi_s}^{(1)}\right]^T\left[E^s\right]\left[B_{\chi_s}^{(2)}\right]+\left[B_{\chi_s}^{(2)}\right]^T\left[D^s\right]\left[B_{\gamma_s}^{(0)}\right]+\left[B_{\chi_s}^{(2)}\right]^T\left[E^s\right]\left[B_{\chi_s}^{(1)}\right]\\&+\left[B_{\chi_s}^{(2)}\right]^T\left[F^s\right]\left[B_{\chi_s}^{(2)}\right]\Big)dA\end{aligned} \tag{25}$$

For the two face sheets, the element stiffness matrix can be written as:

a. Top face sheet:

$$\left[K_e^{(t)}\right]=\iint\Bigg(\underbrace{\left[B_m^t\right]^T\left[A^{(t)}\right]\left[B_m^t\right]}_{membrane}+\underbrace{\left[B_m^t\right]^T\left[B^{(t)}\right]\left[B_f^t\right]}_{coupling\ membrane\text{-}bending}+\underbrace{\left[B_f^t\right]^T\left[B^{(t)}\right]\left[B_m^t\right]}_{coupling\ bending\text{-}membrane} \tag{26}$$

$$+\underbrace{\left[B_f^t\right]^T\left[D^{(t)}\right]\left[B_f^t\right]}_{bending}+\underbrace{\left[B_s^t\right]^T\left[A_c^{(t)}\right]\left[B_s^t\right]}_{shear}\Bigg)dA$$

b. Bottom face sheet:

$$\left[K_e^{(b)}\right]=\iint\Bigg(\underbrace{\left[B_m^b\right]^T\left[A^{(b)}\right]\left[B_m^b\right]}_{membrane}+\underbrace{\left[B_m^b\right]^T\left[B^{(b)}\right]\left[B_f^b\right]}_{coupling\ membrane\text{-}bending}+\underbrace{\left[B_f^b\right]^T\left[B^{(b)}\right]\left[B_m^b\right]}_{coupling\ bending\text{-}membrane}$$
$$+\underbrace{\left[B_f^b\right]^T\left[D^{(b)}\right]\left[B_f^b\right]}_{bending}+\underbrace{\left[B_s^b\right]^T\left[A_c^{(b)}\right]\left[B_s^b\right]}_{shear}\Bigg)dA \tag{27}$$

The total element mass matrix, for the three-layer sandwich plate, can be written as

$$\left[M_e\right]=\iint\left(\left[N\right]^T\left[m^{(t)}\right]\left[N\right]+\left[N\right]^T\left[m^{(c)}\right]\left[N\right]+\left[N\right]^T\left[m^{(b)}\right]\left[N\right]\right)dA \tag{28}$$

where $\left[m^{(t)}\right]$, $\left[m^{(c)}\right]$ and $\left[m^{(b)}\right]$ are the consistent mass matrices of the top face sheet, core and the bottom face sheet, respectively, containing inertia terms. Now, after evaluating the stiffness and mass matrices for all elements, the governing equations for free vibration analysis can be stated in the form of generalized eigenvalue problem.

$$\left[K\right]\{\chi\}-\omega^2\left[M\right]\{\chi\}=0 \tag{29}$$

where, ω denote the natural frequency, $\left[K\right]$ is he global stiffness matrix, $\left[M\right]$ is the global mass matrix, $\{\chi\}$ are the vectors defining the mode shapes.

5 NUMERICAL RESULTS AND DISCUSSIONS

In this section, several examples on the free vibration analysis of laminated composite and sandwich plates will be analyzed to demonstrate the performance and the versatility of the developed finite element model. The MATLAB programming language is used to solve the eigenvalue problem. The obtained numerical results are compared with the analytical solutions and others finite elements numerical results found in the literature.

Table 1 shows the boundary conditions, for which the numerical results have been obtained, where CCCC, SSSS, CSCS and CFCF respectively indicate: fully clamped, fully simply supported, two opposite edges clamped and other two simply supported, two opposite edges clamped and other two free. Table 2 shows the material models (MM) considered for different numerical evaluation.

Boundary conditions	Abbreviations	Restrained edges
Simply supported	SSSS	$w_0=\psi_x^c=\eta_x^c=\zeta_x^c=\psi_x^t=\psi_x^b=0$ at $x=\pm a/2$ $w_0=\psi_y^c=\eta_y^c=\zeta_y^c=\psi_y^t=\psi_y^b=0$ at $y=\pm b/2$
Clamped	CCCC	$w_0=\psi_x^c=\psi_y^c=\eta_x^c=\eta_y^c=\zeta_x^c=\zeta_y^c=\psi_x^t=\psi_y^t=\psi_x^b=\psi_y^b=0$
Simply supported- Clamped	SCSC	Simply supported at $x=\pm a/2$ Clamped at $y=\pm b/2$
Clamped-free	CFCF	Clamped at $x=\pm b/2$ Free at $y=\pm b/2$

Table 1: Boundary conditions used in this study.

