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Mode Shape Analysis of Multiple Cracked Functionally Graded Timoshenko Beams

Abstract

The present paper addresses free vibration of multiple cracked Timoshenko beams made of Functionally Graded Material (FGM). Cracks are modeled by rotational spring of stiffness calculated from the crack depth and material properties vary according to the power law throughout the beam thickness. Governing equations for free vibration of the beam are formulated with taking into account actual position of the neutral plane. The obtained frequency equation and mode shapes are used for analysis of the beam mode shapes in dependence on the material and crack parameters. Numerical results validate usefulness of the proposed herein theory and show that mode shapes are good indication for detecting multiple cracks in Timoshenko FGM beams.

Keywords

FGM; Cracked Timoshenko beam; Modal analysis; Mode shape.

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1 INTRODUCTION

Since the FGMs are increasingly implemented in the practice of industries, dynamics of cracked structures made of FGM gets an enormous attention of researches and engineers. Also, a number of methods has been proposed to vibration analysis of the structures such as Galerkin's method, Finite Element Method (FEM), Dynamic Stifness Method (DSM), ... However, for the simple structures such as beams the analytical approach shows to be most accurate and efficient. Yan et al. (2008) calculated natural frequencies and mode shapes of cracked FGM Euler-Bernoulli beam. Ke et al. (2009) studied effects of open edge cracks to vibration of FGM Timoshenko beam with different boundary conditions. Aydin (2013) established frequency equations of free vibration for FGM Euler-Bernoulli beam with arbitrary number of cracks. Likely, the authors (J. Yang, Y. Chen, Y. Xiang, X.L. Jia 2008; J. Yang, Y. Chen 2008) obtained frequency equation of the beam in the form of third order determinant without using transfer matrix method at crack positions. Wei et al. (2012) established equations of motion of FGM Timoshenko beam with rotary inertia and shear deformation included. Because of ignoring axial inertia, the bending vibration is independent from axial vibration. The authors used transfer matrix method to obtain frequency equations and mode shapes of beam with arbitrary number of open edge cracks only in the form of third-order determinant. This is remarkable achievement in free vibration analysis of multiple cracked FGM beam. Sherafatnia et al. (2014) analyzed natural frequencies and mode shapes of cracked beam using different theories of beam. It was demonstrated by the authors that fundamental mode shape is the same for all the beam theories, but notable difference between the mode shapes is revealed for the higher modes.

Using Galerkin's procedure, Yan et al. (2011) obtained dynamic deflections of cracked FGM beam on elastic foundation under a transverse moving load. Kitipornchai et al. (2009) used Ritz's method to analyze nonlinear vibration of an open edge crack FGM Timoshenko beam. Wattanasakulpong et al. (2012) investigated free vibration of FGM beams with general elastically end constraints by differential transformation method. The FEM has been used to calculate frequencies and mode shapes of FGM beam (Z.G. Yu, F.L. Chu 2009; S.D. Akbas 2013; A. Banerjee, B. Panigrahi, G. Pohit 2015). As the FEM is formulated by using frequency independent polynomial shape function, this method cannot capture all necessary high frequencies of interest. This limitation of the FEM could be resolved by using DSM (H. Su, A. Banerjee 2015; N.T. Khiem, N.D. Kien, N.N. Huyen 2014; T.V. Lien, N.T. Duc and N.T. Khiem 2016; N.T. Khiem, T.V. Lien 2002). Since the dynamic stiffness method uses the frequency-dependent shape functions obtained from the exact solution of the governing differential equations of free vibration, natural frequencies and mode shapes obtained by the method should be more accurate. Nevertheless, there is very little effort devoted to develop the DSM for vibration analysis of cracked FGM Timoshenko beams.

On the other hand, because of grading material properties the neutral plane and mid plane of the FGM beams are different and effect of the neutral plane position on static and dynamic behavior of the beam is investigated in some studies. Eltaher et al. (2013) showed that the natural frequencies calculated by using the mid-plane theory of FGM beam are higher than those obtained by taking into account actual position of neutral plane. Furthermore, study by Huyen and Khiem (2016) revealed that taking into account actual position of neutral plane simplifies the governing differential equations of FGM beam and allows one to find a condition for uncoupling of axial and flexural vibrations likely to the homogeneous beam.

This paper is devoted first to establish governing equations for vibration of multiple cracked FGM Timoshenko beam based on the power law of material grading; actual position of neutral plane and rotational spring model of cracks. From the governing equations, a novel form of frequency equation and an explicit expression of mode shapes for multiple cracked FGM Timoshenko beam are obtained. Using the obtained equations, mode shapes of cracked FGM Timoshenko beam are examined along the material and crack parameters. Numerical results show that the proposed herein theory is useful not only for modal analysis of cracked FGM Timoshenko beams but also for crack detection in the beam by mode shape measurement.

2 GOVERNING EQUATIONS

Consider a FGM beam of length L, cross sectional area $A=b\times h$. It is assumed that the material properties of FGM beam vary along the thickness direction by the power law distribution as follows

$$\begin{Bmatrix} E(z) \\ G(z) \\ \rho(z) \end{Bmatrix} = \begin{Bmatrix} E_b \\ G_b \\ \rho_b \end{Bmatrix} + \begin{Bmatrix} E_t - E_b \\ G_t - G_b \\ \rho_t - \rho_b \end{Bmatrix} \left(\frac{z}{h} + \frac{1}{2} \right)^n, -h/2 \le z \le h/2, \tag{1}$$

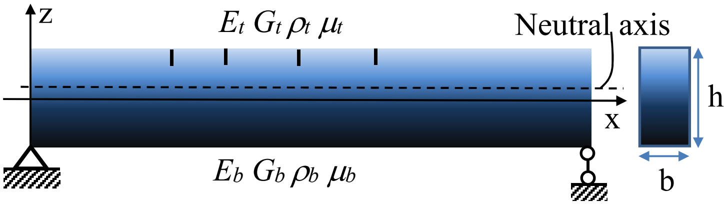


Figure 1: A multiple cracked FGM beam.

where E, G and ϱ stand for Young's, shear modulus and material density, n is power law exponent, z is co-ordinate of point from the mid plane at high $h/2$ (Fig. 1). Based on the Hamilton's principle, we can get free vibration equations of FGM Timoshenko beam in frequency domain in the form (T.V. Lien, N.T. Duc and N.T. Khiem 2016)

$$\tilde{\mathbf{A}}\mathbf{z}'' + \tilde{\mathbf{\Pi}}\mathbf{z}' + \tilde{\mathbf{C}}\mathbf{z} = 0 \tag{2}$$

where $\mathbf{z} = \{U, \Theta, W\}^T$ is amplitude of axial displacement, rotation and deflection

$$\{U, \Theta, W\} = \int_{-\infty}^{\infty} \{u_0(x,t), \theta(x,t), w_0(x,t)\} e^{-i\omega t} dt \tag{3}$$

and matrices (see Appendix A1)

$$\tilde{\mathbf{A}} = \begin{bmatrix} A_{11} & -A_{12} & 0 \\ -A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}; \tilde{\mathbf{\Pi}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{33} \\ 0 & -A_{33} & 0 \end{bmatrix}; \tilde{\mathbf{C}}(\omega) = \begin{bmatrix} \omega^2 I_{11} & -\omega^2 I_{12} & 0 \\ -\omega^2 I_{12} & \omega^2 I_{22} - A_{33} & 0 \\ 0 & 0 & \omega^2 I_{11} \end{bmatrix} \tag{4}$$

