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# Dynamic Analysis of Imperfect FGM Circular Cylindrical Shells Reinforced by FGM Stiffener System Using Third Order Shear Deformation Theory in Term of Displacement Components

**Abstract**

This paper presents dynamic analysis of an eccentrically stiffened imperfect circular cylindrical shells made of functionally graded materials (FGM), subjected to axial compressive load and filled inside by elastic foundations in thermal environments by analytical method. Shells are reinforced by FGM stringers and rings taking into account thermal elements. The stability equations in terms of displacement components for stiffened shells are derived by using the third-order shear deformation theory and smeared stiffeners technique.The closed-form expressions for determining the natural frequency, nonlinear frequency-amplitude curve and nonlinear dynamic response are obtained by using Galerkin method and fourth-order Runge-Kutta method. The effects of stiffeners, foundations, imperfection, material and dimensional parameters pre-existent axial compressive and thermal load on dynamic responses of shells are considered.

**Keywords**

Analytical; Dynamic analysis; Elastic foundation; Functionally graded material; Stiffened cylindrical shell; Vibration.

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## 1 INTRODUCTION

In recent decades, functionally graded material stiffened shells are more widely used in modern engineering structures as tunnels, pipelines, pressure vessels, storage tanks and in other applications. The structures are often strongly acted by forces depending on time leading to instability of work. Thus, their nonlinear dynamic stability analysis is one of interesting and important problems and has received considerable attention of researchers.

For un-stiffened shells, many researches focused on the vibration analysis of un-stiffened shells. Bich and Nguyen (2012) presented nonlinear vibration of functionally graded circular cylindrical shells based on improved Donnell equations. Loy et al. (1994 and 2001) considered vibration of functionally graded and laminated cylindrical shells. Lam and Loy (2000) researched vibration of thin rotating laminated composite cylindrical shells. Sheng and Wang (2008 and 2010) considered the effect of thermal load on buckling, vibration and dynamic buckling of FGM cylindrical un-stiffened shells embedded in a linear elastic medium based on the first-order shear deformation theory (FSDT) taking into account the rotary inertia and transverse shear strains. Some investigations on the vibration analysis of FGM un-stiffened cylindrical shells resting on the Pasternak elastic foundation have been published by Sofiyev et al. (2013). Huang and Han (2010) presented nonlinear dynamic buckling of functionally graded cylindrical shells subjected to time dependent axial load. Bahadori and Najafizadeh (2015) showed free vibration analysis of two-dimensional functionally graded axisymmetric cylindrical shell on Winkler–Pasternak elastic foundation by first-order shear deformation theory and using Navier-differential quadrature solution methods. Sofiyev et al. (2013, 2015) gave influences of shear stresses on the dynamic instability of exponentially graded sandwich cylindrical shells by using the shear deformation theory and classical shell theory. The same author analyzed torsional vibration and stability of functionally graded orthotropic cylindrical shells on elastic foundations. Shen and Wang (2014) presented nonlinear vibration of shear deformable FGM cylindrical panels resting on elastic foundations in thermal environments. Sofiyev et al. (2015) showed stability and vibration of sandwich cylindrical shells containing a functionally graded material core with transverse shear stresses and rotary inertia effects. Besides, Sofiyev (2015) also studied influences of shear stresses on the dynamic instability of exponentially graded sandwich cylindrical shells. Bahadori and Najafizadeh (2015) analyzed free vibration analysis of two-dimensional functionally graded axisymmetric cylindrical shell on Winkler–Pasternak elastic foundation by First-order Shear Deformation Theory and using Navier-differential quadrature solution methods.

As can be seen the above introduced results only relate to un-stiffened structures. However, in practice, plates and shells including cylindrical shells usually are reinforced by stiffeners system to provide the benefit of added load carrying capability with a relatively small additional weight. Thus, the study on dynamic behavior of those structures is significant practical problem.

For stiffened shells, many studies were carried out with eccentrically stiffened shells made of homogenous materials. Najafizadeh and Isvandzibaei (2007) showed vibration of functionally graded cylindrical shells based on higher order shear deformation plate theory with ring support. These authors (2009) also presented vibration of functionally graded cylindrical shells based on different shear deformation shell theories with ring support under various boundary conditions. Bich et al (2013) studied the nonlinear static and dynamical buckling analysis of imperfect eccentrically stiffened functionally graded circular cylindrical thin shells under axial compression. Lei et al (2014) presented dynamic stability analysis of carbon nanotube-reinforced functionally graded cylindrical panels using the element free kp-Ritz method. Nonlinear dynamic analysis of eccentrically stiffened functionally graded circular cylindrical thin shells under external pressure and surrounded by an elastic medium was analyzed by Dung and Nam (2014). Dung and Hoa (2015) presented a semi-analytical method for analyzing the nonlinear dynamic behavior of FGM cylindrical shells surrounded by an elastic medium under time-dependent torsional loads based on the classical shell theory with the deflection function correctly represented by three terms. The material properties of shell and stiffeners are assumed to be continuously graded in the thickness direction. Duc and Quan (2015) studied nonlinear dynamic analysis of imperfect FGM double curved thin shallow shells with temperature-dependent properties on elastic foundation. Nonlinear dynamic response and vibration of shear deformable imperfect eccentrically stiffened S-FGM circular cylindrical shells surrounded on elastic foundations was reseached by Duc and Thang (2015). Duc (2016) studied nonlinear thermal dynamic analysis of eccentrically stiffened S-FGM circular cylindrical shells surrounded on elastic foundations using the Reddy's third-order shear deformation shell theory through stress function in thermal environment.

With the plates or other kinds of shells, there are many available results. Sofiyev (2009) analyzed the vibration and stability behavior of freely supported un-stiffened FGM conical shells subjected to external pressure by Galerkin method. The same author (2012) analyzed the nonlinear vibration of un-stiffened FGM truncated conical shells by analytical approach. Based on the First order shear deformation theory (FSDT), Malekzadeh and Heydarpour (2013) studied effects of centrifugal and Coriolis, of geometrical and material parameters on the free vibration behavior of rotating FGM un-stiffened truncated conical shells subjected to different boundary conditions. Lei et al (2015) investigated free vibration analysis of laminated functionally graded carbon nanotube (FG-CNT) reinforced composite rectangular plates using the kp-Ritz method. By using the element-free kp-Ritz method, these authors (2016) also presented analysis of laminated CNT reinforced functionally graded plates. Dung and Vuong (2016) showed nonlinear analysis on dynamic buckling of eccentrically stiffened functionally graded material toroidal shell segment surrounded by elastic foundations in thermal environment and under time-dependent torsional loads. Dung et al. (2014 and 2016) investigated the static buckling and vibration of FGM conical shells reinforced by FGM stiffeners under axial compressive load and external pressure by analytical method. The change of distance between stringers is considered in these work.

A novelty of the present study is to present an analytical method for investigate dynamic response of imperfect FGM circular cylindrical shells reinforced by FGM stiffener system and filled inside by an elastic foundations, in thermal environments. Theoretical formulations in terms of displacement components according to Reddy's third-order shear deformation shell theory (2004) and the smeared stiffeners technique are derived. The thermal elements of shells and stiffeners are taken into account in two cases which are uniform temperature rise law and nonlinear temperature change. The closed-form expressions for determining the natural frequency, nonlinear frequency-amplitude curve and nonlinear dynamic response are obtained by using Galerkin method and fourth-order Runge-Kutta method. The effects of stiffener, temperature, foundation, material and dimensional parameters, pre-existent axial compressive and on the stability of stiffened FGM shells are considered.

## 2 FUNDAMENTAL EQUATIONS OF ECCENTRICALLY STIFENED-FUNCTIONALLY GRADED MATERIAL SHELLS (ES-FGM SHELLS)

### 2.1 Functionally Graded Material Shells

Consider a thin circular cylindrical shell is made of ceramic and metal, with mean radius R, thickness h and length L subjected to axial compressive load P, external uniform pressure q and thermal load. Assume that the shell is simply supported at two butt-ends. The middle surface of the shells is referred to the coordinates x, y, z as shown in Fig. 1. Further, assume that the shell is stiffened by closely spaced circular rings and longitudinal stringers.The quantity $z_1, z_2$ epresents the eccentricity (Figure 1). It means that the distance from the shell middle surface to the stringer centroid $z_1$ (the stringer eccentricity) and the distance from the conical shell middle surface to the ring centroid $z_2$ (the ring eccentricity). Besides, the cylindrical shell is filled with elastic foundations represented by two foundation parameter $K_1$ and $K_2$ which are the Winkler foundation stiffness and shearing layer stiffness of the Pasternak foundation, respectively.

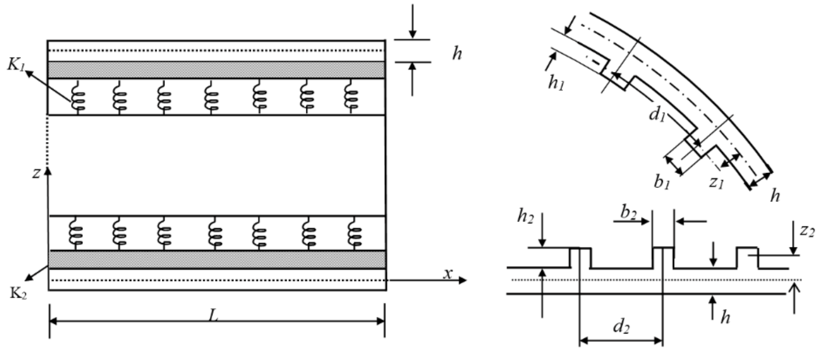


**Figure 1**: Geometry and coordinate system of a stiffened FGM circular cylindrical shell.

Functionally graded material of shell in this paper is assumed to be made of a mixture of ceramic and metal with a power law. Then the Young moduli $E$, thermal expansion coefficient $\alpha$, thermal conductivity coefficient $K$ and density mass $\rho$ can be expressed in the form:

For shells

$$\begin{gathered}
E_{sh}\left(z\right) = E_m + \left(E_c - E_m\right)\left(\frac{2z+h}{2h}\right)^k;\\
\alpha_{sh}\left(z\right) = \alpha_m + \left(\alpha_c - \alpha_m\right)\left(\frac{2z+h}{2h}\right)^k;\\
K_{sh}\left(z\right) = K_m + \left(K_c - K_m\right)\left(\frac{2z+h}{2h}\right)^k, -h/2 \le z \le h/2,\ k \ge 0;\\
\rho_{sh}\left(z\right) = \rho_m + \left(\rho_c - \rho_m\right)\left(\frac{2z+h}{2h}\right)^k;
\end{gathered} \tag{1}$$

For stringers and rings

$$\begin{gathered}
E_s\left(z\right) = E_c + \left(E_m - E_c\right)\left(\frac{2z-h}{2h_1}\right)^{k_2},\ h/2 \le z \le h/2 + h_1;\\
E_r\left(z\right) = E_c + \left(E_m - E_c\right)\left(\frac{2z-h}{2h_2}\right)^{k_3}, h/2 \le z \le h/2 + h_2;
\end{gathered} \tag{2}$$

$$\alpha_s(z) = \alpha_c + (\alpha_m - \alpha_c)\left(\frac{2z-h}{2h_1}\right)^{k_2}, h/2 \le z \le h/2 + h_1;$$

$$\alpha_r(z) = \alpha_c + (\alpha_m - \alpha_c)\left(\frac{2z-h}{2h_2}\right)^{k_3}, h/2 \le z \le h/2 + h_2;$$

$$K_s(z) = K_c + (K_m - K_c)\left(\frac{2z-h}{2h_1}\right)^{k_2}, h/2 \le z \le h/2 + h_1;$$

$$K_r(z) = K_c + (K_m - K_c)\left(\frac{2z-h}{2h_2}\right)^{k_3}, h/2 \le z \le h/2 + h_2;$$

$$\rho_s(z) = \rho_c + (\rho_m - \rho_c)\left(\frac{2z-h}{2h_1}\right)^{k_2}, h/2 \le z \le h/2 + h_1;$$

$$\rho_r(z) = \rho_c + (\rho_m - \rho_c)\left(\frac{2z-h}{2h_2}\right)^{k_3}, h/2 \le z \le h/2 + h_2;$$

where the volume fraction index $k \ge 0$; and $h$ is the thickness of shell; z is the thickness coordinate varing from $-h/2$ to $h/2$; the subscripts $m$ and $c$ refer to the metal and ceramic constituents respectively; the subscripts sh, s, r indicate shell, stringer, ring respectivly; $k_2, k_3$ are volume fractions indexes of stringer and ring, respectively.

Note $k_2 = k_3 = 1/k$, when $k_2 \to \infty, k_3 \to \infty$ lead to homogeneous stiffener.

The Poisson's ratio $\nu$ is assumed to be constant: $\nu(z) = \nu = const.$

As can be seen with the mentioned laws, the continuity between shell and stiffeners is guaranteed.

### 2.2 Constitutive Equations

According to the third-order shear deformation theory with von Karman geometrical nonlinearity, the strain components of the shell at a distance $z$ from the middle surface are of the form as Reddy (2004)

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + zk_x^1 + z^3 k_x^{(3)}, \quad \varepsilon_y = \varepsilon_y^0 + zk_y^1 + z^3 k_y^{(3)}; \\ \gamma_{xy} &= \gamma_{xy}^0 + zk_{xy}^1 + z^3 k_{xy}^{(3)}, \quad \gamma_{xz} = \gamma_{xz}^0 + z^2 k_{xz}^{(2)}, \gamma_{yz} = \gamma_{yz}^0 + z^2 k_{yz}^{(2)}; \end{aligned} \tag{3}$$

in which

$$\begin{aligned} &\varepsilon_x^0 = u_{,x} + \frac{1}{2}w_{,x}^2, \quad \varepsilon_y^0 = v_{,y} - \frac{w}{R} + \frac{1}{2}w_{,y}^2; \\ &\gamma_{xy}^0 = u_{,y} + v_{,x} + \mathrm{w}_{,x}\mathrm{w}_{,y}, \gamma_{xz}^0 = \phi_x + \mathrm{w}_{,x}, \gamma_{yz}^0 = \phi_y + \mathrm{w}_{,y}; \\ &k_x^1 = \phi_{x,x}, \quad k_{xy}^1 = \phi_{x,y} + \phi_{y,x}, \quad k_x^{(3)} = \frac{-4}{3h^2}\left(\phi_{x,x} + \mathrm{w}_{,xx}\right); \end{aligned} \tag{4}$$

$$k_y^{(3)} = \frac{-4}{3h^2}\left(\phi_{y,y} + w_{,yy}\right), \qquad k_{xy}^{(3)} = \frac{-4}{3h^2}\left(\phi_{x,y} + \phi_{y,x} + 2w_{,xy}\right);$$
$$k_{xz}^{(2)} = \frac{-4}{h^2}\left(\phi_x + \mathrm{w}_{,x}\right), \qquad k_{yz}^{(2)} = \frac{-4}{h^2}\left(\phi_y + \mathrm{w}_{,y}\right);$$

where $u = u(x,y)$, $v = v(x,y)$ and $w = w(x,y)$ are displacement components of the middle surface points along the $x, y$ and $z$ directions, and $\phi_x$, $\phi_y$ represent the transverse normal rotations about the $y$ and $x$ axes, respectively. $\gamma_{xy}$ is the shear strain and $\gamma_{xz}, \gamma_{yz}$ are the transverse shear deformations.