Elastic properties	Unit	MM1	MM2	MM3	MM4	MM5	MM6	MM7	MM8
E_{11}	GPa	24.51	0.1036	276.0	0.5776	40	Open	131.0	0.00690
E_{22}	GPa	7.77	0.1036	6.9	0.5776	E_{22}	E_{22}	10.34	0.00690
G_{12}	GPa	3.34	0.05	6.9	0.1079	$0.6E_{22}$	$0.6E_{22}$	6.9	0.00344
G_{13}	GPa	3.34	0.05	6.9	0.1079	$0.5E_{22}$	$0.6E_{22}$	6.2	0.00344
G_{23}	GPa	1.34	0.05	6.9	0.2221	$0.5E_{22}$	$0.5E_{22}$	6.9	0.00345
ν_{12}	-	0.078	0.32	0.25	0.0025	0.25	0.25	0.22	10^{-5}
ρ	Kg/m^3	1800	130	681.8	1000	1	1	1627	97

Table 2: Material models (MM) considered for different laminated and sandwich plate.

5.1 Convergence Study

In the first example, the convergence of the developed quadrilateral element is studied for a seven-layer simply supported square sandwich plate. Two sandwich plates with various lay-ups on face sheets [0/90/0/core/0/90/0] and [45/-45/45/core/-45/45/-45] are considered. The core is made of HEREX-C70.130 PVC foam (MM1) and the face sheets are made of glass polyester resin (MM2). The geometrical properties of the plate are ($a/h = 10$, a/b = 1, $hc/h = 0.88$) where h is the total thickness of the plate. The convergence of the non-dimensional results of natural frequencies, for the first four modes, is shown in Table 3 with different mesh sizes (6×6, 8×8, 10×10, 12×12, 14×14 and 16×16). The comparison was made with the analytical solutions based on LW approach (Jam et al. 2010, Rahmani et al. 2010), the 3D-finite element models also based on LW approach (FEM-3D-LW) (Malekzadeh and Sayyidmousavi 2009, Burlayenko et al. 2015), the FEM-Q9 and Q4 solution based on HSDT (Nayak et al. 2002) and other analytical solution based on HSDT (Meunier and Shenoi 1999). The results of the comparison show the performances and convergence of the present formulation.

References	FE Models	Various lay-ups on face sheets	Frequencies (Hz)			
			Mode 1	Mode 2	Mode 3	Mode 4
Present element (6×6)	QSFT52	Case 1[a]	14.736*	28.207	28.802	37.584
Present element (8×8)	QSFT52		14.583	27.499	28.115	36.627
Present element (10×10)	QSFT52		14.513	27.173	27.796	36.167
Present element (12×12)	QSFT52		14.477	26.999	27.626	35.954
Present element (14×14)	QSFT52		14.452	26.893	27.524	35.777
Present element (16×16)	QSFT52		14.440	26.826	27.456	35.706
Burlayenko et al. (2015)	FEM-3D-LW		14.620	26.800	27.400	35.550
Rahmani et al. (2010)	Analytical-LW		14.270	26.310	27.040	34.950
Malekzadeh, K et al. (2009)	FEM-3D-LW		14.740	26.830	27.530	35.600
Jam et al. (2010) % error[a]	Analytical-LW		15.040	26.733	27.329	35.316
			(-3.989)	(0.347)	(0.464)	(1.104)
Meunier and Shenoi (1999)	Analytical-HSDT		15.280	28.690	30.010	38.860
Nayak et al. (2002)	FEM-Q9-HSDT		15.040	28.100	29.200	37.760
Nayak et al. (2002)	FEM-Q4-HSDT		15.340	30.180	31.960	40.940
Present element (6×6)	QSFT52	Case 2[b]	15.674	28.756	28.756	38.363
Present element (8×8)	QSFT52		15.536	28.069	28.069	37.478
Present element (10×10)	QSFT52		15.473	27.754	27.754	37.053
Present element (12×12)	QSFT52		15.437	27.587	27.587	36.805
Present element (14×14)	QSFT52		15.419	27.485	27.485	36.698
Present element (16×16)	QSFT52		15.405	27.417	27.417	36.592
Burlayenko et al. (2015)	FEM-3D-LW		15.420	27.170	27.460	36.240
Jam et al. (2010) % error	Analytical-LW		15.786	27.316	27.316	36.216
			(-2.445)	(0.270)	(0.270)	(1.038)
Malekzadeh, K et al. (2009)	FEM-3D-LW		15.810	27.230	27.230	36.260
Meunier and Shenoi (2000)	Analytical-HSDT		16.380	29.650	29.650	40.000
Nayak et al. (2002)	FEM-Q9-HSDT		16.090	28.930	28.930	38.760
Nayak et al. (2002)	FEM-Q4-HSDT		16.430	31.170	31.170	42.780

* The natural frequencies are expressed as: $\bar{\omega} = \omega \frac{a^2}{h}\sqrt{\rho_c / E_c}$

a Percentage error = [(Present result - Analytical result) / Analytical result] × 100

b Various lay-ups on face sheets : Case 1: [0/90/0/core/0/90/0] and Case 2 : [45/-45/45/core/-45/45/-45].

Table 3: Non-dimensional natural frequencies for a square multi-layered sandwich plate with various lay-ups on face sheets.