Seeking solution of equation (2) in the form $\mathbf{z}_0 = \mathbf{d}e^{\lambda x}$, we obtain so-called characteristic equation

$$\det[\lambda^2 \tilde{\mathbf{A}} + \lambda \tilde{\mathbf{\Pi}} + \tilde{\mathbf{C}}] = 0 \tag{5}$$

This is a cubic algebraic equation respect to $\eta=\lambda^2$ that can be elementarily solved and gives three roots η_1, η_2, η_3 (see Appendix A2). Therefore, we get

$$\lambda_{1,4} = \pm k_1; \lambda_{2,5} = \pm k_2; \lambda_{3,6} = \pm k_3; k_j = \sqrt{\eta_j}, j = 1,2,3 \tag{6}$$

General continuous solution of Eq. (2) can be now represented as

$$\{\mathbf{z}_0\} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{16} \\ d_{21} & d_{22} & \dots & d_{26} \\ d_{31} & d_{32} & \dots & d_{36} \end{bmatrix} \cdot \begin{Bmatrix} e^{\lambda_1 x} \\ \vdots \\ e^{\lambda_6 x} \end{Bmatrix} \tag{7}$$

Taking into account the first and last equations in (2) one gets

$$\begin{bmatrix} d_{11} & d_{12} & \dots & d_{16} \\ d_{21} & d_{22} & \dots & d_{26} \\ d_{31} & d_{32} & \dots & d_{36} \end{bmatrix} = \begin{bmatrix} \alpha_1 C_1 & \alpha_2 C_2 & \dots & \alpha_6 C_6 \\ C_1 & C_2 & \dots & C_6 \\ \beta_1 C_1 & \beta_2 C_2 & \dots & \beta_6 C_6 \end{bmatrix}$$

where $C_1,...,C_6$ are constants and

$$\alpha_j = \frac{\omega^2 I_{12}}{\omega^2 I_{11} + \lambda_j^2 A_{11}}; \beta_j = \frac{\lambda_j A_{33}}{(\omega^2 I_{11} + \lambda_j^2 A_{33})}; j = 1,2,...,6 .$$

Using the notations introduced in (6) it is easily to verify that

$$\alpha_4 = \alpha_1; \alpha_5 = \alpha_2; \alpha_6 = \alpha_3; \beta_4 = -\beta_1; \beta_5 = -\beta_2; \beta_6 = -\beta_3 .$$

Therefore, expression (7) can be now rewritten in the form

$$\mathbf{z}_0(x,\omega) = \mathbf{G}(x,\omega)\mathbf{C} , \tag{8}$$

where $\mathbf{C} = (C_1,...,C_6)^T$ and $\mathbf{G}(x,\omega) = [\mathbf{G}_1(x,\omega) \ \ \mathbf{G}_2(x,\omega)]$ are function matrices

$$\mathbf{G}_1(x,\omega) = \begin{bmatrix} e^{k_1 x} & e^{k_2 x} & e^{k_3 x} \\ \alpha_1 e^{k_1 x} & \alpha_2 e^{k_2 x} & \alpha_3 e^{k_3 x} \\ \beta_1 e^{k_1 x} & \beta_2 e^{k_2 x} & \beta_3 e^{k_3 x} \end{bmatrix}; \mathbf{G}_2(x,\omega) = \begin{bmatrix} e^{-k_1 x} & e^{-k_2 x} & e^{-k_3 x} \\ \alpha_1 e^{-k_1 x} & \alpha_2 e^{-k_2 x} & \alpha_3 e^{-k_3 x} \\ -\beta_1 e^{-k_1 x} & -\beta_2 e^{-k_2 x} & -\beta_3 e^{-k_3 x} \end{bmatrix} \tag{9}$$

It is assumed that the beam has been cracked at different positions $e_1,...,e_n$. Based on fracture mechanics (F. Erdogan, B.H. Wu 1997), the stiffness reduction of FGM beam caused by presence of the cracks can be modeled by equivalent springs of stiffness K_j (Ke et al., 2009). Therefore, conditions that must be satisfied at the cracks are (T.V. Lien, N.T. Duc and N.T. Khiem 2016)

$$\begin{gathered} U(e_j+0) = U(e_j-0); \Theta(e_j+0) = \Theta(e_j-0) + M(e_j)/K_j; W(e_j+0) = W(e_j-0) \\ N(e_j) = N(e_j+0) = N(e_j-0); Q(e_j+0) = Q(e_j-0); M(e_j+0) = M(e_j-0) = M(e_j) \end{gathered} \tag{10}$$

where N, Q and M are internal axial, shear forces and bending moment respectively (N.T. Khiem, N.D. Kien, N.N. Huyen 2014)

$$N = A_{11}U'_x - A_{12}\Theta'; M = A_{12}U'_x - A_{22}\Theta'_x; Q = A_{33}(W'_x - \Theta) \tag{11}$$

Substituting (11) into (10), one can rewrite the conditions (10) as follows

$$\begin{gathered} U(e_j+0)=U(e_j-0);\ \Theta(e_j+0)=\Theta(e_j-0)+\gamma_j\Theta'_x(e_j);\ W(e_j+0)=W(e_j-0) \\ U'_x(e_j+0)=U'_x(e_j-0);\ \Theta'_x(e_j+0)=\Theta'_x(e_j-0);\ W'_x(e_j+0)=W'_x(e_j-0)+\gamma_j\Theta'_x(e_j) \\ \gamma_j=A_{22}/K_j;\ j=1,2,3,\ldots,n \end{gathered} \tag{12}$$

So called crack magnitudes γ_j defined in (12) are function of the material properties such as Young's modulus, power law exponent n and cross sectional dimensions. For the FGM beam, the crack magnitude can be calculated as

$$\begin{gathered} \gamma_j=\gamma_b\theta_2(R_E,n) \\ \gamma_b=E_bI/R; I=bh^3/12\ ;\ \theta_2(R_E,n)=12\left(\frac{3R_E+n}{3(3+n)}-\frac{2R_E+n}{(2+n)}\alpha+\frac{R_E+n}{(1+n)}\alpha^2\right) \end{gathered} \tag{13}$$

In case of homogenous beam $E_t=E_b=E_0$ ($R_E=1$), the crack magnitudes can be calculated from crack depth a_j as

$$\begin{aligned} &\gamma_{j0}=E_0I/R_0=6\pi(1-\nu^2).h.f(z)\ ;\ z=a_j/h \\ &f(z)=z^2(0.6272-1.04533z+4.5948z^2-9.9736z^3+20.2948z^4 \\ &\quad -33.0351z^5+47.1063z^6-40.7556z^7+19.6z^8) \end{aligned} \tag{14}$$

Therefore, for modal analysis of cracked FGM beam, the crack magnitudes can be approximated by (Khiem and Huyen 2016)

$$\gamma_j=F(z)=6\pi.(1-\nu^2).h.\theta_2(R_E,n)f(z) \tag{15}$$

These functions would be used below for determining spring stiffness from given crack depth.