Hooke's Law for a shell taken into account temperature effects is defined as:

For shell

$$\left(\sigma_x^{sh}, \sigma_y^{sh}\right) = \frac{E(z)}{1-\nu^2}\left[\left(\varepsilon_x, \varepsilon_y\right) + \nu\left(\varepsilon_y, \varepsilon_x\right)\right] - \frac{E_{sh}(z)}{1-\nu}\alpha_{sh}(z)\Delta T\left(z\right)\left(1,1\right);$$
$$\left(\sigma_{xy}^{sh}, \sigma_{xz}^{sh}, \sigma_{yz}^{sh}\right) = \frac{E_{sh}(z)}{2\left(1+\nu\right)}\left(\gamma_{xy}, \gamma_{xz}, \gamma_{yz}\right); \tag{5}$$

For stiffeners

$$\sigma_x^s = E_s(z)\varepsilon_x - E_s(z)\alpha_s(z)\Delta T\left(z\right);$$
$$\sigma_y^r = E_r(z)\varepsilon_y - E_r(z)\alpha_r(z)\Delta T\left(z\right); \tag{6}$$
$$\sigma_{xz}^s = G_s(z)\gamma_{xz}, \; \sigma_{yz}^r = G_r(z)\gamma_{yz};$$

where $G_s, G_r$ are shear modulus of stringers and ring respectively; $\Delta T\left(z\right) = T\left(z\right) - T_0$ is temperature difference between the surfaces of FGM cylindrical shell and taking $T_0 = T_m$ .

Using the smeared stiffeners technique and calculating the total force resultants, total moment resultants, and transverse force resultants of ES-FGM shells in thermal environment, we obtain

$$N_x = a_{11}\varepsilon_x^0 + a_{12}\varepsilon_y^0 + a_{13}\phi_{x,x} + a_{14}\phi_{y,y} + a_{15}w_{,xx} + a_{16}w_{,yy} + a_{17}\Phi_1 + a_{18}\Phi_{1s};$$
$$N_y = a_{21}\varepsilon_x^0 + a_{22}\varepsilon_y^0 + a_{23}\phi_{x,x} + a_{24}\phi_{y,y} + a_{25}w_{,xx} + a_{26}w_{,yy} + a_{27}\Phi_1 + a_{28}\Phi_{1r}; \tag{7}$$
$$N_{xy} = a_{31}\gamma_{xy}^0 + a_{32}\phi_{x,y} + a_{33}\phi_{y,x} + a_{34}w_{,xy};$$

$$M_x = b_{11}\varepsilon_x^0 + b_{12}\varepsilon_y^0 + b_{13}\phi_{x,x} + b_{14}\phi_{y,y} + b_{15}w_{,xx} + b_{16}w_{,yy} + b_{17}\Phi_2 + b_{18}\Phi_{2s};$$
$$M_y = b_{21}\varepsilon_x^0 + b_{22}\varepsilon_y^0 + b_{23}\phi_{x,x} + b_{24}\phi_{y,y} + b_{25}w_{,xx} + b_{26}w_{,yy} + b_{27}\Phi_2 + b_{28}\Phi_{2r}; \tag{8}$$
$$M_{xy} = b_{31}\gamma_{xy}^0 + b_{32}\phi_{x,\mathrm{y}} + b_{33}\phi_{y,x} + b_{34}\mathrm{w}_{,xy};$$

$$P_x = c_{11}\varepsilon_x^0 + c_{12}\varepsilon_y^0 + c_{13}\phi_{x,x} + c_{14}\phi_{y,y} + c_{15}w_{,xx} + c_{16}w_{,yy} + c_{17}\Phi_4 + c_{18}\Phi_{4s};$$
$$P_y = c_{21}\varepsilon_x^0 + c_{22}\varepsilon_y^0 + c_{23}\phi_{x,x} + c_{24}\phi_{y,y} + c_{25}w_{,xx} + c_{26}w_{,yy} + c_{27}\Phi_4 + c_{28}\Phi_{4r}; \tag{9}$$
$$P_{xy} = c_{31}\gamma_{xy}^0 + c_{32}\phi_{x,y} + c_{33}\phi_{y,x} + c_{34}w_{,xy};$$

$$\begin{aligned} Q_x &= d_{11}\gamma_{xz}^0 + d_{12}\phi_x + d_{13}w_{,x}; \\ Q_y &= d_{21}\gamma_{yz}^0 + d_{22}\phi_y + d_{23}w_{,y}; \end{aligned} \tag{10}$$

$$\begin{aligned} R_x &= e_{11}\gamma_{xz}^0 + e_{12}\phi_x + e_{13}w_{,x}; \\ R_y &= e_{21}\gamma_{yz}^0 + e_{22}\phi_y + e_{23}w_{,y}; \end{aligned} \tag{11}$$

in which $a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}$ $(i = \overline{1 \div 3}, j = \overline{1 \div 8})$ and $\Phi_1, \Phi_2, \Phi_4, \Phi_{1s}, \Phi_{2s}, \Phi_{4s}, \Phi_{1r}, \Phi_{2r}, \Phi_{4r}$ can be found in Appendix A.

Eqs. (7), (8) and (9) are one of new contributions in this work in which the thermal elements of the both shell and stiffener in equations of $N_{ij}, M_{ij}$ and $P_{ij}$ are established.

The nonlinear equations of motion of an imperfect FGM shell filled by elastic foundation based on the third order shear deformation theory are given by Reddy (2004)

$$\begin{aligned}
&N_{x,x} + N_{xy,y} = I_0 \frac{\partial^2 u}{\partial t^2} + J_1 \frac{\partial^2 \phi_x}{\partial t^2} - \lambda I_3 \frac{\partial^3 \mathrm{w}}{\partial x \partial t^2}; \\
&N_{xy,x} + N_{y,y} = I_0 \frac{\partial^2 v}{\partial t^2} + J_1 \frac{\partial^2 \phi_y}{\partial t^2} - \lambda I_3 \frac{\partial^3 \mathrm{w}}{\partial y \partial t^2}; \\
&Q_{x.x} + Q_{y,y} - 3\lambda\left(R_{x.x} + R_{y,y}\right) + \lambda\left(P_{x,xx} + 2P_{xy,xy} + P_{y,yy}\right) + \frac{N_y}{R} + \left(N_x^0 + N_x\right)\left(w_{,xx} + w_{,xx}^*\right) \\
&\quad + 2N_{xy}\left(w_{,xy} + w_{,xy}^*\right) + N_y\left(w_{,yy} + w_{,yy}^*\right) + \left(N_{x,x} + N_{xy,y}\right)\left(w_{,x} + w_{,x}^*\right) \\
&\quad + \left(N_{y,y} + N_{xy,x}\right)\left(w_{,y} + w_{,y}^*\right) - K_1 w + K_2(w_{,xx} + w_{,yy}) + q \\
&\quad = I_0 \frac{\partial^2 w}{\partial t^2} + 2\varepsilon I_0 \frac{\partial w}{\partial t} - \lambda^2 I_6 \left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) \\
&\quad + \lambda I_3 \left(\frac{\partial^3 u}{\partial x \partial t^2} + \frac{\partial^3 v}{\partial y \partial t^2}\right) + \lambda J_4 \left(\frac{\partial^3 \phi_x}{\partial x \partial t^2} + \frac{\partial^3 \phi_y}{\partial y \partial t^2}\right) \\
&M_{x,x} + M_{xy,y} - Q_x + 3\lambda R_x - \lambda(P_{x,x} + P_{xy,y}) = J_1 \frac{\partial^2 u}{\partial t^2} + L_2 \frac{\partial^2 \phi_x}{\partial t^2} - \lambda J_4 \frac{\partial^3 w}{\partial x \partial t^2}; \\
&M_{xy,x} + M_{y,y} - Q_y + 3\lambda R_y - \lambda(P_{xy,x} + P_{y,y}) = J_1 \frac{\partial^2 v}{\partial t^2} + L_2 \frac{\partial^2 \phi_y}{\partial t^2} - \lambda J_4 \frac{\partial^3 w}{\partial y \partial t^2};
\end{aligned} \tag{12}$$

where $\lambda, I_0, I_3, I_4, I_6, J_1$ and $J_4$ are given in Appendix B; $\varepsilon$ is damping coefficient.

Substituting Eqs. $(7 \div 11)$ and $(3 \div 4)$ into Eqs. (12), after some transformations we obtain the equations of motion of ES-FGM cylindrical shell in terms of displacement components as follows

$$\begin{aligned} &L_{11}(u) + L_{12}(v) + L_{13}(w) + L_{14}(\phi_x) + L_{15}(\phi_y) + P_1(w) + Q_1(w, w^*) \\ &\quad = I_0 \frac{\partial^2 u}{\partial t^2} + J_1 \frac{\partial^2 \phi_x}{\partial t^2} - \lambda I_3 \frac{\partial^3 \mathrm{w}}{\partial x \partial t^2}, \end{aligned} \tag{13}$$

$$L_{21}\left(u\right)+L_{22}\left(v\right)+L_{23}\left(w\right)+L_{24}\left(\phi_x\right)+L_{25}\left(\phi_y\right)+P_2\left(w\right)+Q_2\left(w,w^*\right)$$
$$=I_0\frac{\partial^2 v}{\partial t^2}+J_1\frac{\partial^2\phi_y}{\partial t^2}-\lambda I_3\frac{\partial^3 \mathrm{w}}{\partial y\partial t^2},$$
$$L_{31}\left(u\right)+L_{32}\left(v\right)+L_{33}\left(w\right)+L_{34}\left(\phi_x\right)+L_{35}\left(\phi_y\right)+P_3\left(w\right)+R_1(u,w)+R_2(v,w)+R_3(\phi_x,w)$$
$$+R_4\left(\phi_y,w\right)+R_5(u,w^*)+R_6(v,w^*)+R_7(\phi_x,w^*)+R_8\left(\phi_y,w^*\right)+R_9\left(w,w^*\right)$$
$$=I_0\frac{\partial^2 w}{\partial t^2}+2\varepsilon I_0\frac{\partial w}{\partial t}-\lambda^2 I_6\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+\lambda I_3\left(\frac{\partial^3 u}{\partial x\partial t^2}+\frac{\partial^3 v}{\partial y\partial t^2}\right)$$
$$+\lambda J_4\left(\frac{\partial^3\phi_x}{\partial x\partial t^2}+\frac{\partial^3\phi_y}{\partial y\partial t^2}\right);$$
$$L_{41}\left(u\right)+L_{42}\left(v\right)+L_{43}\left(w\right)+L_{44}\left(\phi_x\right)+L_{45}\left(\phi_y\right)+P_4\left(w\right)+Q_4\left(w,w^*\right)$$
$$=J_1\frac{\partial^2 u}{\partial t^2}+L_2\frac{\partial^2\phi_x}{\partial t^2}-\lambda J_4\frac{\partial^3 w}{\partial x\partial t^2};$$
$$L_{51}\left(u\right)+L_{52}\left(v\right)+L_{53}\left(w\right)+L_{54}\left(\phi_x\right)+L_{55}\left(\phi_y\right)+P_5\left(w\right)+Q_5\left(w,w^*\right)$$
$$=J_1\frac{\partial^2 v}{\partial t^2}+L_2\frac{\partial^2\phi_y}{\partial t^2}-\lambda J_4\frac{\partial^3 w}{\partial y\partial t^2};$$

where linear operators $L_{ij}(\ )\left(i,j=\overline{1,5}\right)$, nonlinear operators $P_i(\ ),Q_i(\ )\left(i=\overline{1,5}\right)$ and $R_i\left(i=\overline{1,9}\right)$ are given in Appendix C.

Eqs. (13) is used to analyze dynamic responses of ES- FGM cylindrical shell subjected to combined mechanical and thermal load on elastic foundations.

## 3 TEMPERATURE

### 3.1 Uniform Temperature Rise

Assume the temperature environment uniformly raised from initial value $T_i$ to final one $T_f$ and $\Delta T=T_f-T_i$ is a constant. Substituting Eqs. (1) and (2) into Eq. (A2), after calculating integrals, we obtain the thermal parameters $\Phi_1,\Phi_{1s},\Phi_{1r}$ as

$$\Phi_1=\Phi_1^0\Delta T\,h\ ,\Phi_{1s}=\Phi_{1s}^0\Delta T\frac{b_1h_1}{d_1},\ \Phi_{1r}=\Phi_{1r}^0\Delta T\frac{b_2h_2}{d_2}; \tag{14}$$

where

$$\Phi_1^0=E_m\alpha_m+\frac{E_m\alpha_{cm}+E_{cm}\alpha_m}{k+1}+\frac{E_{cm}\alpha_{cm}}{2k+1},E_{cm}=E_c-E_m,\alpha_{cm}=\alpha_c-\alpha_m; \tag{15}$$

$$\Phi_{1s}^{0}=E_c\alpha_c+\frac{E_c\alpha_{mc}+E_{mc}\alpha_c}{k_2+1}+\frac{E_{mc}\alpha_{mc}}{2k_2+1},E_{mc}=E_m-E_c,\alpha_{mc}=\alpha_m-\alpha_c;$$

$$\Phi_{1r}^{0}=E_c\alpha_c+\frac{E_c\alpha_{mc}+E_{mc}\alpha_c}{k_3+1}+\frac{E_{mc}\alpha_{mc}}{2k_3+1};$$

### 3.2 Nonlinear Temperature Change Across the Thickness z

In this case, the temperature through the thickness of the shell is governed by the one-dimensional Fourier equation of steady-state heat conduction established in cylindrical coordinate whose origin is on the symmetric axis of cylindrical shell as follows

$$\frac{d}{d\bar{z}}\left[K\left(\bar{z}\right)\frac{dT}{d\bar{z}}\right]+\frac{K\left(\bar{z}\right)}{\bar{z}}\frac{dT}{d\bar{z}}=0,\quad T\Big|_{\bar{z}=R-h/2}=T_c,\quad T\Big|_{\bar{z}=R+h/2}=T_m; \tag{16}$$

where $T_m$ and $T_c$ are temperatures at metal-rich and ceramic-rich surfaces, respectively. In Eq. (16), $\bar{z}$ is radial coordinate of a point which is distant from the symmetric axis of cylinder respect to the point of shell i.e.

$$\bar{z}=R-z\ and\ \ R-h/2\le\bar{z}\le R+h/2.$$

According to Eq.(16), we get

**a) For shell**: Eq.(16) is of the form

$$\frac{d}{d\bar{z}}\left[K_{sh}\left(\bar{z}\right)\frac{dT}{d\bar{z}}\right]+\frac{K_{sh}\left(\bar{z}\right)}{\bar{z}}\frac{dT}{d\bar{z}}=0,\ T\Big|_{\bar{z}=R-h/2}=T_c,\ T\Big|_{\bar{z}=R+h/2}=T_m; \tag{17}$$

By solving Eq. (17) with mentioned boundary conditions, the solution for temperature distribution across the shell thickness is obtained

$$T\left(\bar{z}\right)=T_c+\frac{T_{mc}}{\int\limits_{R-h/2}^{R+h/2}\frac{d\bar{z}}{\bar{z}\,K_{sh}\left(\bar{z}\right)}}\int\limits_{R-h/2}^{\bar{z}}\frac{d\xi}{\xi\,K_{sh}\left(\xi\right)}. \tag{18}$$

Due to mathematical difficulty when caculating integral, this section only considers linear distribution of metal and ceramic, that means *k=1*. Substituting expressions (1) into Eq. (18) and caculating integrals, after that substituting $\bar{z}=R-z$, we have an expression

$$T\left(z\right)=T_c+\frac{T_{mc}}{\ln\frac{K_c\left(R/h+1/2\right)}{K_m\left(R/h-1/2\right)}}\times\left[\ln\frac{\left(R-z\right)/h}{R/h-1/2}-\ln\frac{K_m+K_{cm}\left(2z+h\right)/\left(2h\right)}{K_c}\right]. \tag{19}$$

Deduce

$$\Delta T\left(z\right)=T_{cm}+\frac{T_{mc}}{\ln\dfrac{K_c\left(R/h+1/2\right)}{K_m\left(R/h-1/2\right)}}\times\left[\ln\frac{\left(R-z\right)/h}{R/h-1/2}-\ln\frac{K_m+K_{cm}\left(2z+h\right)/\left(2h\right)}{K_c}\right]. \tag{20}$$