5.2 Square Sandwich Plate (0/90/C/90/0) Having Two-Ply Laminated Stiff Sheets at the Faces

In this problem, a simply supported square sandwich plate having two laminated stiff layers is investigated. The thickness of each laminate layer is 0.05h, whereas the thickness of the core is 0.8h. The mechanical properties MM3 and MM4 of table 2 are adopted, respectively, for laminated face sheets and core. The non-dimensional natural frequencies, for the first six modes, are presented in table 4 using a mesh size of 12×12. In the present analysis, different thickness ratios (a/h = 6.67, 10 and 20) are considered. The obtained results are compared with the 3D-elasticity solution given by Kulkarni and Kapuria (2008), analytical results of Wang et al. (2000) using p-Ritz method and some existing finite element results based on HZZT (Chakrabarti and Sheikh 2004, Kulkarni and Kapuria 2008). It is clear, from the table 4, that the results of developed element are in excellent agreement with numerical results found in the literature.

$\frac{a}{h}$	Modes	Present QSFT52	Kulkarni et al. (2008) 3D-Elasticity	Kulkarni et al. (2008) FEM-Q4-TSDT	Wang et al. (2000) Anal-Solution	Chakrabarti et al. (2004) FEM-T6-HZZT
6.67	1st	10.564*	10.524	13.315	11.414	10.560
	2nd	16.329	16.149	21.561	17.552	14.455
	3rd	18.962	18.728	23.177	20.426	18.735
	4th	22.745	22.434	28.713	24.436	20.580
	5th	23.867	23.172	31.140	-	23.373
	6th	28.722	27.879	34.055	-	26.165
10	1st	9.871	9.828	12.088	10.555	10.051
	2nd	15.681	15.505	20.615	16.830	14.409
	3rd	18.310	18.075	22.152	19.648	18.962
	4th	21.999	21.696	27.675	23.616	19.424
	5th	22.816	22.202	30.143	-	21.252
	6th	27.861	26.915	35.329	-	24.496
20	1st	7.742	7. 688	8.721	8.029	7.927
	2nd	14.039	13.845	17.705	14.858	13.042
	3rd	16.196	15.920	18.530	16.984	17.315
	4th	19.992	19.656	24.105	21.111	18.834
	5th	21.278	20.676	27.714	-	20.091
	6th	25.604	24.948	32.136	-	24.139

* The natural frequencies are expressed as: $\overline{\omega} = 100\omega a\sqrt{\rho_c / E_{11f}}$

Table 4: Non-dimensional fundamental frequencies with different modes for simply supported sandwich plate with laminated face sheets (0/90/C/90/0).

Moreover, the same plate is analyzed by considering two different boundary conditions, CCCC and SCSC. The non-dimensional natural frequencies, for the first six modes, are reported in table 5 for different thickness ratios ($a/h = 5$, 10 and 20). The first six mode shapes obtained for SSSS, CFCF and CFFF square laminated sandwich plate with a/h =10 are shown in Figures 4, 5 and 6. It can be observed that, in comparison with the FEM solution based on HZZT (Khandelwal et al. 2013, Chalak et al. 2013, Kulkarni and Kapuria 2008, Chakrabarti and Sheikh 2004), the present element gives more accurate results than the other models.

$\frac{a}{h}$	Boundary condition	Modes	Present QSFT52	Khandelwal et al. (2013) FEM-Q9-HZZT	Chalak et al. (2013) FEM-Q9-HZZT	Kulkarni et al. (2008) FEM-Q4-HZZT	Chakrabarti et al. (2004) FEM-T6-HZZT
5	SCSC	1st	11.445	11.591	11.408	11.516	11.063
		2nd	18.267	17.108	18.014	18.379	15.199
		3rd	19.707	20.409	19.485	19.626	18.255
		4th	24.448	24.079	24.086	24.722	19.781
		5th	27.657	24.593	26.559	27.409	20.127
		6th	29.866	29.989	29.014	29.231	22.907
	CCCC	1st	12.200	12.121	12.138	12.440	11.864
		2nd	18.733	18.453	18.469	19.106	15.672
		3rd	21.120	20.706	20.764	21.442	19.477
		4th	25.614	25.058	25.138	26.691	20.057
		5th	27.959	26.849	26.860	28.043	21.167
		6th	29.866	30.908	31.050	32.257	23.628
10	SCSC	1st	10.346	10.860	10.344	10.378	10.422
		2nd	16.399	16.131	16.310	16.411	15.021
		3rd	18.547	18.962	18.349	18.395	19.135
		4th	22.523	22.438	22.294	22.494	20.372
		5th	24.079	22.628	23.554	23.718	21.642
		6th	27.934	27.532	27.211	27.258	25.215
	CCCC	1st	11.318	11.349	11.356	11.468	11.524
		2nd	16.967	16.900	16.909	17.135	15.691
		3rd	19.332	19.214	19.236	19.494	19.946
		4th	23.187	23.003	23.041	23.659	20.783
		5th	24.428	23.925	23.935	24.253	22.356
		6th	27.934	28.502	28.539	28.918	25.812
20	SCSC	1st	8.635	9.502	8.666	8.649	8.623
		2nd	14.604	14.853	14.645	14.601	13.533
		3rd	16.581	17.233	16.367	16.357	17.601
		4th	20.390	20.708	20.234	20.233	19.441
		5th	22.006	21.310	21.767	21.673	20.416
		6th	26.107	25.768	25.342	25.324	24.620
	CCCC	1st	10.253	10.330	10.336	10.332	10.536
		2nd	15.530	15.598	15.609	15.600	14.709
		3rd	17.605	17.644	17.659	17.674	18.708
		4th	21.217	21.246	21.278	21.404	20.182
		5th	22.568	22.354	22.368	22.323	21.369
		6th	26.107	26.636	26.683	26.893	25.406

Table 5: Non-dimensional fundamental frequencies for laminated sandwich plate (0/90/C/90/0) with different boundary conditions.