First, seeking solution $\mathbf{S}(x)$ of equation (2) satisfying the conditions

$$\mathbf{S}(0)=(0,1,0)^T;\mathbf{S}'(0)=(0,0,1)^T \tag{16}$$

one finds $\mathbf{S}=\{S_1,S_2,S_3\}^T$ where (T.V. Lien, N.T. Duc and N.T. Khiem 2016)

$$\begin{gathered} S_1(x)=\delta_1\cosh k_1x+\delta_2\cosh k_2x+\delta_3\cosh k_3x \\ S_2(x)=\delta_1\alpha_1\cosh k_1x+\delta_2\alpha_2\cosh k_2x+\delta_3\alpha_3\cosh k_3x \\ S_3(x)=\delta_1\beta_1\sinh k_1x+\delta_2\beta_2\sinh k_2x+\delta_3\beta_3\sinh k_3x \end{gathered} \tag{17}$$

$$\begin{gathered} \delta_1=\left[k_2\beta_2-k_3\beta_3+(\alpha_3-\alpha_2)\right]/\delta;\ \delta_2=\left[k_3\beta_3-k_1\beta_1+(\alpha_1-\alpha_3)\right]/\delta; \\ \delta_3=\left[k_1\beta_1-k_2\beta_2+(\alpha_2-\alpha_1)\right]/\delta;\ \delta=k_1\beta_1(\alpha_3-\alpha_2)+k_2\beta_2(\alpha_1-\alpha_3)+k_3\beta_3(\alpha_2-\alpha_1). \end{gathered} \tag{18}$$

Denoting solution of equation (2) in the interval (e_j,e_{j+1}) by $\mathbf{z}_j(x)$, it is easily to verify that

$$\mathbf{z}_j(x) = \mathbf{z}_{j-1}(x) + \gamma_j \Theta'_{j-1}(e_j)\mathbf{S}(x - e_j), \tag{19}$$

where $\mathbf{z}_{j-1}(x)$ is solution in (e_{j-1}, e_j) being continuously expended to the subsequent interval (e_j, e_{j+1}) and $\mathbf{S}(x)$ is defined in the form (17). Namely, since both functions $\mathbf{z}_{j-1}(x), \mathbf{S}(x - e_j)$ are solutions of Eq. (2) in (e_j, e_{j+1}), their combination in (19) would be solution of that equation in the interval. Moreover, solution (19) satisfies also the conditions

$$\begin{gathered} U_j(e_j) = U_{j-1}(e_j);\ \Theta_j(e_j) = \Theta_{j-1}(e_j) + \gamma_j \Theta'_{j-1}(e_j); W_j(e_j) = W_{j-1}(e_j) \\ U'_j(e_j) = U'_{j-1}(e_j); \Theta'_j(e_j) = \Theta'_{j-1}(e_j); W'_j(e_j) = W'_{j-1}(e_j) + \gamma_j \Theta'_j(e_j) \end{gathered} \tag{20}$$

These conditions ensure that solution of Eq. (2) in the form of (19) satisfy condition at crack positions (12). Based on the recurrent connection, one can express general solution of Eq. (2) for FGM beam with n crack in the form

$$\begin{gathered} \mathbf{z}_c(x) = \mathbf{z}_0(x) + \sum_{j=1}^{n} \mu_j \mathbf{K}(x - e_j) \\ \mu_j = \gamma_j \left[\Theta'_0(e_j) + \sum_{k=1}^{j-1} \mu_k S'_2(e_j - e_k) \right], j = 1,2,3,\dots,n \end{gathered} \tag{21}$$

In the later equation, $\mathbf{z}_0(x)$ is continuous solution found above in the form (7) and function $\mathbf{K}$(x) is

$$\mathbf{K}(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ \mathbf{S}(x) & \text{for} \quad x \succ 0 \end{cases}; \mathbf{K}'(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ \mathbf{S}'(x) & \text{for} \quad x \succ 0 \end{cases} \tag{22}$$

Suppose that boundary conditions for solution of Eq. (2), are represented by

$$\mathbf{B}_0 \{\mathbf{z}\}\big|_{x=0} = 0 ; \mathbf{B}_\mathrm{L} \{\mathbf{z}\}\big|_{x=L} = 0 \tag{23}$$

where $\mathbf{B}_0$, $\mathbf{B}_\mathrm{L}$ are differential matrix operators of dimension 3×3 given in Appendix A3. Since the second term of solution (21) satisfy any trivial condition at $x = 0$, the first condition in (23) is only applied for $\mathbf{z}_0(x)$. Splitting the constant vector $\mathbf{C} = \{\mathbf{C}_1, \mathbf{C}_2\}^T$ into $\mathbf{C}_1 = \{C_1, C_2, C_3\}^T ; \mathbf{C}_2 = \{C_4, C_5, C_6\}^T$, the boundary condition at the left end of the beam can be rewritten as

$$\begin{gathered} \mathbf{B}_{01}\mathbf{C}_1 + \mathbf{B}_{02}\mathbf{C}_2 = 0 \\ \mathbf{B}_{01}(\omega) = \mathbf{B}_0 \{\mathbf{G}_1(x, \omega)\}\big|_{x=0} ; \mathbf{B}_{02}(\omega) = \mathbf{B}_0 \{\mathbf{G}_2(x, \omega)\}\big|_{x=0} \end{gathered} \tag{24}$$

Eq. (24) allows eliminating one of the vectors $\mathbf{C}_1, \mathbf{C}_2$ and as result the solution $\mathbf{z}_0(x)$ can be reassembled as $\mathbf{z}_0(x, \omega) = \mathbf{G}_0(x, \omega)\mathbf{D}$ with $\mathbf{G}_0(x, \omega)$ is *3×3* dimension matrix function and arbitrary constant vector $\mathbf{D} = \{D_1, D_2, D_3\}^T$. In particular, one has

$$\Theta_0(x) = [g_{21}(x,\omega)D_1 + g_{22}(x,\omega)D_2 + g_{23}(x,\omega)D_3] \tag{25}$$

where $g_{2k}(x,\omega), k = 1,2,3$ are elements on second row of matrix $\mathbf{G}_0(x,\omega)$. So, solution (20) can be now expressed as

$$\mathbf{z}_c(x) = \mathbf{G}_0(x,\omega)\mathbf{D} + \sum_{j=1}^{n} \mu_j \mathbf{K}(x - e_j) \tag{26}$$

Satisfying boundary condition at right end of the beam leads to

$$\begin{gathered}[\mathbf{B}_{L0}(\omega)]\{\mathbf{D}\} + \sum_{j=1}^{n} \mu_j \{\mathbf{b}(e_j)\} = 0 \\ \mathbf{B}_{\mathbf{L}0}(\omega) = \mathbf{B}_{\mathbf{L}} \{\mathbf{G}_0(x,\omega)\}\big|_{x=L} ; \mathbf{b}_c(e_j) = \mathbf{B}_L \{\mathbf{S}(x - e_j)\}\big|_{x=L}\end{gathered} \tag{27}$$

In the case of intact beam, when $\mu_j = 0, j = 1,...,n$, equation (27) is reduced to

$$[\mathbf{B}_{L0}(\omega)]\{\mathbf{D}\} = 0 \tag{28}$$

that enables to determine undamaged natural frequencies by solving the equation

$$L_0(\omega) = \det[\mathbf{B}_{L0}(\omega)] = 0 \tag{29}$$

Each roots ω_j^0 of this equation is related to mode shape

$$\Phi_j^0(x) = C_j^0 \mathbf{G}_0(x,\omega_j^0)\overline{\mathbf{D}}_j \tag{30}$$

where C_j^0 is an arbitrary constant and $\overline{\mathbf{D}}_j$ is the normalized solution of (28) corresponding to ω_j^0.