Substituting Eq. (1) and (20) into expression (A2) and accounting, we have

$$\Phi_1=\Phi_1^1\Delta T\,h, \tag{21}$$

where

$$\begin{aligned}
&\Delta T=T_c-T_m;\\
&\Phi_1^1=E_m\alpha_m+\frac{E_m\alpha_{cm}+E_{cm}\alpha_m}{2}+\frac{E_{cm}\alpha_{cm}}{3}-\frac{E_m\alpha_mI_0+\left(E_m\alpha_{cm}+E_{cm}\alpha_m\right)I_1+E_{cm}\alpha_{cm}I_2}{\ln\left[K_c\left(R/h+1/2\right)/K_m/\left(R/h-1/2\right)\right]};\\
&I_0=\left(\frac{R}{h}+\frac{1}{2}\right)\ln\frac{R/h+1/2}{R/h-1/2}-\frac{K_m}{K_{cm}}\ln\frac{K_c}{K_m};\\
&I_1=\frac{-1}{4}\left(1+\frac{2R}{h}\right)+\frac{1}{8}\left(1+\frac{2R}{h}\right)^2\ln\frac{R/h+1/2}{R/h-1/2}-\frac{K_m}{2K_{cm}}+\frac{1}{2}\left(\frac{K_m}{K_{cm}}\right)^2\ln\frac{K_c}{K_m};\\
&I_2=\frac{-1}{18}\left(3+\frac{9R}{h}+\frac{6R^2}{h^2}\right)+\frac{1}{24}\left(1+\frac{2R}{h}\right)^3\ln\frac{R/h+1/2}{R/h-1/2}-\frac{K_m}{6K_{cm}}+\frac{1}{3}\left(\frac{K_m}{K_{cm}}\right)^2-\frac{1}{3}\left(\frac{K_m}{K_{cm}}\right)^3\ln\frac{K_c}{K_m}.
\end{aligned} \tag{22}$$

**b) For stringer stiffeners:**

Eq.(16) leads to

$$\begin{aligned}
&\frac{d}{d\bar z}\left[K_s\left(\bar z\right)\frac{dT}{d\bar z}\right]+\frac{K_s\left(\bar z\right)}{\bar z}\frac{dT}{d\bar z}=0,\qquad R-\frac{h}{2}-h_1\le\bar z\le R-\frac{h}{2};\\
&T\Big|_{\bar z=R-h/2}=T_c,\qquad T\Big|_{\bar z=R-h/2-h_1}=T_m.
\end{aligned} \tag{23}$$

Similar to the case of shell, according to expression (2) and Eq. (23), we obtain

$$\Phi_{1s}=\Phi_{1s}^1\Delta T\frac{b_1h_1}{d_1}; \tag{24}$$

where

$$\begin{aligned}
&\Phi_{1s}^1=\frac{\left[E_c\alpha_cJ_0+\left(E_c\alpha_{mc}+E_{mc}\alpha_c\right)J_1+E_{mc}\alpha_{mc}J_2\right]}{\ln\left[K_m\left(h-2R\right)/K_c/\left(h-2R+2h_1\right)\right]};\\
&J_0=\frac{h-2R}{2h_1}\ln\frac{h-2R+2h_1}{h-2R}-\frac{K_c}{K_{mc}}\ln\frac{K_m}{K_c};
\end{aligned} \tag{25}$$

$$J_1 = \frac{-1}{8}\left(\frac{h-2R}{h_1}\right)^2 \ln\frac{h-2R+2h_1}{h-2R} + \frac{h-2R}{4h_1} + \frac{1}{2}\left(\frac{K_c}{K_{mc}}\right)^2 \ln\frac{K_m}{K_c} - \frac{K_c}{2K_{mc}};$$

$$J_2 = \frac{1}{24}\left(\frac{h-2R}{h_1}\right)^3 \ln\frac{h-2R+2h_1}{h-2R} - \frac{1}{12}\left(\frac{h-2R}{h_1}\right)^2$$
$$+\frac{h-2R}{12h_1} - \frac{1}{3}\left(\frac{K_c}{K_{mc}}\right)^3 \ln\frac{K_m}{K_c} - \frac{1}{3}\left(\frac{K_c}{K_{mc}}\right)^2 - \frac{K_c}{6K_{mc}}.$$

**c) For ring stiffeners:**

Similarly, in this case, we also obtain

$$\Phi_{1r} = \Phi_{1r}^1 \Delta T \frac{b_2 h_2}{d_2}; \tag{26}$$

where

$$\Phi_{1r}^1 = \frac{\left[E_c\alpha_c F_0 + \left(E_c\alpha_{mc} + E_{mc}\alpha_c\right)F_1 + E_{mc}\alpha_{mc}F_2\right]}{\ln\left[K_m\left(h-2R\right)/K_c/\left(h-2R+2h_2\right)\right]};$$

$$F_0 = \frac{h-2R}{2h_2}\ln\frac{h-2R+2h_2}{h-2R} - \frac{K_c}{K_{mc}}\ln\frac{K_m}{K_c};$$

$$F_1 = \frac{-1}{8}\left(\frac{h-2R}{h_2}\right)^2 \ln\frac{h-2R+2h_2}{h-2R} + \frac{h-2R}{4h_2} + \frac{1}{2}\left(\frac{K_c}{K_{mc}}\right)^2 \ln\frac{K_m}{K_c} - \frac{K_c}{2K_{mc}}; \tag{27}$$

$$F_2 = \frac{1}{24}\left(\frac{h-2R}{h_2}\right)^3 \ln\frac{h-2R+2h_2}{h-2R} - \frac{1}{12}\left(\frac{h-2R}{h_2}\right)^2 + \frac{h-2R}{12h_2}$$
$$-\frac{1}{3}\left(\frac{K_c}{K_{mc}}\right)^3 \ln\frac{K_m}{K_c} - \frac{1}{3}\left(\frac{K_c}{K_{mc}}\right)^2 - \frac{K_c}{6K_{mc}}.$$

## 4 NONLINEAR DYNAMICAL ANALYSIS

In this section, an analytical approach is given to analyze nonlinear dynamic responses of ES-FGM shells filled by elastic foundations. Assume the shell subjected to axial compressive load $p$, external uniform pressure $q$ and thermal load. So

$$N_x^0 = -ph. \tag{28}$$

Consider cylindrical shell is simply supported at two butt-ends, the corresponding boundary conditions

$$v = w = \phi_y = 0,\ M_x = 0 \text{ at } x = 0 \text{ and } x = L. \tag{29}$$

With the boundary conditions (29) we choose solution as

$$
\begin{aligned}
&u = Ucos\frac{m\pi x}{L}\sin\frac{ny}{R}, \qquad v = V\sin\frac{m\pi x}{L}cos\frac{ny}{R};\\
&w = W\sin\frac{m\pi x}{L}\sin\frac{ny}{R}, \quad w^{*} = W_0\sin\frac{m\pi x}{L}\sin\frac{ny}{R};\\
&\phi_x = \phi_1 cos\frac{m\pi x}{L}\sin\frac{ny}{R},\ \phi_y = \phi_2\sin\frac{m\pi x}{L}\cos\frac{ny}{R};
\end{aligned}
\tag{30}
$$

where m is numbers of half waves in $x$-direction, $n$-wave number in circumferential direction and $U,V,W,\phi_1,\phi_2$ are constant coefficients.

Substituting Eqs. (30) into Eqs. (13) and then applying Galerkin method to obtain nonlinear algebraic equations for $U,V,W,\phi_1,\phi_2$ as follows

$$t_{11}U + t_{12}V + t_{13}W + t_{14}\phi_1 + t_{15}\phi_2 + t_{16}W\left(W + 2W_0\right) = I_0\frac{d^2U}{dt^2} - \lambda I_3\alpha\frac{d^2W}{dt^2} + J_1\frac{d^2\phi_1}{\partial t^2}; \tag{31a}$$

$$t_{21}U + t_{22}V + t_{23}W + t_{24}\phi_1 + t_{25}\phi_2 + t_{26}W\left(W + 2W_0\right) = I_0\frac{d^2V}{dt^2} - \lambda I_3\beta\frac{d^2W}{dt^2} + J_1\frac{d^2\phi_2}{\partial t^2}; \tag{31b}$$

$$
\begin{aligned}
&t_{31}U + t_{32}V + t_{33}W + t_{34}\phi_1 + t_{35}\phi_2 + t_{36}W^2 + t_{37}WW_0 + t_{38}U\left(W + W_0\right) + t_{39}V\left(W + W_0\right)\\
&\quad + t_{311}\phi_2\left(W + W_0\right) + t_{312}W\left(W + W_0\right)\left(W + 2W_0\right) + \Phi_{1T}\left(W + W_0\right) + \Phi_{2T} + \frac{4\delta_m\delta_n}{mn\pi^2}q\\
&= \left[I_0 + \lambda^2 I_6\left(\alpha^2 + \beta^2\right)\right]\frac{d^2W}{dt^2} + 2\varepsilon I_0\frac{dW}{dt} - \alpha\lambda I_3\frac{d^2U}{dt^2} - \beta\lambda I_3\frac{d^2V}{dt^2} - \alpha\lambda J_4\frac{d^2\phi_1}{dt^2} - \beta\lambda J_4\frac{d^2F}{dt^2};
\end{aligned}
\tag{31c}
$$

$$t_{41}U + t_{42}V + t_{43}W + t_{44}\phi_1 + t_{45}\phi_2 + t_{46}W\left(W + 2W_0\right) = J_1\frac{d^2U}{dt^2} + L_2\frac{d^2\phi_1}{dt^2} - \lambda J_4\alpha\frac{d^2W}{\partial t^2}; \tag{31d}$$

$$t_{51}U + t_{52}V + t_{53}W + t_{54}\phi_1 + t_{55}\phi_2 + t_{56}W\left(W + 2W_0\right) = J_1\frac{d^2V}{dt^2} + L_2\frac{d^2\phi_2}{dt^2} - \lambda J_4\beta\frac{d^2W}{\partial t^2}; \tag{31e}$$

where $t_{ij}$ are defined in Appendix D and $\alpha = (m\pi)/L, \beta = n/R$.

The system of five equation (31) is used to analyze dynamic responses of ES-FGM cylindrical shells. However, because it is difficult to find an analytical solution of this system, so it is solved numerically by four-order Runge-Kutta method.

After here some cases that we can obtain analytical solution are presented.

Using Volmir's assumption (1972) we can consider four right sides of the four equations (31 a, b, d, e) equal zero i.e.

$$I_0\frac{d^2U}{dt^2} - \lambda I_3\alpha\frac{d^2W}{dt^2} + J_1\frac{d^2\phi_1}{\partial t^2} = 0; \tag{32}$$

$$I_0\frac{d^2V}{dt^2}-\lambda I_3\beta\frac{d^2W}{dt^2}+J_1\frac{d^2\phi_2}{\partial t^2}=0;$$
$$J_1\frac{d^2U}{dt^2}+L_2\frac{d^2\phi_1}{dt^2}-\lambda J_4\alpha\frac{d^2W}{\partial t^2}=0;$$
$$J_1\frac{d^2V}{dt^2}+L_2\frac{d^2\phi_2}{dt^2}-\lambda J_4\beta\frac{d^2W}{\partial t^2}=0.$$

From Eqs.(32) expressing $\ddot{U},\ddot{V},\ddot{\phi}_1,\ddot{\phi}_2$ through $\ddot{W}$, after substituting the obtained results into the third equation of Eqs. (31), we obtain

$$\begin{gathered}
t_{11}U+t_{12}V+t_{13}W+t_{14}\phi_1+t_{15}\phi_2+t_{16}W\left(W+2W_0\right)=0;\\
t_{21}U+t_{22}V+t_{23}W+t_{24}\phi_1+t_{25}\phi_2+t_{26}W\left(W+2W_0\right)=0;\\
t_{31}U+t_{32}V+t_{33}W+t_{34}\phi_1+t_{35}\phi_2+t_{36}W^2+t_{37}WW_0+t_{38}U\left(W+W_0\right)+t_{39}V\left(W+W_0\right)\\
+t_{310}\phi_1\left(W+W_0\right)+t_{311}\phi_2\left(W+W_0\right)+t_{312}W\left(W+W_0\right)\left(W+2W_0\right)\\
+\Phi_{1T}\left(W+W_0\right)+\Phi_{2T}+\frac{4\delta_m\delta_n}{mn\pi^2}q\\
=g_5\frac{d^2W}{dt^2}+2\varepsilon I_0\frac{dW}{dt};\\
t_{41}U+t_{42}V+t_{43}W+t_{44}\phi_1+t_{45}\phi_2+t_{46}W\left(W+2W_0\right)=0;\\
t_{51}U+t_{52}V+t_{53}W+t_{54}\phi_1+t_{55}\phi_2+t_{56}W\left(W+2W_0\right)=0;
\end{gathered}\tag{33}$$

From the first two equations of Eqs. (33), we express $U,V$ through $W,\phi_1,\phi_2$ after substituting obtained results into the last two equations of Eqs. (33) to solve $\phi_1,\phi_2$ through $W$. Combining with the third equation of Eqs. (33) and after some transformations, we can obtain

$$\begin{gathered}
g_5\frac{d^2W}{dt^2}+2\varepsilon I_0\frac{dW}{dt}-g_1W-\left(\Phi_{1T}-\alpha^2N_x^0\right)\left(W+W_0\right)-t_{36}W^2-t_{37}WW_0-g_2W\left(W+2W_0\right)\\
-g_3W\left(W+W_0\right)-g_4W\left(W+W_0\right)\left(W+2W_0\right)-\Phi_{2T}=\frac{4\delta_m\delta_n}{mn\pi^2}q;
\end{gathered}\tag{34}$$

where $\Phi_1,\Phi_{1s},\Phi_{1r}$ showed as Eq. (14) with uniform temperature rise case; and as Eqs. (21)-(24)-(26) with nonlinear temperature change. And $g_i\left(i=\overline{1,6}\right)$ are given in Appendix D.

Using the fourth-order Runge–Kutta method for Eq. (34) with known initial conditions, we can analyze nonlinear dynamic responses of ES-FGM cylindrical shells.

### 4.1 Natural frequencies

In order to establish explicit expression of natural frequency $\omega$ of the shell, we choose

$$U = U_0 e^{i\omega t},\ V = V_0 e^{i\omega t},\ W = W_0 e^{i\omega t},\ \phi_1 = \phi_{10} e^{i\omega t},\ \phi_2 = \phi_{20} e^{i\omega t}, \tag{35}$$

Substituting Eqs. (35) into Eqs. (33), then omitting imperfection, temperrature and nonlinear parts leads to a system of five homogeneous equations for $U_0$, $V_0$, $W_0$, $\phi_{10}$ and $\phi_{20}$. Because the solutions (33) are nontrivial, the determinant of coefficient matrix of resulting equation must be zero. Conclusion

$$\begin{vmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{21} & t_{23} & t_{24} & t_{25} \\ t_{31} & t_{32} & t_{33} + g_5\omega^2 & t_{34} & t_{35} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} \end{vmatrix} = 0. \tag{36}$$

Solving Eqs. (36) yieds frequencies of cylindrical shell.

In other hand, from Eq. (34) the fundamental frequencies of the shell can be determined approximately by explicit expression

$$\omega_{mn} = \sqrt{\frac{g_6}{g_5}}. \tag{37}$$

### 4.2 Frequency-Amplitude Curve

Consider nonlinear vibration of a cylindrical shell under an uniformly distributed transverse load $q = H\ \sin\Omega t$. Assuming pre-loaded compression p, Eq. (34) has of the form

$$\frac{d^2W}{dt^2} + \frac{2\varepsilon I_0}{g_5}\frac{dW}{dt} + \omega_{mn}^2\left(W + H_1 W^2 + H_2 W^3\right) - H_3 \sin\Omega t = 0, \tag{38}$$

where $H_1 = -\left(t_{36} + g_2 + g_3\right) / g_6$, $H_2 = -g_4 / g_6$, $H_3 = 4\delta_m \delta_n H / \left(mn\pi^2 g_5\right)$.