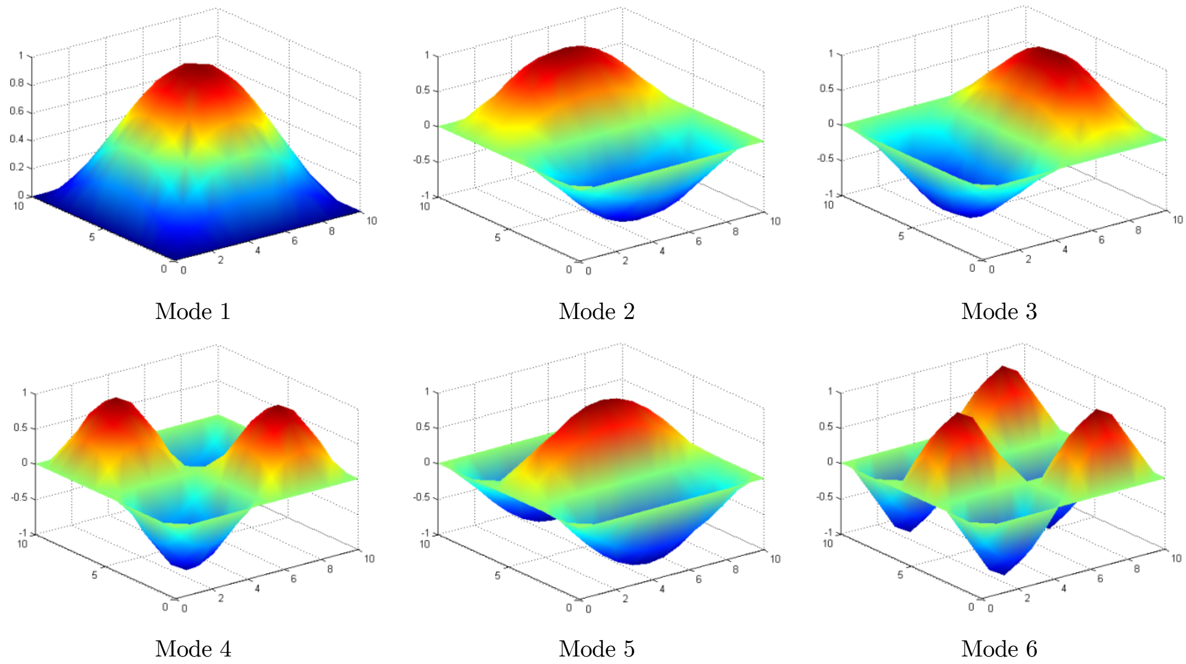

Figure 4: First six mode shapes of SSSS square laminated sandwich plate (0/90/C/90/0) with a/h =10.