For cracked beam, the constant vector $\mathbf{D}$ is sought in the form $\mathbf{D} = \sum_{j=1}^{n} \mu_j \mathbf{D}_j$ that leads the equation (27) to $[\mathbf{B}_{L0}(\omega)]\{\mathbf{D}_j\} = -\{\mathbf{b}_c(e_j)\}$ from that one is able to calculate

$$\mathbf{D}_j = -[\mathbf{B}_{L0}(\omega)]^{-1}\{\mathbf{b}_c(e_j)\} = -(1/L_0)\{\overline{\mathbf{b}}_c(e_j)\} \tag{31}$$

Therefore, solution (26) gets the form

$$\mathbf{z}_c(x) = (1/L_0)\sum_{j=1}^{n} \mu_j [L_0(\omega)\mathbf{K}(x - e_j) - \mathbf{G}_0(x,\omega)\overline{\mathbf{b}}_c(e_j)] \tag{32}$$

The solution (32) still contains unknown parameters $\mu_j, j = 1,...,n$. Substituting (25) together with constant vector $\mathbf{D}$ found above into (20), one gets

$$\mu_j = (\gamma_j / L_0)\sum_{k=1}^{n} \mu_k [L_0(\omega) K_2'(e_j - e_k) - g'_{jk}],\ j = 1,2,3,...,n$$
$$g'_{jk} = g'_{21}(e_j,\omega)\bar{b}_{c1}(e_k) + g'_{22}(e_j,\omega)\bar{b}_{c2}(e_k) + g'_{23}(e_j,\omega)\bar{b}_{c3}(e_k)$$

The above equation can be rewritten in the matrix form

$$\begin{gathered}[L_0(\omega)\mathbf{I} - \Gamma(\boldsymbol{\gamma})\mathbf{A}(\mathbf{e},\omega)]\{\boldsymbol{\mu}\} = 0 \\ \Gamma(\boldsymbol{\gamma}) = diag\{\gamma_1,...,\gamma_n\};\mathbf{A}(\mathbf{e},\omega) = [a_{jk} = L_0(\omega).K_2'(e_j - e_k) - g'_{jk};\ j,k = 1,2,...,n] \\ \boldsymbol{\mu} = \{\mu_1,...,\mu_n\}^T;\boldsymbol{\gamma} = \{\gamma_1,...,\gamma_n\}^T;\mathbf{e} = \{e_1,...,e_n\}^T\end{gathered} \tag{33}$$

Condition for existence of non-trivial solution of Eq. (33) is

$$f(\omega,\boldsymbol{\gamma},\mathbf{e}) \equiv \det[L_0(\omega)\mathbf{I} - \Gamma(\boldsymbol{\gamma})\mathbf{A}(\mathbf{e},\omega)] = 0 \tag{34}$$

This is frequency equation for FGM beam with arbitrary number of cracks, solution of which gives natural frequencies (ω_j, $j=1,2,3,..$). In particularity, when vector $\boldsymbol{\gamma} = 0$, Eq. (34) becomes Eq. (29).

Once a natural frequency has been found, we can determine vector $\boldsymbol{\mu}_j$ from the Eq. (33) as an eigenvector $\boldsymbol{\mu}_j = \{\mu_{j1},...,\mu_{jn}\}^T$ that enables to determine mode shape related to the natural frequency ω_j as

$$\mathbf{\Phi}_j(x) = \frac{1}{L_0(\omega_j)}\sum_{k=1}^{n} \mu_{jk}[L_0(\omega_j)\mathbf{K}(x - e_k) - \mathbf{G}_0(x,\omega_j)\bar{\mathbf{b}}_c(e_k)] \tag{35}$$

Since the eigenvector $\boldsymbol{\mu}_j$ is determined with an arbitrary constant that could be specified by using a normality condition, for instance

$$\max\{\mathbf{\Phi}_j(x)\} = 1 \tag{36}$$

3 ANALYSIS OF MODE SHAPES OF MULTIPLE CRACKED FGM TIMOSHENKO BEAM

3.1 Comparison in Particular Cases

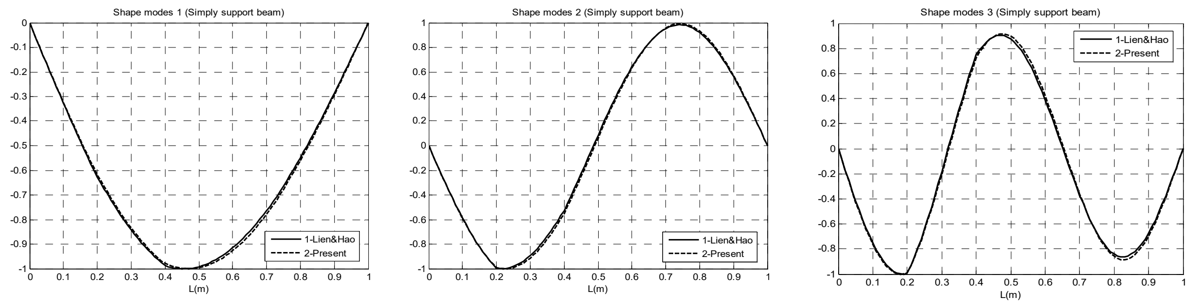
For validation of the obtained above equations in this subsection we compare natural frequencies and mode shapes computed by the equations (33) and (35) with those obtained by using other methods in the cases of homogeneous beam and uncracked FGM beam.

a) *Homogeneous beam with open edge cracks:* A simple support homogeneous beam with material and geometric parameters as follows: $E=210GPa$, $\rho=7800kg/m^3$, $\mu=0.25$, $L=1.0m$, $b=0.1m$, $h=0.1m$. There are two cracks in the beam at position $0.2m$ and $0.4m$ with the depth $a/h=30\%$ (Tran Van Lien, Trinh Anh Hao 2013).

Natural frequencies and mode shapes of three lowest modes computed for the beam and compared to those obtained by Lien and Hao (2013) are shown in Table 1 and Fig. 2. It is easily to see a good agreement of the obtained results.