For seeking amplitude–frequency characteristics of nonlinear vibration, substituting $W = \Psi \sin\Omega t$ into Eq. (38), leads to

$$X \equiv \Psi\left(\omega_{mn}^2 - \Omega^2\right)\sin\Omega t + \frac{2\varepsilon I_0 \Psi\Omega}{g_5}\cos\Omega t + \omega_{mn}^2 H_1 \Psi^2 \sin^2\Omega t + \omega_{mn}^2 H_2 \Psi^3 \sin^3\Omega t - H_3 \sin\Omega t = 0 \tag{39}$$

Integrating over a quarter of vibration period $\int_0^{\pi/2\Omega} X \sin\Omega t\, dt = 0$, we obtain

$$\Omega^2 - \frac{4\varepsilon I_0}{g_5\pi}\Omega = \omega_{mn}^2\left(1 + \frac{8}{3\pi}H_1\Psi + \frac{3H_2}{4}\Psi^2\right) - \frac{H_3}{\Psi}. \tag{40}$$

By taking $\gamma^2 = \dfrac{\Omega^2}{\omega_{mn}^2}$ Eq.(40) is rewritten as

$$\gamma^2 - \frac{4\varepsilon I_0}{g_5\pi}\gamma = 1 + \frac{8}{3\pi}H_1\Psi + \frac{3H_2}{4}\Psi^2 - \frac{H_3}{\Psi\omega_{mn}^2}; \tag{41}$$

without damping

$$\gamma^2 = 1 + \frac{8}{3\pi}H_1\Psi + \frac{3H_2}{4}\Psi^2 - \frac{H_3}{\Psi\omega_{mn}^2}. \tag{42}$$

The frequency–amplitude relation of free nonlinear vibration is obtained

$$\omega_{NL}^2 = \omega_{mn}^2\left(1 + \frac{8}{3\pi}H_1\Psi + \frac{3H_2}{4}\Psi^2\right); \tag{43}$$

where $\omega_{NL}$ is the nonlinear vibration frequency of the shell.

# 5 NUMERICAL RESULTS AND DISCUSSION

## 5.1 Comparison Results

To validate the present approach, in the first comparison this paper compares the natural frequencies of the cylindrical shell obtained from expression (36) with the results given by Eq. (25) Bich and Nguyen (2012) using Donnell shallow shell theory for un-stiffened isotropic FGM shells without elastic foundations (in table 1). It is seen that good agreements are obtained in this comparison.

| $m$ | Natural frequencies in Eq (25 ) from Bich-Nguyen (2012) | Present |
|---|---|---|
| 1 | 384.3080 | 384.3054 |
| 2 | 490.4446 | 490.4304 |
| 3 | 517.1251 | 517.0916 |
| 4 | 527.7445 | 527.6839 |
| 5 | 533.8962 | 533.8004 |
| 6 | 539.0572 | 538.9176 |
| 7 | 544.8023 | 544.6097 |
| 8 | 552.1373 | 551.8812 |
| 9 | 561.8667 | 561.5346 |
| 10 | 574.7031 | 574.2800 |

$n = 1$, $E_m = E_c = E = 7 \times 10^{10}$ N/m$^2$, $\rho_m = \rho_c = \rho = 2702$ Kg/m$^3$, $\nu = 0.3$, $R = 1.5$ m, $L = 2 \times R$, $h = R/200$

**Table 1**: Comparison of natural frequencies (Hz) for a simply supported isotropic cylindrical shell.

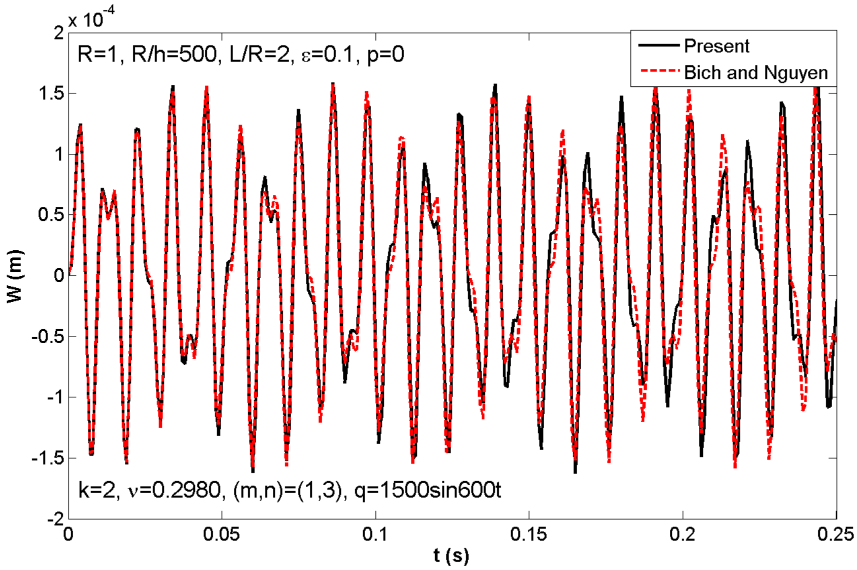


**Figure 2**: The comparison of dynamic respones results with those of Bich and Nguyen (2012).

In the second comparison, Fig. 2 shows the comparison of the nonlinear response of the shell calculated by the approximate Eq. (34) in this paper and Eqs. (32) in Bich and Nguyen (2012) with input parameters as: $E_c = 154.3211\times 10^9$ (Pa), $\rho_c = 5700$ (kg/m$^3$), $E_m = 105.6960\times 10^9$ (Pa), $\rho_m = 4429$ (kg/m$^3$), $\nu = 0.2980$, $k=2$, $k_2=1/k$, $k_3=1/k$, $R=1$(m), $L=2R$, $h=R/500$. It is seen that these results (in Fig.2) are in good agreement to these one of Bich and Nguyen (2012).

From Fig. 2 and Table 1, we conclude that the Volmir's assumption (1972) can be used for nonlinear dynamical analysis with an acceptable accuracy.

In the following subsections, this study will examine the effects of input parameters on nonlinear dynamical response of cylindrical shell with the material properties and the geometric properties of shell are $\nu = 0.3$, $E_m = 70\ GPa$, $\rho_m = 2702\,\mathrm{kg/m^3}$, $E_c = 380\ GPa$, $\rho_c = 3800$ kg/m$^3$, $\alpha_m = 23\times 10^{-6}\ {}^0C^{-1}$, $\alpha_c = 7.4\times 10^{-6}\ {}^0C^{-1}$, $K_m = 204\ W/mK$, $K_c = 10.4\,W/mK$, $d_1 = 2\pi R/n_1$, $d_2 = L/n_2$, $n_1$ and $n_2$ are number of stringer and rings, respectively.

## 5.2 Effect of inside and outside FGM stiffeners

The effects of stiffeners on nonlinear dynamical response of FGM cylindrical shells are given in Fig.3 with $k=1$, $k_2 = k_3 = 1/k$, $\Delta T = 0$, $n_1 = 63$, $n_2 = 15$, $R = 1.5$ (m), $L = 2R$, $h = R/200$, $(m,n) = (1,3)$, $K_1 = 10^8$ (N/m$^3$), $K_2 = 5\times 10^5$ (N/m), $H_3 = 1200$ (N/m$^2$). From obtained results as can be seen with the same stiffener numbers, the time - deflection curve of outside stiffened shell is higher than one of inside stiffened shell. This clearly shows the inside stiffeners are more effective than outside those in this case.

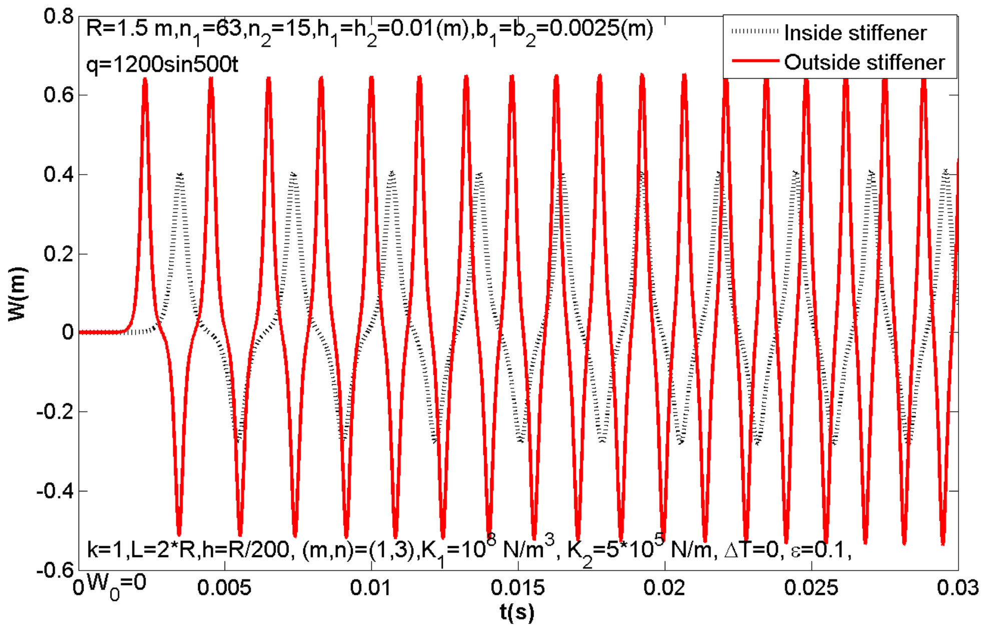


**Figure 3**: Effect of inside and outside FGM stiffeners on nonlinear dynamical response of FGM cylindrical shells.

## 5.3 Effect of Imperfection

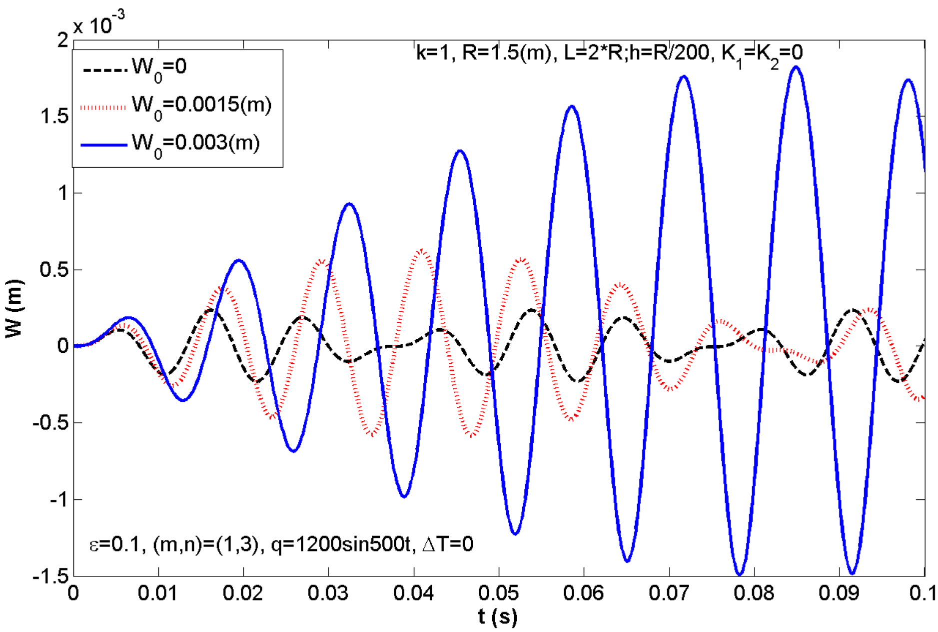


**Figure 4a**: Effect of imperfection $W_0$ on nonlinear responses of FGM cylindrical shells.

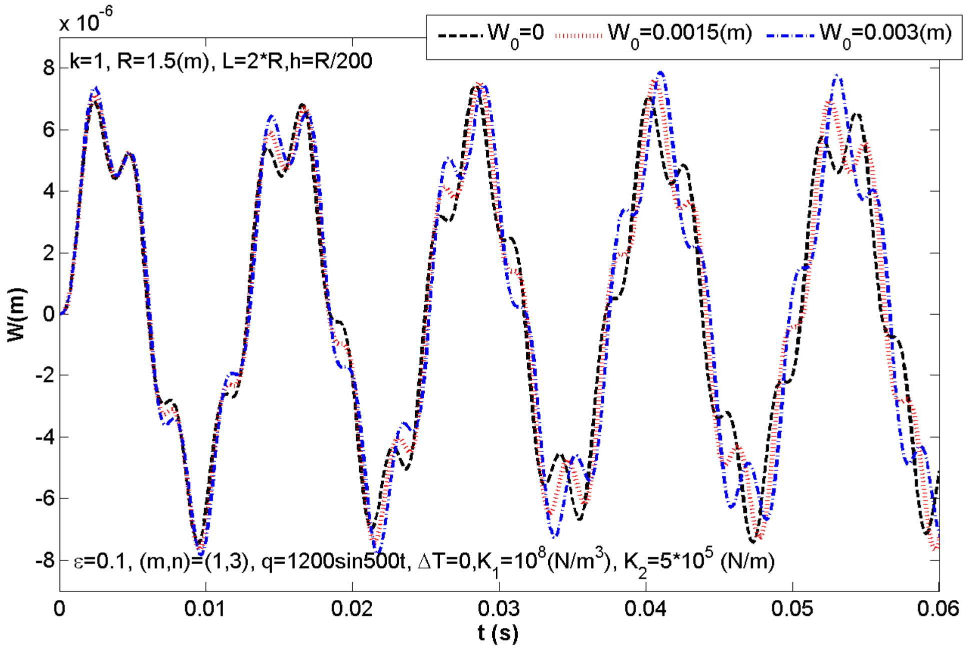


**Figure 4b**: Effect of imperfection $W_0$ on nonlinear responses of FGM cylindrical shells.

Figs. 4a and 4b consider effects of imperfection on nonlinear responses FGM cylindrical shell with two case: without foundation (Fig.4a) and with foundation (Fig.4b). Graphs are plotted with $W_0 = 0,\ 0.0015\,(m),\ 0.003\,(m)$ and $K_1 = K_2 = 0\,(\text{Fig.4a}), K_1 = 10^8\,(N/m^3), K_2 = 10^5\,(N/m)$. It is found that, nonlinear responses curves are higher with the increase of initial amplitude $W_0$. The time - deflection curve with $W_0 = 0.003\,(\mathrm{m})$ is the highest and with $W_0 = 0\,(\mathrm{m})$ it is the shortest. This clearly the known initial amplitude slightly influences on nonlinear dynamic response curves of the FGM shells.

## 5.4 Effect of Foundation Parameters

Fig. 5 describes the effects of foundation parameters on time - deflection curves of FGM cylindrical shell. It can be observed that if the foundation parameters $K_1$ and $K_2$ are larger, the curves are lower. Especially, the amplitude of time - deflection curve of shell without foundation is the highest and the amplitude of time - deflection curve corresponding to the presence of the both foundation parameters $K_1$ and $K_2$ is the smallest. This shows advantage of foundation parameters in vibration of FGM cylindrical shell.