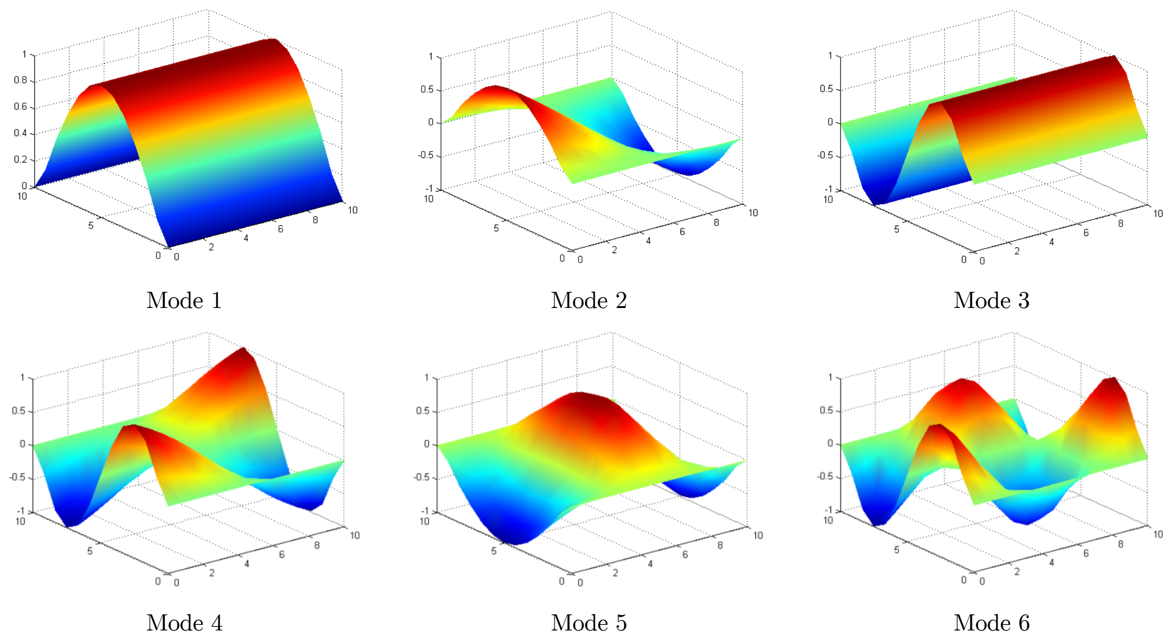

Figure 5: First six mode shapes of CFCF square laminated sandwich plate (0/90/C/90/0) with a/h =10.

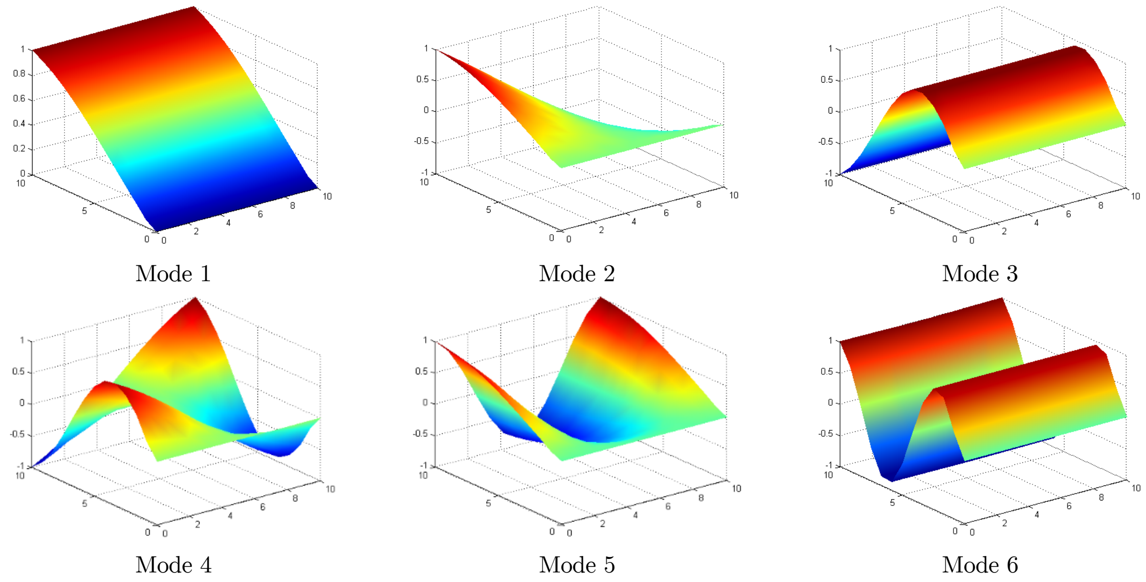

Figure 6: First six mode shapes of CFFF square laminated sandwich plate (0/90/C/90/0) with a/h =10.

5.3 Skew Laminated Plates

In order to evaluate the performance of the developed element for the study of free vibration response of irregular plates, a five layer symmetric cross-ply skew laminated plates (90/0/90/0/90) with simply supported edges is considered. The geometry of the skew plates is shown in Figure 7. The material properties MM5 of Table 2 is used for this analysis. The skew angle α is varied from 0°, 15°, 30°, 45° and 60°. The non-dimensional natural frequencies for the first four modes are reported in Table 6, considering the thickness ratios (a/h) as 10. A mesh size of 12×12 is considered for the analysis. The first six flexural mode shapes obtained for $\alpha = 45°$ are shown in Figure. 8. The comparison was made with the analytical solutions of Wang (1997) using B-spline Rayleigh-Ritz method, the solution of Ferreira et al. (2005) based on Radial Basic Function (RBF), as well as with the finite element models of Nguyen-Van (2009) and Garg et al. (2006). The results of the comparison show the effectiveness of the present element in the analysis of this type of structures.

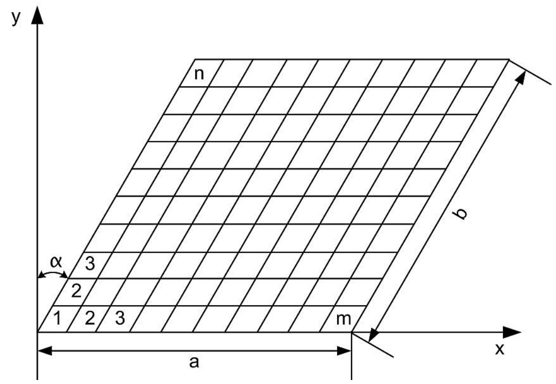

Figure 7: A skew plate with mesh arrangement (mesh size: m × n).