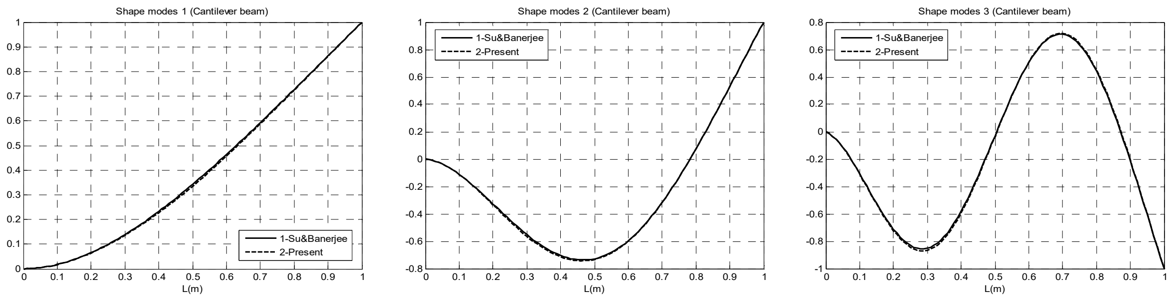
b) Intact FGM beam: An intact cantilever FGM Timoshenko beam with aluminum Al_2O_3 in the top: $E_t=390GPa$, $\rho_t=3960kg/m^3$, $\mu_t=0.25$ and steel at the bottom: $E_b=210GPa$, $\rho_b=7800kg/m^3$, $\mu_b=0.31$ is examined for power law exponent $n=1$ and geometry dimensions $L=1.0m$, $b=0.1m$, $h=0.1m$ (H. Su, J.R. Banerjee 2015).

Three lowest frequencies and mode shapes of the FGM beam obtained by using the presented above theory are compared with those given in H. Su, J.R. Banerjee (2015) and illustrated in Table 1 and Fig. 3. It can be observed that the calculated results are very close to the results of Su & Banerjee (2015).



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 2: Comparison of three lowest mode shapes for simple support Timoshenko beam with 2 cracks at position *0.2m* and *0.4m*, crack depth *a/h=30%*.



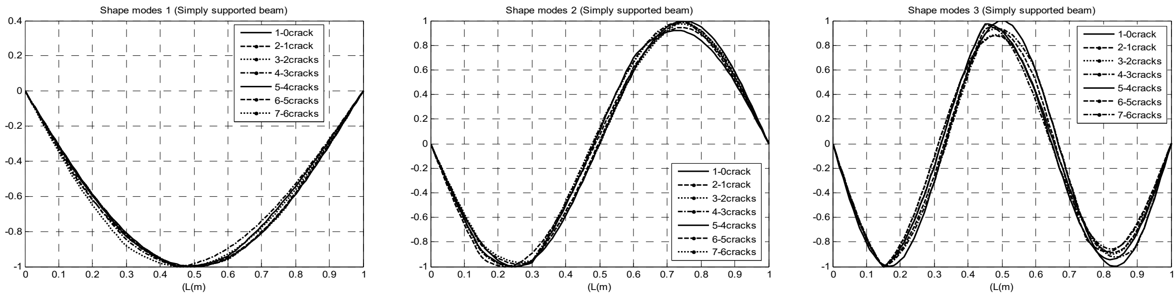
a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 3: Comparison of the first three mode shapes with Su & Banerjee (2015) for intact cantilever FGM Timoshenko beam.

Figs	Cases	Freq. 1 (rad/s)	Freq. 2 (rad/s)	Freq. 3 (rad/s)
2	Present	1316.9	5131.4	10743
	Lien & Hao	1330.7	5162.5	10803
3	Present	708.5	4248.7	11086
	Su & Banerjee	719.0	4314.0	11220
4	0 crack	990.53	3782.6	7963.2
	1 crack	972.22	3584.8	7464.5
	2 cracks	919.77	3403.7	7463.9
	3 cracks	856.15	3402.4	7016.1
	4 cracks	808.42	3274.1	6971.5
	5 cracks	785.25	3073.8	6662.5
	6 cracks	781.0	3016.6	6439.2
5	a/h=0%	990.53	3782.6	7963.2
	a/h=10%	973.11	3758.3	7918.2
	a/h=20%	927.29	3689.6	7806.4
	a/h=30%	858.76	3572.1	7656.0
	a/h=40%	771.73	3393.3	7491.0
	a/h=50%	669.14	3131.8	7328.9
6	a/h=10%	987.12	3751.1	7903.8
	a/h=20%	977.39	3664.6	7752.3
	a/h=30%	960.34	3526.3	7540.6
7	a/h=10%	2141.4	5472.1	9855.6
	a/h=20%	2140.5	5430.4	9702.7
	a/h=30%	2139.0	5365.0	9481.5
8	a/h=10%	352.26	2131.6	5548.5
	a/h=20%	342.07	2131.3	5502.9
	a/h=30%	325.72	2130.8	5418.2
9	a/h=10%	966.69	3697.1	7798.0
	a/h=20%	906.19	3478.2	7367.6
	a/h=30%	820.88	3165.0	6736.2
10	a/h=10%	2120.6	5417.1	9758.9
	a/h=20%	2064.6	5237.8	9357.3
	a/h=30%	1979.2	4982.5	8781.0
11	a/h=10%	350.39	2095.6	5477.2
	a/h=20%	335.52	2002.4	5234.9
	a/h=30%	312.86	1866.2	4874.6
12	1 crack	960.34	3526.3	7540.6
	2 cracks	892.98	3478.9	7256.2
	3 cracks	839.06	3353.7	7185.8
	4 cracks	820.88	3165.0	6736.2
13	1 crack	2139.0	5365.0	9481.5
	2 cracks	2052.0	5247.7	9229.8
	3 cracks	1987.4	5054.7	9181.3
	4 cracks	1979.2	4982.5	8781.0
14	1 crack	325.72	2130.8	5418.2
	2 cracks	315.48	2014.1	5297.4
	3 cracks	313.05	1889.2	5093.8
	4 cracks	312.86	1866.2	4874.6

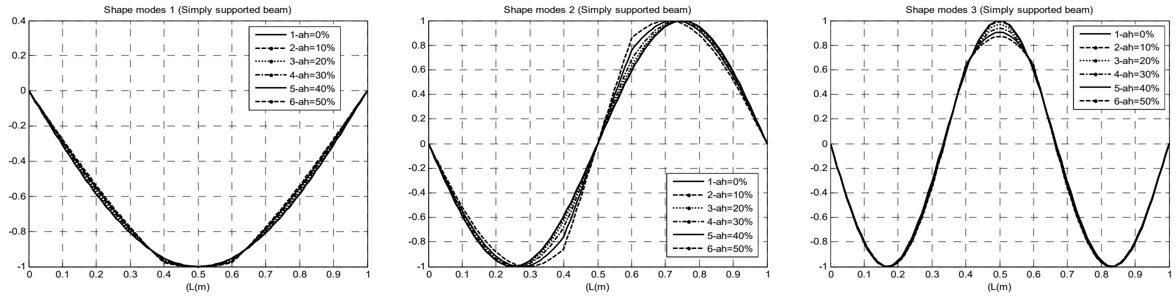
Table 1: Natural frequencies corresponding to the mode shapes illustrated in the Figures 2-14.