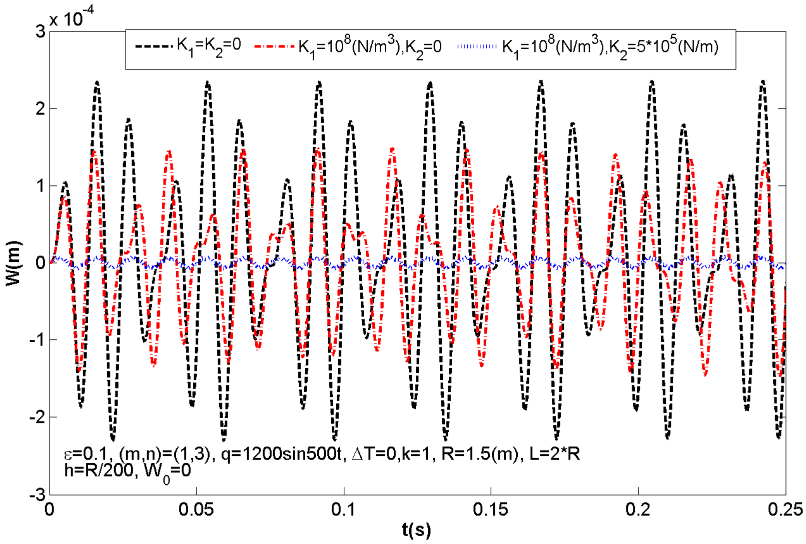


**Figure 5**: Effect of foundation parameters on nonlinear responses of FGM cylindrical shells.

## 5.5 Effects of the Volume Fraction Index *k*

Fig. 6 considers the effects of volume fraction indexes $k$ on the time - deflection ($W$ - $t$) curves of the shell with $k$= 0; 1; 5. It is found that, the height of time - deflection curve decreases with the increase of $k$. The amplitude of the oscillation of FGM cylindrical shells with $k$=0 is the smallest and it is the biggest with $k$= 5 . In addition, the vibration strength of FGM shell is more than fully metal shell and less than that of fully ceramic shell. This property is suitable to the real property of material, because the higher value of $k$ corresponds to a metal-richer shell which usually has less stiffness than a ceramic-richer one.

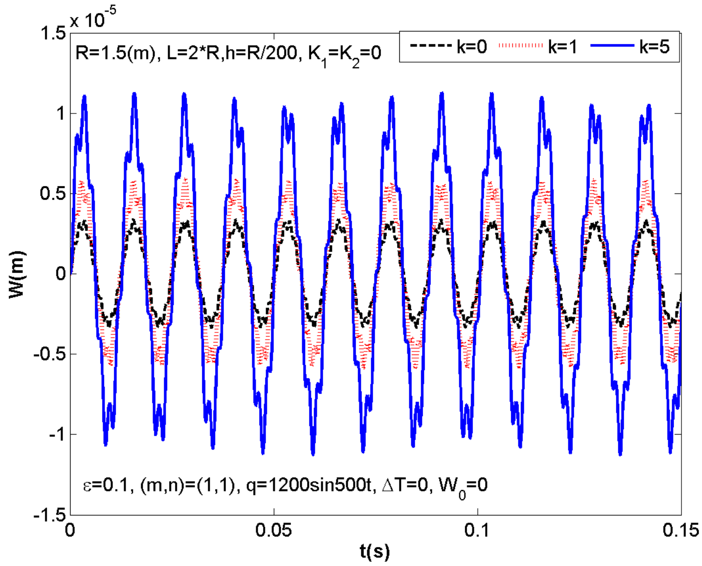


**Figure 6**: Effect of power law index $k$ on nonlinear responses of FGM cylindrical shells.

### 5.6 Effect of Temperature

Figs. 7a and 7b give the effect of temperature field on nonlinear responses of FGM cylindrical shells with $k = 2,\ \varepsilon = 0.1,\ R = 1.5\ (\mathrm{m}), L = 2R,\ h = R/200,\ (m,n) = (1,3),\ W_0 = 0,\ p = 4\ (\mathrm{GPa}),$ $K_1 = K_2 = 0,\ H_3 = 1200\ (\mathrm{N/m^2})$. It can be seen that the vibration of shell raises when $\Delta T$ increases. For example in Fig. 7a, with $\Delta T = 400\,K$, the time – deflection curve is bigger than the time – deflection curve corresponding to $\Delta T = 0\,K$ and $\Delta T = 200\,K$.

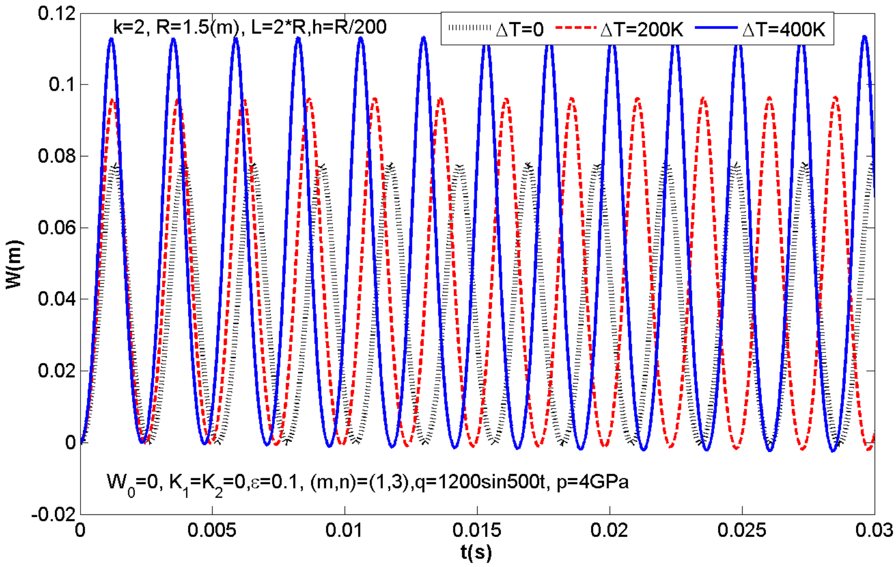


**Figure 7a**: Effect of temperature environment on nonlinear responses of FGM cylindrical shells.

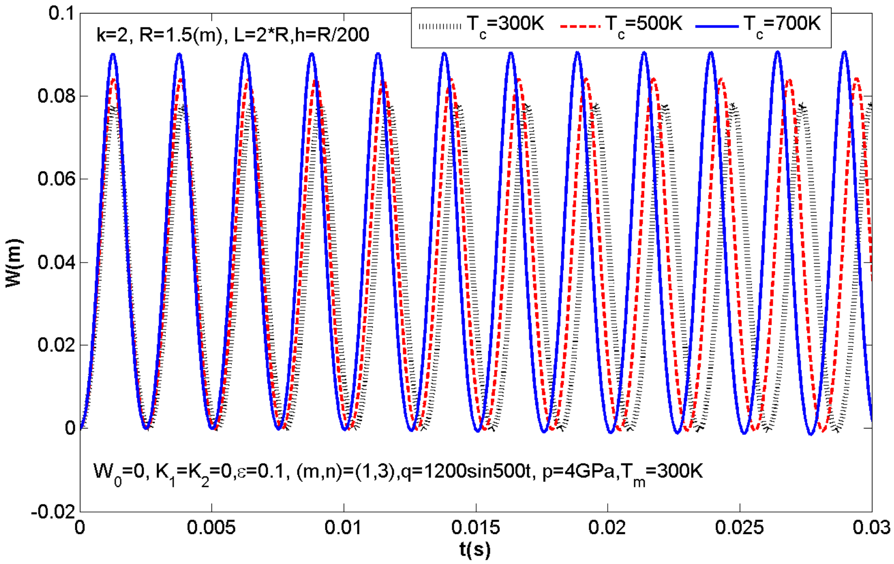


**Figure 7b**: Effect of temperature gradient on nonlinear responses of FGM cylindrical shells.

## 5.7 Effect of Ratio L/R

Fig. 8 gives the effects of the length-to-width ratio $L/R$ on the time – deflection curve with $L/R$= 1; 1.5; 2. It can be seen that the amplitude of vibration of shell is increased considerably when L/$R$ ratio increases.

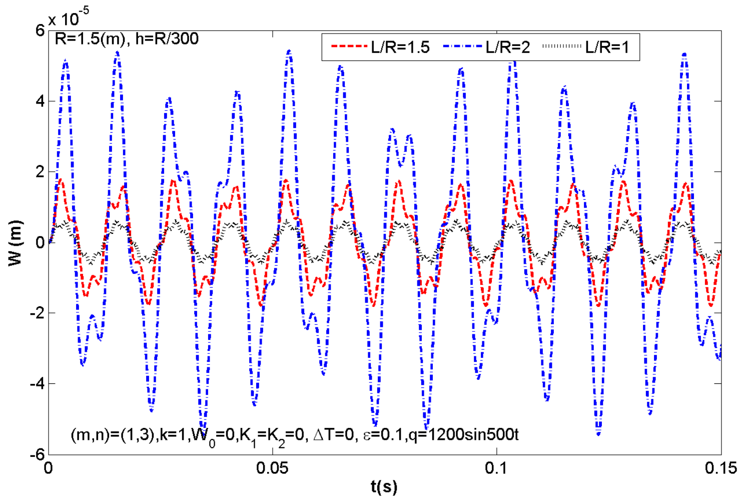


**Figure 8**: Effect of ratio $L/R$ on nonlinear responses of FGM cylindrical shells.

## 5.8 Effect of Ratio R/h Ratio

Fig. 9 illustrates the effects of the width-to-thickness ratio $R/h$ on nonlinear responses of FGM cylindrical shells with $R/h$=100; 200; 250. The obtained results show that the amplitude of vibration of shell is increased considerably when $R/h$ ratio increases. This result agrees with the actual property of structure i.e. because a thicker shell tends to dampen vibration more than a thinner shell.

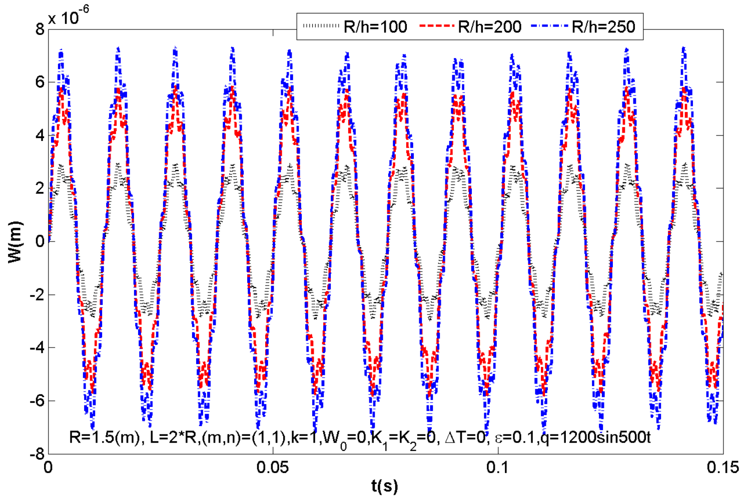


**Figure 9**: Effect of ratio $R/h$ on nonlinear responses of FGM cylindrical shells.

### 5.9 Effect of Damping $\varepsilon$

Fig. 10 considers the effects of damping $\varepsilon$ on the nonlinear response with $\varepsilon = 0$ and $\varepsilon = 5$. It can be seen that damping influences on the time - deflection ($W$ - $t$) curves of the shell are inconsiderable in the first vibration periods of vibration.

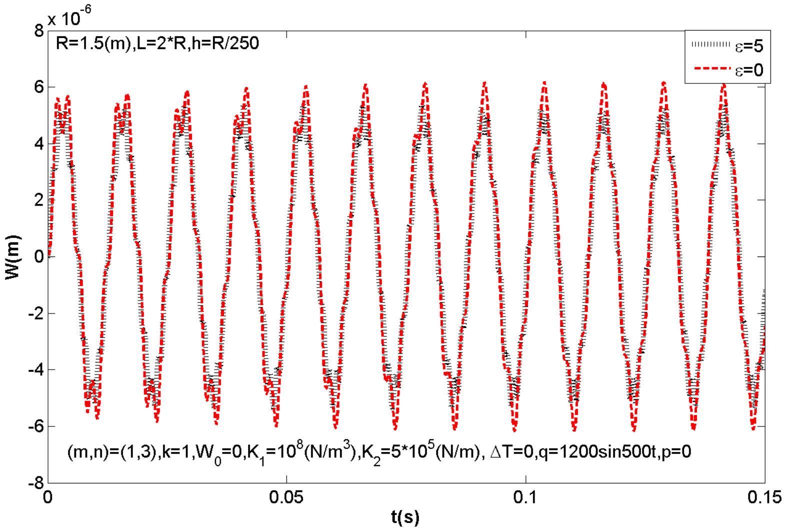


**Figure 10**: Effect of damping on nonlinear responses of FGM cylindrical shells.

### 5.10 Effect of Pre-Loaded Axial Compression

Fig. 11 shows the effects of pre-loaded axial compression on the time - deflection ($W$ - $t$) curves of FGM cylindrical shells with *P=0; 400 MPa; 800 MPa*. The obtained results show that the amplitude of vibration of the shells increases when the value of axial compressive load increases.

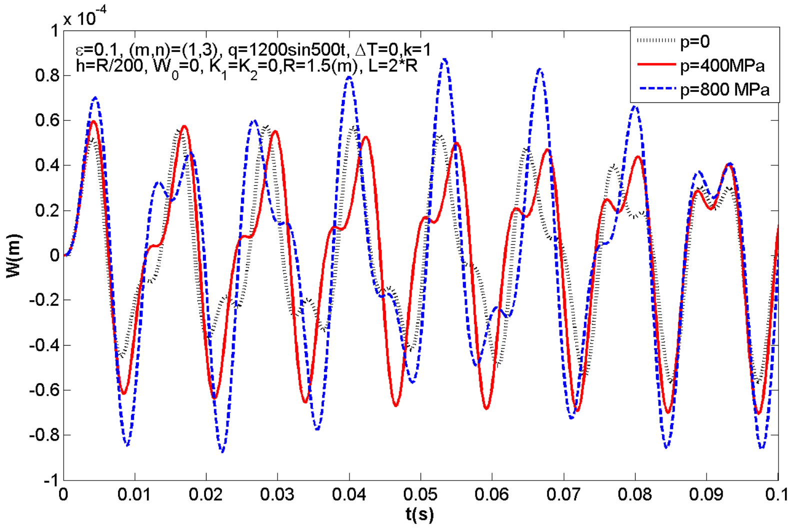


**Figure 11**: Effect of pre-loaded axial compression on nonlinear responses of FGM cylindrical shells.

### 5.11 Frequency – Amplitude Curve

Fig. 12 examines the effects of pre-loaded axial compression on the frequency–amplitude curve of nonlinear free vibration of the shell. It is found that the nonlinear frequency depends apparently on the amplitude and when the pre-loaded axial compression increases, the lowest frequency decreases.

Fig. 13 illustrates the effects of amplitude of external force on frequency–amplitude curve of FGM cylindrical shells with input parameter $k=1$, $k_2 = k_3 = 1/k$, $\varepsilon = 0$, $R = 1.5\,(\mathrm{m})$, $L = 2R$, $h = R/200$, $(m,n) = (1,3)$, $h = R/200$, $(m,n) = (1,1)$, $K_1 = 10^8\,(\mathrm{N/m^3})$, $K_2 = 5\times 10^5$ (N/m). $W_0 = 0$, $p = 0$, $h_1 = h_2 = 0.01\,(\mathrm{m})$, $b_1 = b_2 = 0.0025\,(\mathrm{m})$, $n_1 = 63$, $n_2 = 15$. As can be seen that when the amplitude of external force increasing, the frequency–amplitude curve towards further from the curve of the free vibration case.

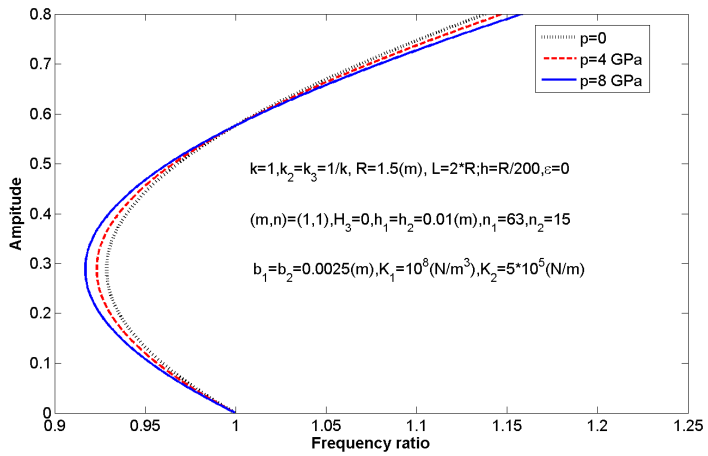


**Figure 12**: Effect of pre-loaded axial compression on frequency–amplitude curve of FGM cylindrical shells in case of free vibration and no damping.