Skew angle	References	FE Models	Frequencies (Hz)			
			Mode 1	Mode 2	Mode 3	Mode 4
0°	Present element	QSFT52	1.5816*	3.0082	3.8583	4.6476
	Nguyen-Van (2009)	MISQ20	1.5733	-	-	-
	Ferreira et al. (2005)	RBF	1.5791	-	-	-
	Wang (1997)	Anal Solution	1.5699	3.0371	3.7324	4.5664
	Garg et al. (2006)	FEM-Q9-TSDT	1.5703	2.8917	3.8041	4.5314
	Garg et al. (2006)	FEM-Q9-FSDT	1.5699	3.0372	3.7325	4.5664
15°	Present element	QSFT52	1.6799	3.0467	4.1075	4.7327
	Nguyen-Van (2009)	MISQ20	1.6896	-	-	-
	Ferreira et al. (2005)	RBF	1.6917	-	-	-
	Garg et al. (2006)	FEM-Q9-TSDT	1.6877	3.0458	4.0264	4.4818
	Garg et al. (2006)	FEM-Q9-FSDT	1.6874	3.1413	3.9600	4.6073
30°	Present element	QSFT52	2.1074	3.2736	4.9556	5.0316
	Nguyen-Van (2009)	MISQ20	2.0820	-	-	-
	Ferreira et al. (2005)	RBF	2.0799	-	-	-
	Wang (1997)	Anal Solution	2.0844	3.5127	4.6997	4.8855
	Garg et al. (2006)	FEM-Q9-TSDT	2.0840	3.4023	4.7176	4.7674
	Garg et al. (2006)	FEM-Q9-FSDT	2.0884	3.5147	4.7033	4.8864
45°	Present element	QSFT52	2.7691	3.7711	5.2930	6.6081
	Nguyen-Van (2009)	MISQ20	2.8855	-	-	-
	Ferreira et al. (2005)	RBF	2.8228	-	-	-
	Wang (1997)	Anal Solution	2.8825	4.2823	5.5868	6.1808
	Garg et al. (2006)	FEM-Q9-TSDT	2.8925	4.1906	5.4149	6.2868
	Garg et al. (2006)	FEM-Q9-FSDT	2.8932	4.2852	5.5886	6.1874
60°	Present element	QSFT52	4.5523	5.2950	6.4896	8.0651
	Nguyen-Van (2009)	MISQ20	4.5412	-	-	-
	Ferreira et al. (2005)	RBF	4.3761	-	-	-

* The natural frequencies are expressed as: $\bar{\omega} = \omega \dfrac{b^2}{\pi^2}\sqrt{\rho/E_2}$

Table 6: Non-dimensional natural frequencies for symmetric cross-ply skew composite laminates (90/0/90/0/90) with a/h = 10.

5.4 Simply Supported Cross-Ply Multilayered Composite Plate (0/90/…/0)

In this example, the effects of number of layers (n) and modulus ratio (E_{11}/E_{22}) on fundamental frequencies ($\bar{\omega}_1$) are studied. Simply supported square cross ply laminated composite plate of equal thickness is considered. In the present analysis, different number of layers with various modular ratios (E_{11}/E_{22}) are adopted. The material properties MM6 of Table 2 are used for this analysis. The results are obtained for thickness ratio $a/h = 5$. The non-dimensional results of natural frequencies are reported in Table 4 using a 12×12 mesh.

It is clear, from the table 7, that the results obtained from developed element are in excellent agreement when compared with those obtained from the 3D-elasticity solution given by Noor (1973), the FEM-Q9 and Q4 solution based on LW (Marjanović and Vuksanović 2014), the FEM-Q8 solution based on GLHSDT (Zhen et al. 2010), the FEM-Q9 and Q4 solution based on HSDT (Nayak et al. 2002) and other analytical results (Owen and Li 1987, Vuksanović 2000, Matsunaga 2000). From Figure 9, it can be seen that the values of natural frequencies of laminated plate increase with increasing in E_{11}/E_{22} modular ratios, whatever the number of layer.

References	FE Models	No. of layers	$\frac{E_{11}}{E_{22}}$			
			3	10	20	30
Present element	QSFT52	3 Layers[a]	0.2646*	0.3252	0.3640	0.3840
Noor (1973)	3D-Elasticity		0.2647	0.3284	0.3842	0.4109
Owen and Li (1987)	RHSDT		0.2695	0.3392	0.3898	0.4194
Vuksanović (2000)	HSDT		0.2673	0.3318	0.3749	0.4015
Marjanović and Vuksanović (2014)	FEM-Q9-LW		0.2621	0.3262	0.3691	0.3927
Marjanović and Vuksanović (2014)	FEM-Q4-LW		0.2683	0.3297	0.3685	0.3886
Nayak et al. (2002)	FEM-Q9-HSDT		0.2623	0.3264	0.3667	0.3941
Zhen et al. (2010)	FEM-Q8-GLHSDT		0.2620	0.3258	0.3688	0.3928
Matsunaga (2000)	Analytical solution		0.2627	0.3266	0.3696	0.3936
Present element	QSFT52	5 Layers[b]	0.2662	0.3404	0.3966	0.4300
Noor (1973)	3D-Elasticity		0.2659	0.3409	0.3979	0.4314
Owen and Li (1987)	RHSDT		0.2699	0.3453	0.4030	0.4370
Vuksanović (2000)	HSDT		0.2684	0.3442	0.3939	0.4269
Marjanović and Vuksanović (2014)	FEM-Q9-LW		0.2618	0.3330	0.3858	0.4166
Marjanović and Vuksanović (2014)	FEM-Q4-LW		0.2683	0.3396	0.3918	0.4219
Nayak et al. (2002)	FEM-Q9-HSDT		0.2636	0.3372	0.3929	0.4257
Matsunaga (2000)	Analytical solution		0.2638	0.3362	0.3901	0.4215
Present element	QSFT52	9 Layers[c]	0.2672	0.3459	0.4086	0.4467
Noor (1973)	3D-Elasticity		0.2664	0.3443	0.4054	0.4421
Nayak et al. (2002)	FEM-Q9-HSDT		0.2636	0.3372	0.3929	0.4257
Nayak et al. (2002)	FEM-Q4-HSDT		0.2641	0.3378	0.3935	0.4263
Zhen et al. (2010)	FEM-Q8-GLHSDT		0.2636	0.3402	0.4002	0.4362
Matsunaga (2000)	Analytical solution		0.2645	0.3414	0.4015	0.4376