3.2 Change in Mode Shapes of Multiple Cracked FGM Timoshenko Beam



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

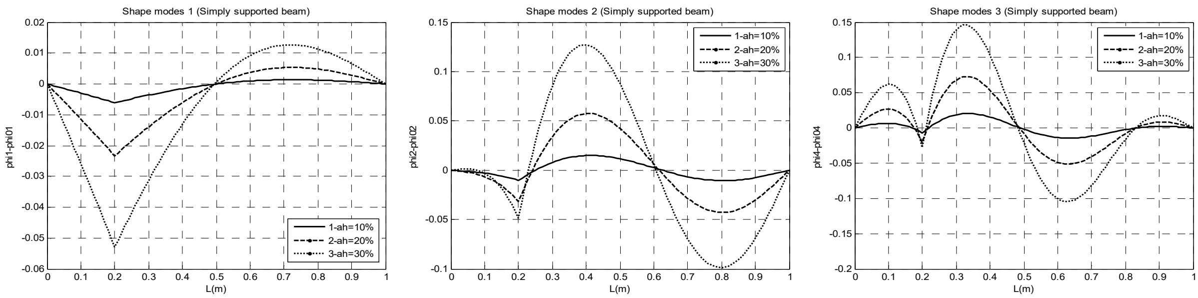
Figure 4: Change in first three mode shapes of simple support FGM Timoshenko beam which has 0 to 6 equidistant cracks (*Δ=0.15m*) of the same depth *a/h=30%*.



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

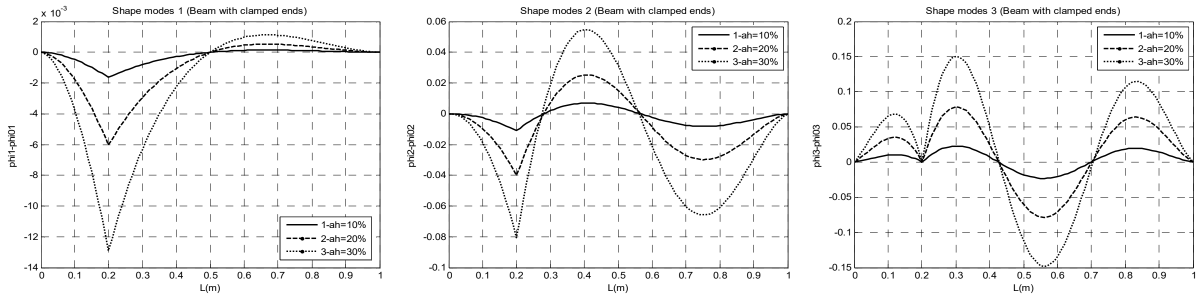
Figure 5: Change in first three mode shapes of simple support FGM beam with 2 cracks at 0.4m and 0.6m and depth varying from *a/h=0%* to *50%*.

In this section, the cracked FGM beam of material properties: $E_t=70GPa$, $\rho_t=2780kg/m^3$, $\mu_t=0.33$, $E_b/E_t=0.5$; $\rho_b=7850kg/m^3$, $\mu_b=0.33$, $n=0.5$ and geometric parameters: $L=1.0m$, $b=0.1m$, $h=0.1m$ is investigated. Three lowest mode shapes of a simply supported FGM Timoshenko beam with various number (from 0 to 6) equidistant cracks of the same depth *a/h=30%* are shown in Fig. 4 and the mode shapes in case of 2 cracks (at *0.4m* and *0.6m*) with equal depth varying from *0%* to *50%* are presented in Fig. 5. It is observed variation of the mode shapes caused by number of cracks and their depth. However, location of the cracks is difficult to detect by observing only the mode shape graphs, it could be identified by comparison with mode shapes of uncracked beam that is demonstrated in the subsequent Figures.



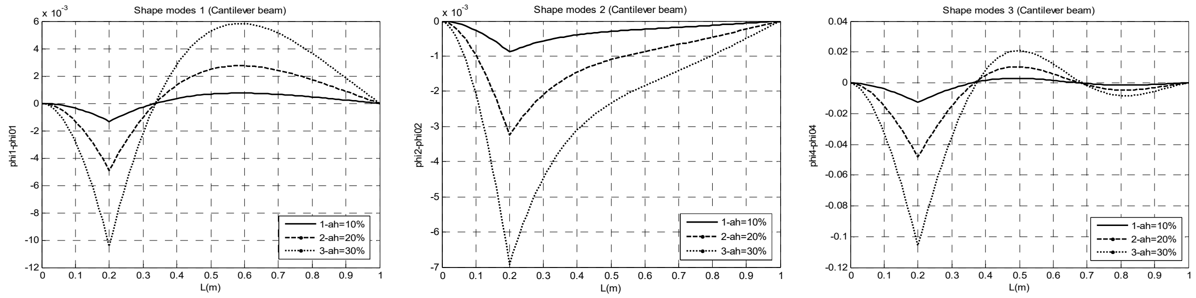
a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 6: Change in first three mode shapes of simple support FGM beam with a crack at 0.2m and depth varying from *a/h=10%* to *30%*.



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

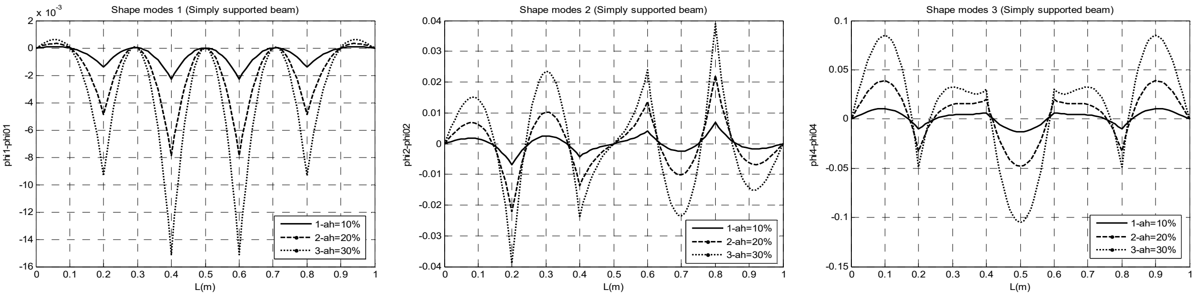
Figure 7: Change in first three mode shapes of clamped end FGM beam with a crack at 0.2m and depth varying from *a/h=10%* to *30%*.



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

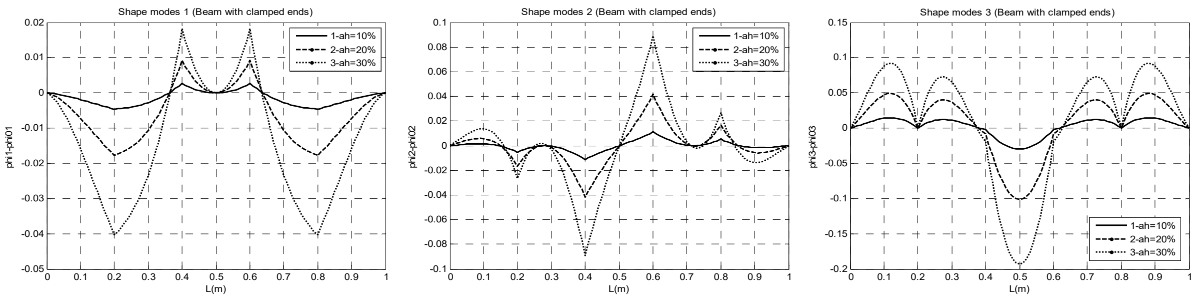
Figure 8: Change in first three mode shapes of cantilever FGM beam with a crack at 0.2m and depth varying from *a/h=10%* to *30%*.