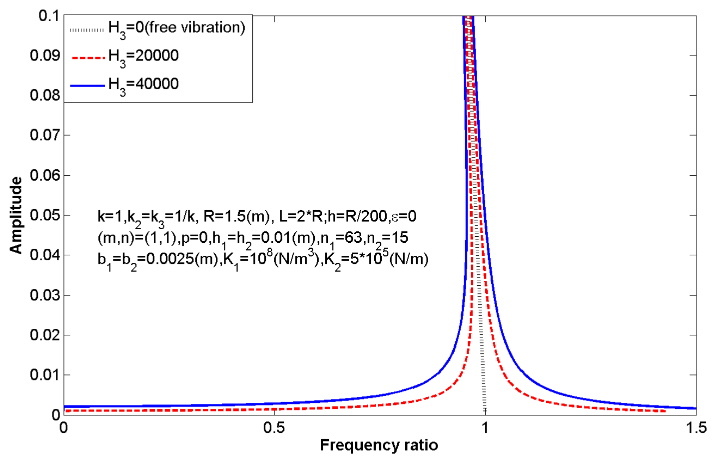


**Figure 13**: Effect of amplitude of external force on frequency–amplitude curve of FGM cylindrical shells.

### 5.12 Beat Phenomenon

Fig. 14 gives nonlinear dynamic response curve of the FGM cylindrical shell when the frequency of the exciting force is near to the natural frequency of the shell with $k = 1$, $k_2 = k_3 = 1/k$, $\varepsilon = 0.1$, $R = 1.5$ (m), $L = 2R$, $h = R/200$, $\left(m,n\right) = \left(1,3\right)$, $K_1 = 10^8$ (N/m$^3$), $K_2 = 5\times10^5$ (N/m). and the natural frequency $\omega = 2156.6(\mathrm{s}^{-1})$. From the graph we can see the beat phenomenon.

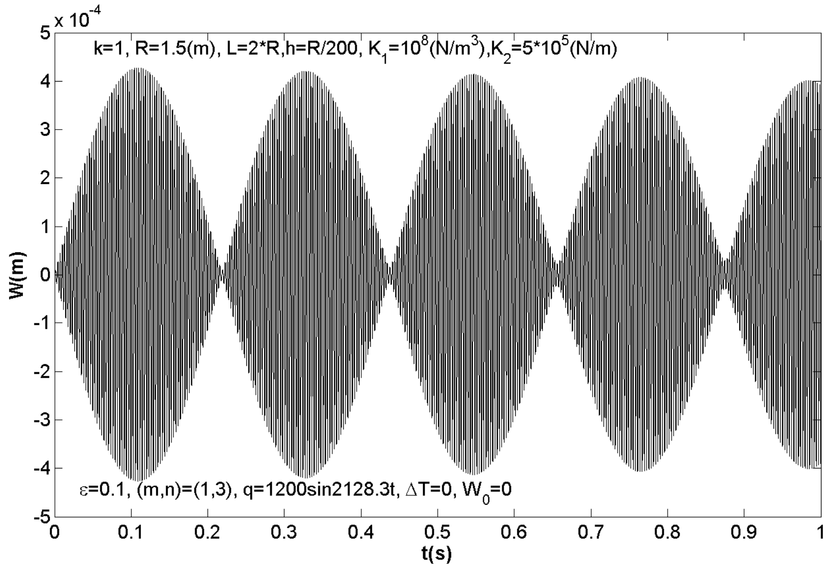


**Figure 14**: Nonlinear responses of FGM cylindrical shells when the frequency of the excitation is near to the natural frequencies.

## 6 CONCLUDING REMARKS

This paper presents dynamic analysis of an eccentrically stiffened imperfect FGM circular cylindrical shells, subjected to axial compressive load and filled inside by elastic foundations in thermal environments by analytical method. Some remarks are deduced from present study and are suitable to the real property of material:

i) According to the third-order shear deformation theory with von Karman geometrical nonlinearity nonlinear dynamic response are considered.
ii) The thermal element in shell and stiffened are taken into account.
iii) Using displacement function, Galerkin method, Volmir'assumption and RungeeKutta method in this study, the closed-form expressions of natural frequency, nonlinear frequency-amplitude curve and nonlinear dynamic response are determined.
iv) Thermal element, elastic foundation, imperfection, damping, pre-existent axial compressive and thermal load and geometrical parameters affect strongly to the nonlinear responses of FGM cylindrical shells.

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# APPENDIX

## Appendix A

The coefficients in Eqs. $(9 \div 13)$ are expressed as

$$\begin{gathered}
a_{11}=\left(\frac{E_1}{1-\nu^2}+\frac{b_1E_{1s}}{d_1}\right),\quad a_{12}=\frac{E_1\nu}{1-\nu^2},\ a_{13}=\frac{E_2}{1-\nu^2}+\frac{b_1E_{2s}}{d_1}-\lambda\left(\frac{E_4}{1-\nu^2}+\frac{b_1E_{4s}}{d_1}\right),\\
a_{14}=\frac{E_2\nu}{1-\nu^2}-\lambda\frac{E_4\nu}{1-\nu^2},a_{15}=-\lambda\left(\frac{E_4}{1-\nu^2}+\frac{b_1E_{4s}}{d_1}\right),\ a_{16}=-\lambda\frac{E_4\nu}{1-\nu^2},a_{17}=-\frac{1}{1-\nu},a_{18}=-\frac{b_1}{d_1},\\
a_{21}=\frac{E_1\nu}{1-\nu^2},\ a_{22}=\frac{E_1}{1-\nu^2}+\frac{b_2E_{1r}}{d_2},\ a_{23}=\frac{E_2\nu}{1-\nu^2}-\frac{4E_4\nu}{3h^2\left(1-\nu^2\right)},\\
a_{24}=\frac{E_2}{1-\nu^2}+\frac{b_2E_{2r}}{d_2}-\lambda\left(\frac{E_4}{1-\nu^2}+\frac{b_2E_{4r}}{d_2}\right),\ a_{25}=-\lambda\frac{E_4\nu}{1-\nu^2},\ a_{26}=-\lambda\left(\frac{E_4}{1-\nu^2}+\frac{b_2E_{4r}}{d_2}\right),\\
a_{27}=-\frac{1}{1-\nu}=a_{17},a_{28}=-\frac{b_2}{d_2},\ a_{31}=\frac{E_1}{2\left(1+\nu\right)},\quad a_{32}=\frac{E_2}{2\left(1+\nu\right)}-\lambda\frac{E_4}{2\left(1+\nu\right)},\quad a_{33}=a_{32},
\end{gathered} \tag{A1}$$

$$a_{34}=-\lambda\frac{E_4}{1+\nu},\ b_{11}=\frac{E_2}{1-\nu^2}+\frac{b_1E_{2s}}{d_1},\ b_{12}=\frac{E_2\nu}{1-\nu^2},\ b_{13}=\frac{E_3}{1-\nu^2}+\frac{b_1E_{3s}}{d_1}-\lambda\left(\frac{E_5}{1-\nu^2}+\frac{b_1E_{5s}}{d_1}\right),$$

$$b_{14}=\frac{E_3\nu}{1-\nu^2}-\lambda\frac{E_5\nu}{1-\nu^2},\quad b_{15}=-\lambda\left(\frac{E_5}{1-\nu^2}+\frac{b_1E_{5s}}{d_1}\right),\ b_{16}=-\lambda\frac{E_5\nu}{1-\nu^2},$$

$$b_{17}=-\frac{1}{1-\nu}=\mathrm{a}_{17},b_{18}=-\frac{b_1}{d_1}=\mathrm{a}_{18},\quad b_{21}=\frac{E_2\nu}{1-\nu^2}=b_{12},\ b_{22}=\frac{E_2}{1-\nu^2}+\frac{b_2E_{2r}}{d_2},$$

$$b_{23}=\frac{E_3\nu}{1-\nu^2}-\lambda\frac{E_5\nu}{1-\nu^2},\ b_{24}=\frac{E_3}{1-\nu^2}+\frac{b_2E_{3r}}{d_2}-\lambda\left(\frac{E_5}{1-\nu^2}+\frac{b_2E_{5r}}{d_2}\right),\ \mathrm{b}_{25}=-\lambda\frac{E_5\nu}{1-\nu^2}=b_{16},$$

$$b_{26}=-\lambda\left(\frac{E_5}{1-\nu^2}+\frac{b_2E_{5r}}{d_2}\right),\ \mathrm{b}_{27}=-\frac{1}{1-\nu}=b_{17},b_{28}=-\frac{b_2}{d_2},$$

$$b_{31}=\frac{E_2}{2\left(1+\nu\right)},b_{32}=\frac{E_3}{2\left(1+\nu\right)}-\lambda\frac{E_5}{2\left(1+\nu\right)},\quad b_{33}=b_{32},\ b_{34}=-\lambda\frac{E_5}{1+\nu},\ c_{11}=\frac{E_4}{1-\nu^2}+\frac{b_1E_{4s}}{d_1},$$

$$c_{12}=\frac{E_4\nu}{1-\nu^2},\ c_{13}=\frac{E_5}{1-\nu^2}+\frac{b_1E_{5s}}{d_1}-\lambda\left(\frac{E_7}{1-\nu^2}+\frac{b_1E_{7s}}{d_1}\right),\ c_{14}=\frac{E_5\nu}{1-\nu^2}-\lambda\frac{E_7\nu}{1-\nu^2},$$

$$c_{15}=-\lambda\left(\frac{E_7}{1-\nu^2}+\frac{b_1E_{7s}}{d_1}\right),c_{16}=-\lambda\frac{E_7\nu}{1-\nu^2},\ c_{17}=-\frac{1}{1-\nu}=\mathrm{a}_{17},c_{18}=-\frac{b_1}{d_1},$$

$$c_{21}=\frac{E_4\nu}{1-\nu^2}=c_{12},\ c_{22}=\frac{E_4}{1-\nu^2}+\frac{b_2E_{4r}}{d_2},\ c_{23}=\frac{E_5\nu}{1-\nu^2}-\lambda\frac{E_7\nu}{1-\nu^2},$$

$$c_{24}=\frac{E_5}{1-\nu^2}+\frac{b_2E_{5r}}{d_2}-\lambda\left(\frac{E_7}{1-\nu^2}+\frac{b_2E_{7r}}{d_2}\right),\ \mathrm{c}_{25}=-\lambda\frac{E_7\nu}{1-\nu^2}=b_{16},\ c_{26}=-\lambda\left(\frac{E_7}{1-\nu^2}+\frac{b_2E_{7r}}{d_2}\right),$$

$$\mathrm{c}_{27}=-\frac{1}{1-\nu}=a_{17},\ c_{28}=-\frac{b_2}{d_2},\ c_{31}=\frac{E_4}{2\left(1+\nu\right)},\quad c_{32}=\frac{E_5}{2\left(1+\nu\right)}-\lambda\frac{E_7}{2\left(1+\nu\right)},$$

$$c_{33}=c_{32},\ c_{34}=-\lambda\frac{E_7}{1+\nu},\ d_{11}=\frac{E_1}{2\left(1+\nu\right)}+\frac{b_1}{d_1}\frac{E_{1s}}{2\left(1+\nu\right)},$$

$$d_{12}=d_{13}=-3\lambda\left[\frac{E_3}{2\left(1+\nu\right)}+\frac{b_1}{d_1}\frac{E_{3s}}{2\left(1+\nu\right)}\right],$$

$$d_{21}=\frac{E_1}{2\left(1+\nu\right)}+\frac{b_2}{d_2}\frac{E_{1r}}{2\left(1+\nu\right)},\quad d_{22}=d_{23}=-3\lambda\left[\frac{E_3}{2\left(1+\nu\right)}+\frac{b_2}{d_2}\frac{E_{3r}}{2\left(1+\nu\right)}\right],$$

$$e_{11}=\frac{E_3}{2\left(1+\nu\right)}+\frac{b_1}{d_1}\frac{E_{3s}}{2\left(1+\nu\right)},\quad e_{12}=e_{13}=-3\lambda\left[\frac{E_5}{2\left(1+\nu\right)}+\frac{b_1}{d_1}\frac{E_{5s}}{2\left(1+\nu\right)}\right],$$

$$e_{21}=\frac{E_3}{2\left(1+\nu\right)}+\frac{b_2}{d_2}\frac{E_{3r}}{2\left(1+\nu\right)},\ e_{22}=e_{23}=-3\lambda\left[\frac{E_5}{2\left(1+\nu\right)}+\frac{b_2}{d_2}\frac{E_{5r}}{2\left(1+\nu\right)}\right],\ \lambda=4/3h^2,$$

where $d_1$ and $d_2$ are denoted the distances between two stringers and rings, respectively; $b_1$, $b_2$ and $h_1$, $h_2$ are the width and thickness of stringer and ring respectively. And

$$\begin{aligned}
&(E_1,E_2,E_3,E_5,E_7)=\int_{-h/2}^{h/2}(1,z,z^2,z^4,z^6)E_{sh}(z)dz,\\
&(E_{1s},E_{2s},E_{3s},E_{5s},E_{7s})=\int_{-h/2}^{h/2+h_1}(1,z,z^2,z^4,z^6)E_s(z)dz,\\
&(E_{1r},E_{2r},E_{3r},E_{5r},E_{7r})=\int_{-h/2}^{h/2+h_2}(1,z,z^2,z^4,z^6)E_r(z)dz,\\
&(\Phi_1,\Phi_2,\Phi_4)=\int_{-h/2}^{h/2}(1,z,z^3)E_{sh}(z)\alpha_{sh}(z)\Delta T(z)dz,\\
&(\Phi_{1s},\Phi_{2s},\Phi_{4s})=\int_{h/2}^{h/2+h_1}(1,z,z^3)E_s(z)\alpha_s(z)\Delta T(z)dz,\\
&(\Phi_{1r},\Phi_{2r},\Phi_{4r})=\int_{h/2}^{h/2+h_2}(1,z,z^3)E_r(z)\alpha_r(z)\Delta T(z)dz,
\end{aligned}\tag{A2}$$

$$\begin{aligned}
&E_1=\left(E_m+\frac{E_c-E_m}{k+1}\right)h,\ E_2=\frac{\left(E_c-E_m\right)kh^2}{2\left(k+1\right)\left(k+2\right)},\\
&E_3=\frac{1}{12}E_mh^3+\left(E_c-E_m\right)\left(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{4k+4}\right)h^3,\\
&E_4=\frac{\left(E_c-E_m\right)h^4}{k+1}\left[\frac{1}{8}-\frac{3}{4\left(k+2\right)}+\frac{3}{\left(k+3\right)\left(k+4\right)}\right],\\
&E_5=\frac{E_mh^5}{80}+\left(E_c-E_m\right)h^5\left[\frac{1}{16\left(k+1\right)}-\frac{1}{2\left(k+2\right)}+\frac{3}{2\left(k+3\right)}-\frac{2}{k+4}+\frac{1}{k+5}\right],\\
&E_7=\frac{E_mh^7}{448}+\left(E_c-E_m\right)h^7\left[\frac{1}{64\left(k+1\right)}-\frac{3}{16\left(k+2\right)}+\frac{15}{16\left(k+3\right)}-\frac{5}{2\left(k+4\right)}+\frac{15}{4\left(k+5\right)}-\frac{3}{k+6}\right.\\
&\qquad\left.+\frac{1}{k+7}\right],\\
&E_{1s}=E_ch_1+E_{mc}h_1\frac{1}{k_2+1},\ E_{2s}=\frac{E_c}{2}h_1\left(h+h_1\right)+E_{mc}h_1^2\left(\frac{1}{k_2+2}+\frac{h}{2h_1}\frac{1}{k_2+1}\right),
\end{aligned}\tag{A3}$$