* The natural frequencies are expressed as: $\overline{\omega} = \omega h\sqrt{\rho/E_2}$

[b] Number of layers : a: [0/90/0], b: [0/90/0/90/0] and c : [0/90/0/90/0/90/0/90/0].

Table 7: Non-dimensional fundamental frequencies for simply supported square cross-ply multilayered composite plate (0/90/.../0) using different E11/E22 ratios.

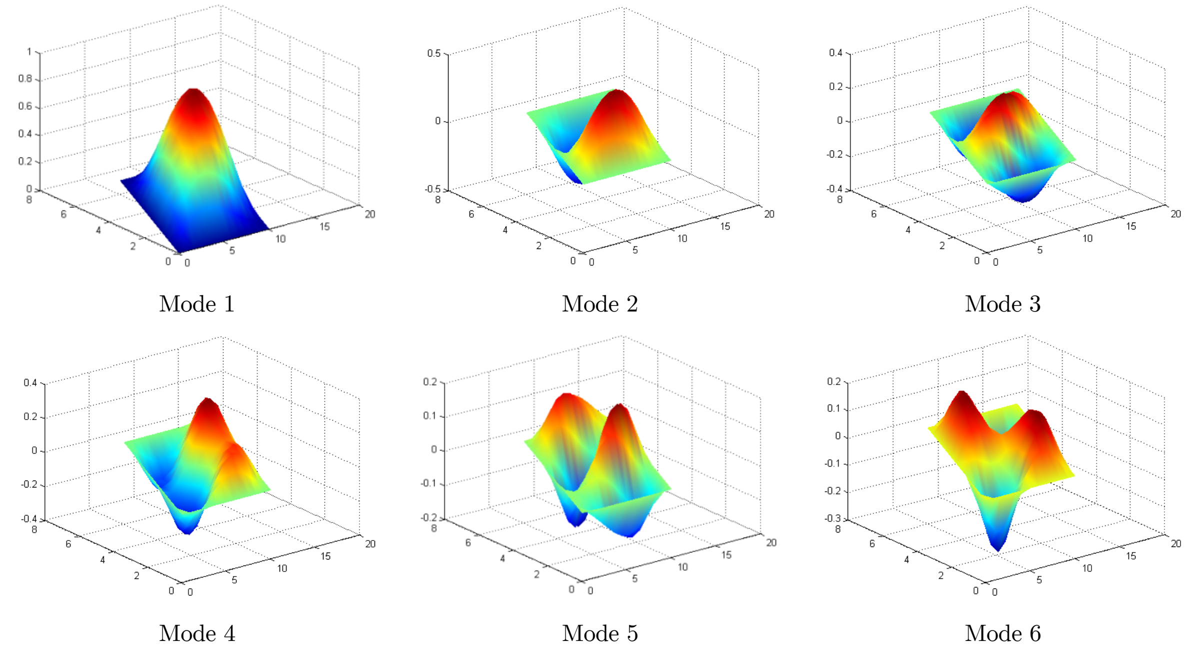

Figure 8: First six mode shapes of simply supported square skew laminated plate for $\alpha = 45°$.

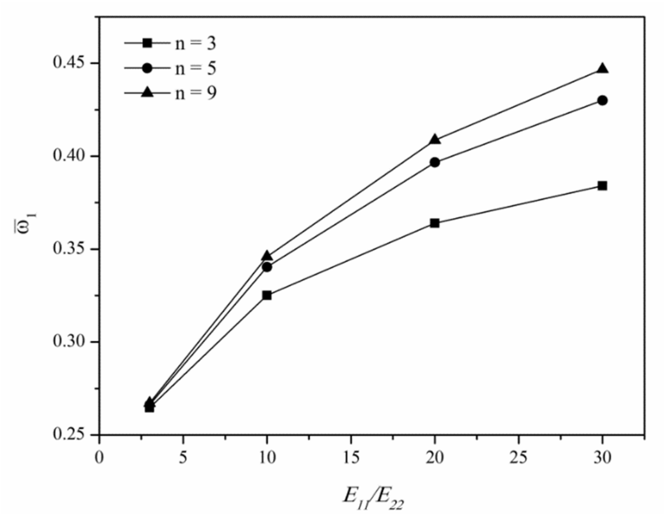

Figure 9: Variation of natural frequencies with respect to modular ratio (E_{11}/E_{22}) for laminated composite square Plates.