Deviation of the cracked mode shapes from the intact ones are computed for the beam with single crack at 0.2m (Figs. 6-8) and 4 equidistant cracks (Figs. 9-11) of various depth (10, 20, 30%) in different cases of boundary conditions. Natural frequencies corresponding to the mode shapes presented in the Figures are tabulated in Table 1. The deviations of mode shapes are computed also for the beam with various number of cracks (from 1 to 4) with the same depth 30% and shown in Figs. 12-14.



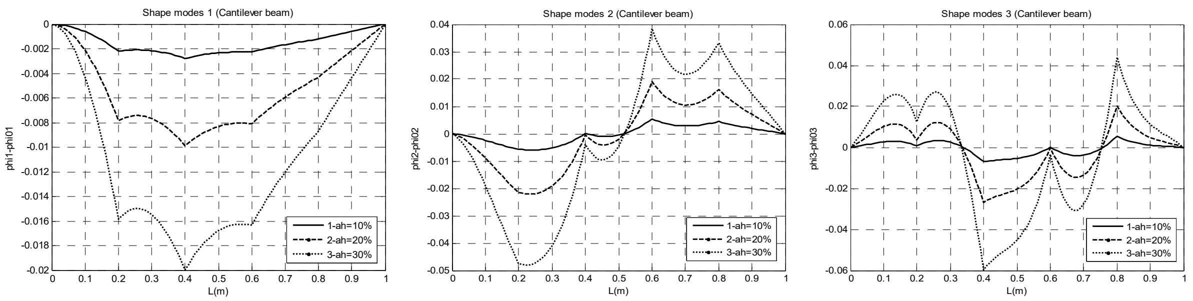
a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 9: Change in first three mode shapes of simple support FGM beam with 4 equidistant cracks and depth *a/h=10%, 20%, 30%.*



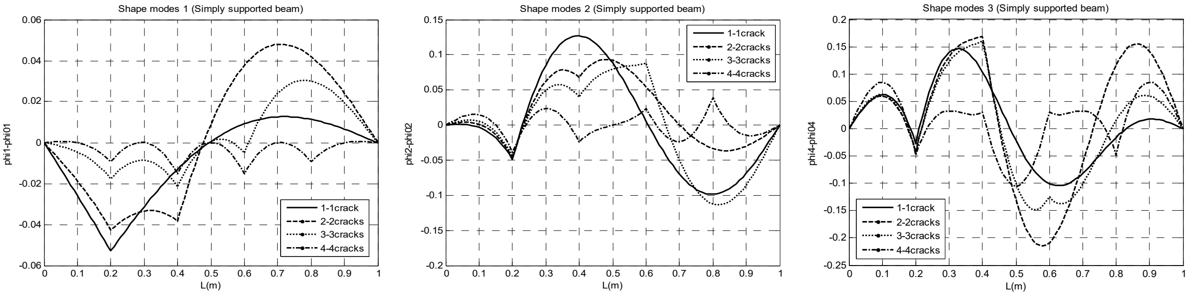
a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 10: Change in first three mode shapes of clamped end FGM beam with 4 equidistant cracks of the depth *a/h=10%, 20%, 30%.*



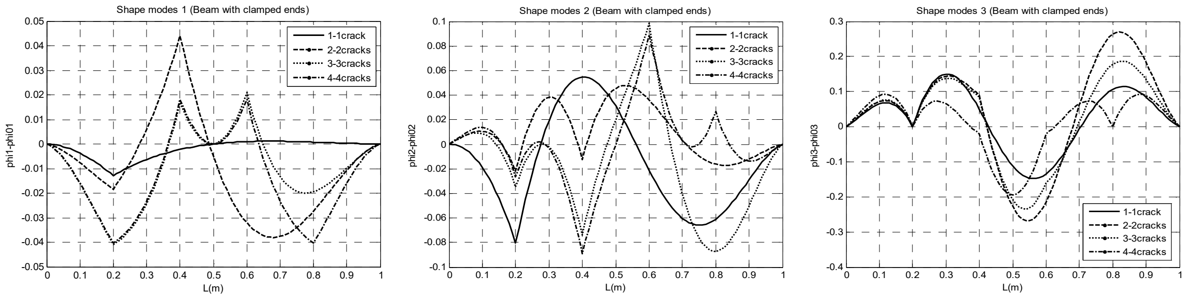
a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 11: Change in first three mode shapes of cantilever FGM beam with 4 equidistant cracks of the depth *a/h=10%, 20%, 30%.*



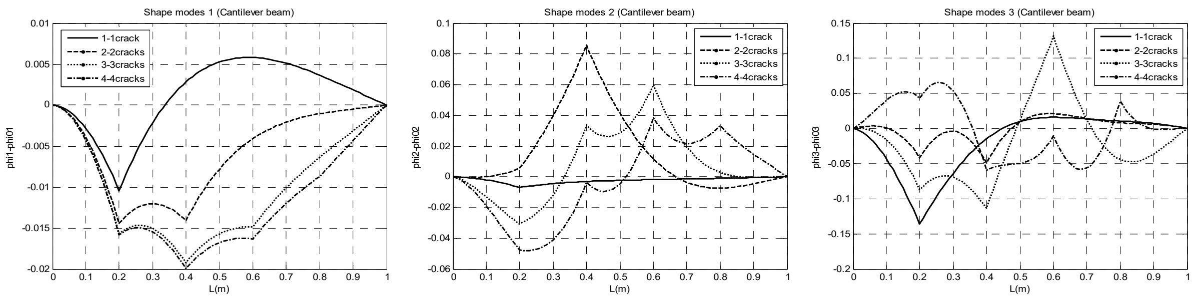
a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 12: Change in first three mode shapes of simple support FGM beam with various number of cracks (from 1 to 4) and depth *a/h=30%*.



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 13: Change in first three mode shapes of clamped end FGM beam with various number of cracks (from 1 to 4) and depth *a/h=30%*.



a. Mode shape 1 b. Mode shape 2 c. Mode shape 3

Figure 14: Change in first three mode shapes of cantilever FGM beam with various number of cracks (from 1 to 4) and depth *a/h=30%*.

Observing graphs in the Figures allows one to make some remarks as follows:

a) Deviation of mode shapes is typically non-smooth (sharp peak) at crack positions, so that crack location could be easily discriminated by using the wavelet analysis of mode shapes.

b) Height of the sharp peaks in the graphs of mode shape deviation is monotonically increasing with crack depth. This enables to estimate also crack depth by the wavelet coefficient of mode shape at the crack location;

c) Effect of symmetric cracks on the mode shape is the same for beam with symmetric boundary conditions and symmetric cracks make no change in mode shape at the beam middle.

d) There are some positions on beam that presence of crack at these points makes no change in a mode shape. Such the points on beam are called invariable for the mode shape. For instance, midpoint of simply supported beam (x=0.5m) is invariable point for the fundamental mode shape (Fig.6a) or x=0.356m and x=0.67m – are invariable for the third mode shape of cantilever beam (Fig.8c).