$$E_{3s}=\frac{E_c}{3}\left[\left(\frac{h}{2}+h_1\right)^3-\frac{h^3}{8}\right]+E_{mc}h_1^3\left(\frac{1}{k_2+3}+\frac{h}{h_1}\frac{1}{k_2+2}+\frac{h^2}{4h_1^2}\frac{1}{k_2+1}\right),$$

$$E_{4s}=\frac{E_c}{4}\left[\left(\frac{h}{2}+h_1\right)^4-\frac{h^4}{16}\right]+E_{mc}h_1^4\left(\frac{1}{k_2+4}+\frac{3h}{2h_1}\frac{1}{k_2+3}+\frac{3h^2}{4h_1^2}\frac{1}{k_2+2}+\frac{h^3}{8h_1^3}\frac{1}{k_2+1}\right),$$

$$E_{5s}=\frac{E_c}{5}\left[\left(\frac{h}{2}+h_1\right)^5-\frac{h^5}{32}\right]+E_{mc}h_1^5\left(\frac{1}{k_2+5}+\frac{2h}{h_1}\frac{1}{k_2+4}+\frac{3h^2}{2h_1^2}\frac{1}{k_2+3}+\frac{h^3}{2h_1^3}\frac{1}{k_2+2}+\frac{h^4}{16h_1^4}\frac{1}{k_2+1}\right)$$

$$E_{7s}=\frac{E_c}{7}\left[\left(\frac{h}{2}+h_1\right)^7-\frac{h^7}{128}\right]+E_{mc}h_1^7\left(\frac{1}{k_2+7}+\frac{3h}{h_1}\frac{1}{k_2+6}+\frac{15h^2}{4h_1^2}\frac{1}{k_2+5}+\frac{5h^3}{2h_1^3}\frac{1}{k_2+4}\right.$$
$$\left.+\frac{15h^4}{16h_1^4}\frac{1}{k_2+3}+\frac{3h^5}{16h_1^5}\frac{1}{k_2+2}+\frac{h^6}{64h_1^6}\frac{1}{k_2+1}\right),$$

$$E_{1r}=E_ch_2+E_{mc}h_2\frac{1}{k_3+1},\quad E_{2r}=\frac{E_c}{2}h_2\left(h+h_2\right)+E_{mc}h_2^2\left(\frac{1}{k_3+2}+\frac{h}{2h_2}\frac{1}{k_3+1}\right),$$

$$E_{3r}=\frac{E_c}{3}\left[\left(\frac{h}{2}+h_2\right)^3-\frac{h^3}{8}\right]+E_{mc}h_2^3\left(\frac{1}{k_3+3}+\frac{h}{h_2}\frac{1}{k_3+2}+\frac{h^2}{4h_2^2}\frac{1}{k_3+1}\right),$$

$$E_{5r}=\frac{E_c}{5}\left[\left(\frac{h}{2}+h_2\right)^5-\frac{h^5}{32}\right]+E_{mc}h_2^5\left(\frac{1}{k_3+5}+\frac{2h}{h_2}\frac{1}{k_3+4}+\frac{3h^2}{2h_2^2}\frac{1}{k_3+3}+\frac{h^3}{2h_2^3}\frac{1}{k_3+2}+\frac{h^4}{16h_2^4}\frac{1}{k_3+1}\right),$$

$$E_{4r}=\frac{E_c}{4}\left[\left(\frac{h}{2}+h_2\right)^4-\frac{h^4}{16}\right]+E_{mc}h_2^4\left(\frac{1}{k_3+4}+\frac{3h}{2h_2}\frac{1}{k_3+3}+\frac{3h^2}{4h_2^2}\frac{1}{k_3+2}+\frac{h^3}{8h_2^3}\frac{1}{k_3+1}\right),$$

$$E_{7r}=\frac{E_c}{7}\left[\left(\frac{h}{2}+h_2\right)^7-\frac{h^7}{128}\right]+E_{mc}h_2^7\left(\frac{1}{k_3+7}+\frac{3h}{h_2}\frac{1}{k_3+6}+\frac{15h^2}{4h_2^2}\frac{1}{k_3+5}+\frac{5h^3}{2h_2^3}\frac{1}{k_3+4}\right.$$
$$\left.+\frac{15h^4}{16h_2^4}\frac{1}{k_3+3}+\frac{3h^5}{16h_2^5}\frac{1}{k_3+2}+\frac{h^6}{64h_2^6}\frac{1}{k_3+1}\right),$$

## Appendix B

The coefficients $\lambda, I_0, I_3, I_4, I_6, J_1$ and $J_4$ in Eqs. (14) are defined as

$$\lambda=4/\left(3h^2\right),\ I_i=\int_{-h/2}^{h/2}\rho_{sh}\left(z\right)z^idz+\frac{b_1}{d_1}\int_{h/2}^{h/2+h_1}\rho_s\left(z\right)z^idz+\frac{b_2}{d_2}\int_{h/2}^{h/2+h_2}\rho_r\left(z\right)z^idz,\ \left(i=\overline{0,6}\right),$$

$$J_i=I_i-\lambda I_{i+2},\ L_2=I_2-2\lambda I_4+\lambda^2I_6.$$

$$I_0=\left(\rho_m+\frac{\rho_{cm}}{k+1}\right)h+\left(\rho_c+\frac{\rho_{mc}}{k_2+1}\right)\frac{b_1h_1}{d_1}+\left(\rho_c+\frac{\rho_{mc}}{k_3+1}\right)\frac{b_2h_2}{d_2},$$

$$I_1 = \frac{\rho_{cm} k h^2}{2\left(k+1\right)\left(k+2\right)} + \frac{\rho_c b_1 h_1}{2d_1}\left(h+h_1\right) + \frac{\rho_{mc} b_1 h_1}{d_1}\left[\frac{h_1}{k_2+2} + \frac{h}{2\left(k_2+1\right)}\right] + \frac{\rho_c b_2 h_2}{2d_2}\left(h+h_2\right)$$

$$+\frac{\rho_{mc} b_2 h_2}{d_2}\left[\frac{h_2}{k_3+2} + \frac{h}{2\left(k_3+1\right)}\right],$$

$$I_2 = \frac{\rho_m h^3}{12} + \rho_{cm} h^3\left[\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4\left(k+1\right)}\right]$$

$$+\frac{\rho_c b_1}{3d_1}\left[\left(\frac{h}{2}+h_1\right)^3 - \frac{h^3}{8}\right] + \rho_{mc}\frac{b_1 h_1}{d_1}\left[\frac{h_1^2}{k_2+3} + \frac{h h_1}{k_2+2} + \frac{h^2}{4\left(k_2+1\right)}\right]$$

$$+\frac{\rho_c b_2}{3d_2}\left[\left(\frac{h}{2}+h_2\right)^3 - \frac{h^3}{8}\right] + \frac{\rho_{mc} b_2 h_2}{d_2}\left[\frac{h_2^2}{k_3+3} + \frac{h h_2}{k_3+2} + \frac{h^2}{4\left(k_3+1\right)}\right],$$

$$I_3 = \rho_{cm} h^4\left[\frac{1}{k+4} - \frac{3}{2\left(k+3\right)} + \frac{3}{4\left(k+2\right)} - \frac{1}{8\left(k+1\right)}\right] + \frac{\rho_c b_1}{4d_1}\left[\left(\frac{h}{2}+h_1\right)^4 - \frac{h^4}{16}\right] + \frac{\rho_c b_2}{4d_2}\left[\left(\frac{h}{2}+h_2\right)^4 - \frac{h^4}{16}\right]$$

$$+\frac{\rho_{mc} b_1 h_1}{d_1}\left[\frac{h_1^3}{k_2+4} + \frac{3h_1^2 h}{2\left(k_2+3\right)} + \frac{3h_1 h^2}{4\left(k_2+2\right)} + \frac{h^3}{8\left(k_2+1\right)}\right]$$

$$+\frac{\rho_{mc} b_2 h_2}{d_2}\left[\frac{h_2^3}{k_3+4} + \frac{3h_2^2 h}{2\left(k_3+3\right)} + \frac{3h_2 h^2}{4\left(k_3+2\right)} + \frac{h^3}{8\left(k_3+1\right)}\right],$$

$$I_4 = \rho_{cm} h^5\left[\frac{1}{k+5} - \frac{2}{k+4} + \frac{3}{2\left(k+3\right)} - \frac{1}{2\left(k+2\right)} + \frac{1}{16\left(k+1\right)}\right]$$

$$+\frac{\rho_m h^5}{80} + \frac{\rho_c b_1}{5d_1}\left[\left(\frac{h}{2}+h_1\right)^5 - \frac{h^5}{32}\right] + \frac{\rho_c b_2}{5d_2}\left[\left(\frac{h}{2}+h_2\right)^5 - \frac{h^5}{32}\right]$$

$$+\frac{\rho_{mc} b_1 h_1}{d_1}\left[\frac{h_1^4}{k_2+5} + \frac{2h_1^3 h}{k_2+4} + \frac{3h_1^2 h^2}{2\left(k_2+3\right)} + \frac{h_1 h^3}{2\left(k_2+2\right)} + \frac{h^4}{16\left(k_2+1\right)}\right]$$

$$+\frac{\rho_{mc} b_2 h_2}{d_2}\left[\frac{h_2^4}{k_3+5} + \frac{2h_2^3 h}{k_3+4} + \frac{3h_2^2 h^2}{2\left(k_3+3\right)} + \frac{h_2 h^3}{2\left(k_3+2\right)} + \frac{h^4}{16\left(k_3+1\right)}\right],$$

$$I_6 = \frac{\rho_m h^7}{448} + \frac{\rho_c b_1}{7d_1}\left[\left(\frac{h}{2}+h_1\right)^7 - \frac{h^7}{128}\right] + \frac{\rho_c b_2}{7d_2}\left[\left(\frac{h}{2}+h_2\right)^7 - \frac{h^7}{128}\right]$$

$$+\frac{\rho_{mc} b_1 h_1}{d_1}\left[\frac{h_1^6}{k_2+7} + \frac{3h_1^5 h}{k_2+6} + \frac{15h_1^4 h^2}{4\left(k_2+5\right)} + \frac{5h_1^3 h^3}{2\left(k_2+4\right)}\right.$$

$$+\frac{15h_1^2h^4}{16\left(k_2+3\right)}+\frac{3h_1h^5}{16\left(k_2+2\right)}+\frac{h^6}{64\left(k_2+1\right)}\Bigg]+\frac{\rho_{mc}b_2h_2}{d_2}\Bigg[\frac{h_2^6}{k_3+7}+\frac{3h_2^5h}{k_3+6}+\frac{15h_2^4h^2}{4\left(k_3+5\right)}+\frac{5h_2^3h^3}{2\left(k_3+4\right)}$$
$$+\frac{15h_2^2h^4}{16\left(k_3+3\right)}+\frac{3h_2h^5}{16\left(k_3+2\right)}+\frac{h^6}{64\left(k_3+1\right)}\Bigg].$$

## Appendix C

Linear operators $L_{ij}\left(\ \right)\left(i,j=\overline{1,5}\right)$ and nonlinear operators $P_i\left(\ \right)\left(i=\overline{1,14}\right)$, $R_i\left(,\right)\left(i=\overline{1,9}\right)$ in Eqs. (15) are given as

$$L_{11}\left(\ \right)=a_{11}\frac{\partial^2}{\partial x^2}+a_{31}\frac{\partial^2}{\partial y^2},\ L_{12}\left(\ \right)=\left(a_{12}+a_{31}\right)\frac{\partial^2}{\partial x\partial y},$$
$$L_{13}\left(\ \right)=-\frac{a_{12}}{R}\frac{\partial}{\partial x}+a_{15}\frac{\partial^3}{\partial x^3}+\left(a_{16}+a_{34}\right)\frac{\partial^3}{\partial x\partial y^2},\ L_{14}\left(\ \right)=a_{13}\frac{\partial^2}{\partial x^2}+a_{32}\frac{\partial^2}{\partial y^2},$$
$$L_{15}\left(\ \right)=\left(a_{14}+a_{33}\right)\frac{\partial^2}{\partial x\partial y},$$
$$P_1\left(\ \right)=a_{11}\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}+\left(a_{12}+a_{31}\right)\frac{\partial}{\partial y}\frac{\partial^2}{\partial x\partial y}+a_{31}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2},$$
$$Q_1\left(w,w^*\right)=a_{11}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial\ w^*}{\partial x}+\frac{\partial\ w}{\partial x}\frac{\partial^2 w^*}{\partial x^2}\right)+a_{31}\left(\frac{\partial w}{\partial x}\frac{\partial^2 w^*}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial\ w^*}{\partial x}\right)$$
$$+\left(a_{12}+a_{31}\right)\left(\frac{\partial^2 w}{\partial x\ \partial y}\frac{\partial\ w^*}{\partial y}+\frac{\partial\ w}{\partial y}\frac{\partial^2 w^*}{\partial x\partial y}\right)$$

$$L_{21}\left(\ \right)=\left(a_{31}+a_{21}\right)\frac{\partial^2}{\partial x\partial y},\ L_{22}\left(\ \right)=a_{31}\frac{\partial^2}{\partial x^2}+a_{22}\frac{\partial^2}{\partial y^2},$$
$$L_{23}\left(\ \right)=\left(a_{34}+a_{25}\right)\frac{\partial^3}{\partial x^2\partial y}-\frac{a_{22}}{R}\frac{\partial}{\partial y}+a_{26}\frac{\partial^3}{\partial y^3},\ L_{24}\left(\ \right)=\left(a_{32}+a_{23}\right)\frac{\partial^2}{\partial x\partial y},$$
$$L_{25}\left(\ \right)=a_{33}\frac{\partial^2}{\partial x^2}+a_{24}\frac{\partial^2}{\partial y^2},$$
$$P_2\left(\ \right)=a_{31}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y}+\left(a_{31}+a_{21}\right)\frac{\partial}{\partial x}\frac{\partial^2}{\partial x\partial y}+a_{22}\frac{\partial}{\partial y}\frac{\partial^2}{\partial y^2},$$
$$Q_2\left(w,w^*\right)=a_{31}\left(\frac{\partial^2 w^*}{\partial x^2}\frac{\partial\ w}{\partial y}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial y}\right)+a_{22}\left(\frac{\partial w^*}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{\partial\ w}{\partial y}\frac{\partial^2 w^*}{\partial y^2}\right)$$
$$+\left(a_{31}+a_{21}\right)\left(\frac{\partial\ w^*}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\ w}{\partial x}\frac{\partial^2 w^*}{\partial x\partial y}\right),$$

$$L_{31}\left(\ \right)=\lambda c_{11}\frac{\partial^3}{\partial x^3}+\lambda(2c_{31}+c_{21})\frac{\partial^3}{\partial x\ \partial y^2}+\frac{a_{21}}{R}\frac{\partial}{\partial x},$$

$$L_{32}\left(\ \right)=\lambda(c_{12}+2c_{31})\frac{\partial^3}{\partial x^2\partial y}+\lambda c_{22}\frac{\partial^3}{\partial y^3}+\frac{a_{22}}{R}\frac{\partial}{\partial y},$$

$$L_{33}\left(\ \right)=-(\frac{a_{22}}{R^2}+K_1)w+[d_{11}+d_{13}-3\lambda(e_{11}+e_{13})-\frac{\lambda c_{12}}{R}+\frac{a_{25}}{R}+K_2]\frac{\partial^2}{\partial x^2}$$
$$+[d_{21}+d_{23}-3\lambda(e_{21}+e_{23})-\frac{\lambda c_{22}}{R}+\frac{a_{26}}{R}+K_2]\frac{\partial^2}{\partial y^2}$$
$$+\lambda c_{15}\frac{\partial^4}{\partial x^4}+(\lambda c_{16}+2\lambda c_{34}+\lambda c_{25})\frac{\partial^4}{\partial x^2\partial y^2}+\lambda c_{26}\frac{\partial^4}{\partial y^4},$$