5.5 Unsymmetric Laminated Sandwich Plate (0/90/C/0/90)

To study the effect of the core thickness ratio (h_c/h_f) and aspect ratio (a/b) on the fundamental frequencies, a simply supported square sandwich plate with unsymmetric laminated face sheets and isotropic core is considered. The mechanical properties MM7 and MM8 of table 2 are adopted, respectively, for the laminated face sheets and the core. The thickness ratio (a/h) is taken to be 10. A comparison has been made with 3D-elasticity solution of Rao et al. (2004) to assess the suitability of the present formulation. Figure 10 shows the effect of the core thickness on the fundamental frequency of vibration. It is seen that the values of non-dimensional natural frequencies, $\overline{\omega} = \omega b^2/h\sqrt{\rho_c/E_{22f}}$, increase with the increasing in h_c/h_f ratio.

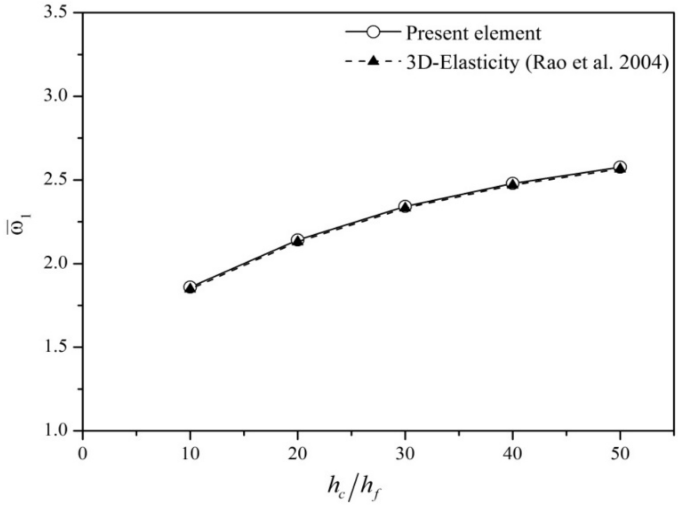

Figure 10: Effect of h_c/h_f ratio on the fundamental frequencies of a simply supported square laminated sandwich plate.

Further, the same sandwich plate was analyzed for different aspect ratios (a/b) keeping the same ratios $a/h = 10$ and $h_c/hf = 10$. Figure 11 shows the effect of aspect ratio on the fundamental frequency. It is found that the variation of the fundamental frequency decrease with increase in aspect ratio. It is concluded that, from Figure 10 and 11, the present results are in very close agreement with the 3D-elasticity solution (Rao et al. 2004).

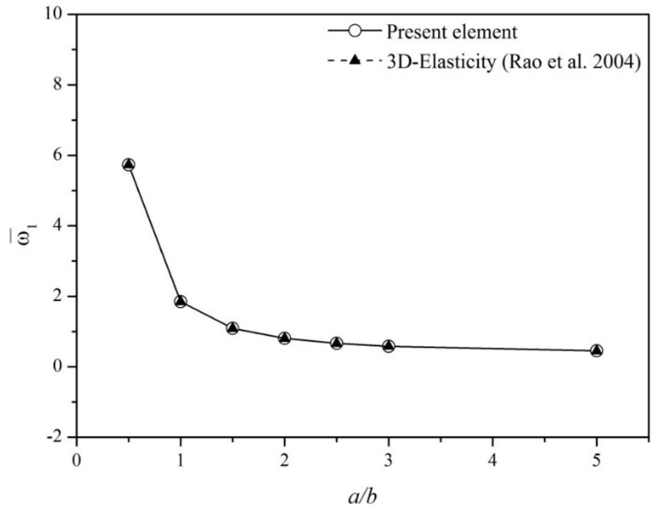

Figure 11: Effect of a/b ratio on the fundamental frequencies of a simply supported laminated sandwich plate.

5 CONCLUSION

A new higher-order layerwise finite element model was proposed for free vibration analysis of laminated composite and sandwich plates. The developed model is based on a proper combination of higher-order and first-order, shear deformation theories. These combined theories satisfy interlaminar displacement continuity. Although the model is a layerwise one, the number of variables is independent of the number of layers. Thus, the plate theory enjoys the advantage of a single-layer plate theory, even though it is based on the concept of a layerwise plate approach. Based on this model, a four-noded C^0 continuous isoparametric element is formulated. The performance and the efficiency of the newly developed FE model are demonstrated by several numerical examples on free vibration analysis of laminated composite, symmetric/unsymmetric sandwich and skew plates, with varying material combinations, aspect ratios, number of layers, geometry and boundary conditions. The results obtained by our model were compared with those obtained by the analytical results and other finite element models found in literature. The comparison showed that the element has an excellent accuracy and a broad range of applicability. It is important to mention here, that the proposed FE formulation is simple and accurate in solving the free vibration problems of laminated composite and sandwich plates.

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