All the mentioned notices are useful indication for crack detection in FGM beam by measurements of mode shapes.

4 CONCLUSIONS

In this paper, the consistent theory of free vibration of multiple cracked FGM Timoshenko beam is formulated on the base of the power law distribution of FGM material, rotation spring model of crack and actual position of neutral axis.

The obtained frequency equation and mode shape of cracked FGM Timoshenko beam provide a simple approach to study not only free vibration of the beam but also the inverse problem of material and crack identification in FGM structures.

Numerical analysis demonatrates that mode shapes of FGM Timoshenko beam are sufficiently sensitive to cracks and dependent on material properties and geometric parameter of the beam.

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APPENDIX

A1. Constants in formula (4)

$$(A_{11}, A_{12}, A_{22}) = \int_A E(z)\left(1, z-h_0, (z-h_0)^2\right)dA;\ A_{33} = \kappa\int_A G(z)dA;$$

$$(I_{11}, I_{12}, I_{22}) = \int_A \rho(z)\left(1, z-h_0, (z-h_0)^2\right)dA.$$

For the power law (1), we get

$$A_{11} = bh\frac{E_t + nE_b}{1+n}\ ;\ A_{12} = bh^2\left(\frac{2E_t + nE_b}{2(2+n)} - \frac{E_t + nE_b}{1+n}\alpha\right);\ \alpha = \frac{1}{2} + \frac{h_0}{h}$$

$$A_{22} = bh^3\left(\frac{3E_t + nE_b}{3(3+n)} - \frac{2E_t + nE_b}{2+n}\alpha + \frac{E_t + nE_b}{1+n}\alpha^2\right);\ A_{33} = \kappa bh\frac{G_t + nG_b}{1+n}\ ;\ I_{11} = bh\frac{\rho_t + n\rho_b}{1+n}$$

$$I_{12} = bh^2\left(\frac{2\rho_t + n\rho_b}{2(2+n)} - \frac{\rho_t + n\rho_b}{1+n}\alpha\right);\ I_{22} = bh^3\left(\frac{3\rho_t + n\rho_b}{3(3+n)} - \frac{2\rho_t + n\rho_b}{2+n}\alpha + \frac{\rho_t + n\rho_b}{1+n}\alpha^2\right)$$

A2. General solution of cub algebraic equation

$$\eta^3 + a\eta^2 + b\eta + c = 0$$

where

$$a = \omega^2\left[\frac{I_{11}}{A_{33}} + \frac{I_{11}A_{22} + I_{22}A_{11}}{A_{11}A_{22}}\right];\ b = \omega^4\left[\frac{I_{11}I_{22} - I_{12}^2}{A_{11}A_{22}} + \frac{I_{11}}{A_{33}}\frac{I_{11}A_{22} + I_{22}A_{11}}{A_{11}A_{22}}\right] - \omega^2\frac{I_{11}}{A_{22}}$$

$$c = \omega^4\left[\frac{\omega^2 I_{11}}{A_{33}}\frac{I_{11}I_{22} - I_{12}^2}{A_{11}A_{22}} - \frac{I_{11}^2}{A_{11}A_{22}}\right]$$

Roots of cub algebraic equation are $\eta_1(\omega), \eta_2(\omega), \eta_3(\omega)$

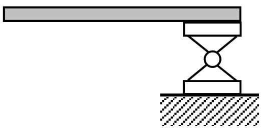
$$\eta_1 = -a/3 + u - b_1/u\ ;\ \eta_{2,3} = -a/3 - (u - b_1/u)/2 \pm i\sqrt{3}(u + b_1/u)/2$$

where

$$u = (a_1 + \sqrt{b_1^3 + c_1^2} - a^3/27)^{1/3}\ ;\ a_1 = ab/6 - c/2\ ;\ b_1 = b/3 - a^2/9; c_1 = a^3/27 - a_1$$

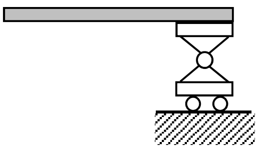
A3. Differential matrix operators

– **S**imply supported (S): $u(x,t) = M(x,t) = w(x,t) = 0$



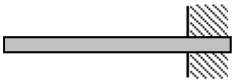
$$[\mathbf{B}_S] = \begin{bmatrix} 1 & 0 & 0 \\ A_{12}\partial_x & A_{22}\partial_x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– **P**ined (P): $N(x,t) = M(x,t) = w(x,t) = 0$



$$[\mathbf{B}_P] = \begin{bmatrix} A_{11}\partial_x & -A_{12}\partial_x & 0 \\ A_{12}\partial_x & A_{22}\partial_x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– **C**lamped (C): $u(x,t) = \theta(x,t) = w(x,t) = 0$



$$\left[\mathbf{B}_C\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– Free (F): $N(x,t) = M(x,t) = Q(x,t) = 0$

$$\left[\mathbf{B}_\mathbf{F}\right] = \begin{bmatrix} A_{11}\partial_x & -A_{12}\partial_x & 0 \\ A_{12}\partial_x & A_{22}\partial_x & 0 \\ 0 & -A_{33} & A_{33}\partial_x \end{bmatrix}$$

- **S**imply supported beam ends (SS): $\left[\mathbf{B}_0\right] = \left[\mathbf{B}_\mathbf{L}\right] = \left[\mathbf{B}_S\right]$
- Pined ends (PP): $\left[\mathbf{B}_0\right] = \left[\mathbf{B}_\mathbf{L}\right] = \left[\mathbf{B}_P\right]$
- Simple beam (SP): $\left[\mathbf{B}_0\right] = \left[\mathbf{B}_\mathbf{S}\right]; \left[\mathbf{B}_\mathbf{L}\right] = \left[\mathbf{B}_P\right]$
- Cantilevered beam (CF): $\left[\mathbf{B}_0\right] = \left[\mathbf{B}_C\right]; \left[\mathbf{B}_\mathbf{L}\right] = \left[\mathbf{B}_F\right]$
- Clamped beam (CC): $\left[\mathbf{B}_0\right] = \left[\mathbf{B}_\mathbf{L}\right] = \left[\mathbf{B}_C\right]$

NOMENCLATURE

E, G, ρ	Young's, shear modulus and materila density
n	Power law exponent
b, h, L	width, height and lenght of the beam
$u_0(x,t), \theta(x,t), w_0(x,t)$	Axial displacement, rotation and deflection
U, Θ,W	Amplitude of axial displacement, rotation and deflection
$\mathbf{z}$	Vector $\mathbf{z} = \{U, \Theta, W\}^T$
η_1, η_2, η_3	Three roots of characteristic equation
N, Q, M	Internal axial, shear forces and bending moment
$e_1, \ldots, e_n$	Crack positions
γ_j	j^{th} crack magnitudes
S(x), **K**(x)	Crack functions
μ_j	Recurrent coefficients
$\mathbf{z}_j$	Solution of Eq. (2) in subsequent interval (e_j, e_{j+1})
$\mathbf{z}_0$, $\mathbf{z}_c$	General solution of Eq. (2) for intact and cracked beam
$\mathbf{B}_0$, $\mathbf{B}_L$	Differential matrix operators of boundary conditions
ω, Φ	Natural frequency, mode shape