$$L_{34}\left(\ \right)=[d_{11}+d_{12}-3\lambda(e_{11}+e_{12})+\frac{a_{23}}{R}]\frac{\partial}{\partial x}+\lambda c_{13}\frac{\partial^3}{\partial x^3}+(2\lambda c_{32}+\lambda c_{23})\frac{\partial^3}{\partial x\ \partial y^2},$$

$$L_{35}\left(\ \right)=[d_{21}+d_{22}-3\lambda(e_{21}+e_{22})+\frac{a_{24}}{R}]\frac{\partial}{\partial y}+(\lambda c_{14}+2\lambda c_{33})\frac{\partial^3}{\partial x^2\partial y}+\lambda c_{24}\frac{\partial^3}{\partial y^3},$$

$$P_3\left(\ \right)=(\lambda c_{11}+a_{15})\left(\frac{\partial^2}{\partial x^2}\right)^2+(2\lambda c_{31}+a_{16}+a_{25})\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}+(\lambda c_{22}+a_{26})\frac{\partial^2}{\partial y^2}-\left(\frac{a_{12}}{R}+\frac{a_{22}}{R}\right)w\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$
$$+\lambda c_{11}\frac{\partial}{\partial x}\frac{\partial^3}{\partial x^3}+(2\lambda c_{31}+\lambda c_{21})\frac{\partial}{\partial x}\frac{\partial^3}{\partial x\partial y^2}+\left(\frac{a_{21}}{2R}-\frac{a_{12}}{R}\right)\left(\frac{\partial}{\partial x}\right)^2$$
$$+a_{15}\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial x}+\left(a_{16}+a_{34}\right)\frac{\partial^3}{\partial x\partial y^2}\frac{\partial}{\partial x}+(\lambda c_{12}+2\lambda c_{31}+\lambda c_{21}+2a_{34})\left(\frac{\partial^2}{\partial x\ \partial y}\right)^2$$
$$+(\lambda c_{12}+2\lambda c_{31})\frac{\partial}{\partial y}\frac{\partial^3}{\partial x^2\partial y}+\left(\lambda c_{22}+a_{26}\right)\frac{\partial}{\partial y}\frac{\partial^3}{\partial y^3}-\frac{a_{22}}{2R}\left(\frac{\partial}{\partial y}\right)^2+\left(a_{25}+a_{34}\right)\frac{\partial^3}{\partial x^2\partial y}\frac{\partial}{\partial y}$$

$$R_1\left(u,w\right)=a_{11}u_{,x}w_{,xx}+a_{21}u_{,x}w_{,yy}+a_{11}u_{,xx}w_{,x}+a_{31}u_{,yy}w_{,x}+2a_{31}u_{,y}w_{,xy}+\left(a_{31}+a_{21}\right)u_{,xy}w_{,y},$$

$$R_2\left(v,w\right)=a_{12}v_{,y}w_{,xx}+a_{22}v_{,y}w_{,yy}+\left(a_{12}+a_{31}\right)v_{,xy}w_{,x}+2a_{31}v_{,x}w_{,xy}+a_{22}v_{,yy}w_{,y}+a_{31}v_{,xx}w_{,y},$$

$$R_3\left(\phi_x,w\right)=a_{13}\phi_{x,x}w_{,xx}+a_{23}\phi_{x,x}w_{,yy}+a_{13}\phi_{x,xx}w_{,x}+a_{32}\phi_{x,yy}w_{,x}+2a_{32}\phi_{x,y}w_{,xy}+\left(a_{23}+a_{32}\right)\phi_{x,xy}w_{,y},$$

$$R_4\left(\phi_y,w\right)=a_{14}\phi_{y,y}w_{,xx}+a_{24}\phi_{y,y}w_{,yy}+\left(a_{14}+a_{33}\right)\phi_{y,xx}w_{,x}+2a_{33}\phi_{y,x}w_{,xy}+a_{24}\phi_{y,yy}w_{,y}+a_{33}\phi_{y,xx}w_{,y},$$

$$R_5\left(u,w^*\right)=a_{11}u_{,x}w^*_{xx}+a_{21}u_{,x}w^*_{,yy}+a_{11}u_{,xx}w^*_{,x}+a_{31}u_{,yy}w^*_{,x}+2a_{31}u_{,y}w^*_{,xy}+\left(a_{31}+a_{21}\right)u_{,xy}w^*_{,y},$$

$$R_6\left(v,w^*\right)=a_{12}v_{,y}w^*_{,xx}+a_{22}v_{,y}w^*_{,yy}+\left(a_{12}+a_{31}\right)v_{,xy}\mathrm{w}^*_{,x}+2a_{31}v_{,x}w^*_{,xy}+a_{22}v_{,yy}w^*_{,y}+a_{31}v_{,xx}\mathrm{w}^*_{,y},$$

$$R_7\left(\phi_x,w^*\right)=a_{13}\phi_{x,x}w^*_{,xx}+a_{23}\phi_{x,x}w^*_{,yy}+a_{13}\phi_{x,xx}w^*_{,x}+a_{32}\phi_{x,yy}w^*_{,x}+2a_{32}\phi_{x,y}w^*_{,xy}+\left(a_{23}+a_{32}\right)\phi_{x,xy}w^*_{,y},$$

$$R_8\left(\phi_y,w^*\right)=a_{14}\phi_{y,y}w^*_{,xx}+a_{24}\phi_{y,y}w^*_{,yy}+\left(a_{14}+a_{33}\right)\phi_{y,xx}w^*_{,x}+2a_{33}\phi_{y,x}w^*_{,xy}+a_{24}\phi_{y,yy}w^*_{,y}+a_{33}\phi_{y,xx}w^*_{,y},$$

$$R_9\left(w,w^*\right)=2\lambda c_{11}w_{,xx}w^*_{,xx}+2\lambda c_{31}(w_{,xx}w^*_{,yy}+w_{,yy}w^*_{,xx})+2\lambda c_{22}w_{,yy}w^*_{,yy}-\left(\frac{a_{12}}{R}+\frac{a_{22}}{R}\right)\mathrm{w}(w^*_{,xx}+w^*_{,yy})$$

$$+\left(a_{15}+a_{25}\right)w_{,xx}\left(w^{*}_{,xx}+w^{*}_{,yy}\right)+\left(a_{16}+a_{26}\right)w_{,yy}\left(w^{*}_{,xx}+w^{*}_{,yy}\right)+\lambda c_{11}\left(w_{,xxx}w^{*}_{,x}+w_{,x}w^{*}_{,xxx}\right)$$

$$+\left(2\lambda c_{31}+\lambda c_{21}\right)\left(w_{,x}w^{*}_{,xyy}+w_{,xyy}w^{*}_{,x}\right)+\left(\frac{a_{21}}{R}-\frac{a_{12}}{R}\right)w_{,x}w^{*}_{,x}+a_{15}w_{,xxx}w^{*}_{,x}+\left(a_{16}+a_{34}\right)w_{,xyy}w^{*}_{,x}$$

$$+\left(2\lambda c_{12}+4\lambda c_{31}+2\lambda c_{21}+2a_{34}\right)w_{,xy}w^{*}_{,xy}+\left(\lambda c_{12}+2\lambda c_{31}\right)\left(w_{,xxy}w^{*}_{,y}+w_{,y}w^{*}_{,xxy}\right)$$

$$+\lambda c_{22}\left(w_{,yyy}w^{*}_{,y}+w_{,y}w^{*}_{,yyy}\right)+\left(a_{25}+a_{34}\right)w_{,xxy}w^{*}_{,y}+a_{26}w_{,yyy}w^{*}_{,y}$$

$$+\left(\frac{a_{11}}{2}w^{2}_{,x}+a_{11}w_{,x}w^{*}_{,x}+\frac{a_{12}}{2}w^{2}_{,y}+a_{12}w_{,y}w^{*}_{,y}\right)\left(w_{,xx}+w^{*}_{,xx}\right)$$

$$+\left(a_{11}w_{,x}w_{,xx}+a_{11}w_{,xx}w^{*}_{,x}+a_{11}w_{,x}w^{*}_{,xx}+a_{31}w_{,x}w_{,yy}+a_{31}w_{,x}w^{*}_{,yy}+a_{31}w_{,yy}w^{*}_{,x}\right)\left(w_{,xx}+w^{*}_{,xx}\right)$$

$$+\left(\frac{a_{21}}{2}w^{2}_{,x}+a_{21}w_{,x}w^{*}_{,x}+\frac{a_{22}}{2}w^{2}_{,y}+a_{22}w_{,y}w^{*}_{,y}\right)\left(w_{,yy}+w^{*}_{,yy}\right)$$

$$+\left(a_{22}w_{,y}w_{,yy}+a_{22}w_{,yy}w^{*}_{,y}+a_{22}w_{,y}w^{*}_{,yy}+a_{31}w_{,xx}w_{,y}+a_{31}w_{,xx}w^{*}_{,y}+a_{31}w_{,y}w^{*}_{,xx}\right)\left(w_{,y}+w^{*}_{,y}\right)$$

$$+2a_{31}\left(w_{,x}w_{,y}+w_{,x}w^{*}_{,y}+w_{,y}w^{*}_{,x}\right)\left(w_{,xy}+w^{*}_{,xy}\right)$$

$$+\left(a_{12}+a_{31}\right)\left(w_{,y}w_{,xy}+w_{,xy}w^{*}_{,y}+w_{,y}w^{*}_{,xy}\right)\left(w_{,x}+w^{*}_{,x}\right)$$

$$+\left(a_{21}+a_{31}\right)\left(w_{,x}w_{,xy}+w_{,xy}w^{*}_{,x}+w_{,x}w^{*}_{,xy}\right)\left(w_{,x}+w^{*}_{,x}\right)$$

$$+\frac{a_{27}}{R}\Phi_{1}+\frac{a_{28}}{R}\Phi_{1r}+\left(a_{27}\Phi_{1}+a_{28}\Phi_{1r}\right)\left(w_{,yy}+w^{*}_{,yy}\right)+\left(a_{17}\Phi_{1}+a_{18}\Phi_{1s}+N^{0}_{x}\right)\left(w_{,xx}+w^{*}_{,xx}\right)+q$$

$$L_{41}\left(\ \right)=\left(b_{11}-\lambda c_{11}\right)\frac{\partial^{2}}{\partial x^{2}}+\left(b_{31}-\lambda c_{31}\right)\frac{\partial^{2}}{\partial y^{2}},\ L_{42}\left(\ \right)=\left(b_{12}+b_{31}-\lambda c_{12}-\lambda c_{31}\right)\frac{\partial^{2}}{\partial x\partial y},$$

$$L_{43}\left(\ \right)=\left(-\frac{b_{12}}{R}-d_{11}-d_{13}+3\lambda\left(e_{11}+e_{13}\right)+\frac{\lambda c_{12}}{R}\right)\frac{\partial}{\partial x}+\left(b_{15}-\lambda c_{15}\right)\frac{\partial^{3}}{\partial x^{3}}$$

$$+\left(b_{16}+b_{34}-\lambda c_{16}-\lambda c_{34}\right)\frac{\partial^{3}}{\partial x\partial y^{2}},$$

$$L_{44}\left(\ \right)=\left(b_{13}-\lambda c_{13}\right)\frac{\partial^{2}}{\partial x^{2}}+\left(b_{32}-\lambda c_{32}\right)\frac{\partial^{2}}{\partial y^{2}}+\left[-d_{11}-d_{12}+3\lambda\left(e_{11}+e_{12}\right)\right]w,$$

$$L_{45}\left(\ \right)=\left(b_{14}+b_{33}-\lambda c_{14}-\lambda c_{33}\right)\frac{\partial^{2}}{\partial x\partial y},$$

$$P_{4}\left(\ \right)=\left(b_{11}-\lambda c_{11}\right)\frac{\partial}{\partial x}\frac{\partial^{2}}{\partial x^{2}}+\left(b_{12}+b_{31}-\lambda c_{12}-\lambda c_{31}\right)\frac{\partial}{\partial y}\frac{\partial^{2}}{\partial x\partial y}+\left(b_{31}-\lambda c_{31}\right)\frac{\partial}{\partial x}\frac{\partial^{2}}{\partial y^{2}},$$

$$Q_{4}\left(w,w^{*}\right)=\left(b_{11}-\lambda c_{11}\right)\left(\frac{\partial^{2}w}{\partial x^{2}}\frac{\partial\ w^{*}}{\partial x}+\frac{\partial\ w}{\partial x}\frac{\partial^{2}w^{*}}{\partial x^{2}}\right)$$

$$+\left(b_{12}+b_{31}-\lambda c_{12}-\lambda c_{31}\right)\left(\frac{\partial w^*}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w^*}{\partial x\partial y}\right)+\left(b_{31}-\lambda c_{31}\right)\left(\frac{\partial w^*}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w^*}{\partial y^2}\right),$$

$$L_{51}\left(\ \right)=\left(b_{31}+b_{21}-\lambda c_{31}-\lambda c_{21}\right)\frac{\partial^2}{\partial x\partial y},\ L_{52}\left(\ \right)=\left(b_{31}-\lambda c_{31}\right)\frac{\partial^2}{\partial x^2}+\left(b_{22}-\lambda c_{22}\right)\frac{\partial^2}{\partial y^2},$$

$$L_{53}\left(\ \right)=\left(b_{34}+b_{25}-\lambda c_{34}-\lambda c_{25}\right)\frac{\partial^3}{\partial x^2\partial y}+\left(-\frac{b_{22}}{R}-d_{21}-d_{23}+3\lambda\left(e_{21}+e_{23}\right)+\frac{\lambda c_{22}}{R}\right)\frac{\partial}{\partial y}$$

$$+\left(b_{26}-\lambda c_{26}\right)\frac{\partial^3}{\partial y^3},$$

$$L_{54}\left(\ \right)=\left(b_{32}+b_{23}-\lambda c_{32}-\lambda c_{23}\right)\frac{\partial^2}{\partial x\partial y},$$

$$L_{55}\left(\ \right)=\left(b_{33}-\lambda c_{33}\right)\frac{\partial^2}{\partial x^2}+\left(b_{24}-\lambda c_{24}\right)\frac{\partial^2}{\partial y^2}+\left[-d_{21}-d_{22}+3\lambda\left(e_{21}+e_{22}\right)\right]w,$$

$$P_5\left(\ \right)=\left(b_{31}-\lambda c_{31}\right)\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y}+\left(b_{31}+b_{21}-\lambda c_{31}-\lambda c_{21}\right)\frac{\partial}{\partial x}\frac{\partial^2}{\partial x\partial y}+\left(b_{22}-\lambda c_{22}\right)\frac{\partial}{\partial y}\frac{\partial^2}{\partial y^2},$$

$$Q_5\left(w,w^*\right)=\left(b_{31}-\lambda c_{31}\right)\left(\frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial y}\right)$$

$$+\left(b_{31}+b_{21}-\lambda c_{31}-\lambda c_{21}\right)\left(\frac{\partial w^*}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w^*}{\partial x\partial y}\right)+\left(b_{22}-\lambda c_{22}\right)\left(\frac{\partial w^*}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w^*}{\partial y^2}\right),$$

## Appendix D

The coefficients $t_{\mathrm{ij}}$ in Eqs. (34) are defined as

$$\begin{aligned}
&t_{11}=-a_{11}\alpha^2-a_{31}\beta^2,\ t_{12}=-\left(a_{12}+a_{31}\right)\alpha\beta,\ t_{13}=\frac{-a_{12}\alpha}{R}-a_{15}\alpha^3-\left(a_{16}+a_{34}\right)\alpha\beta^2,\\
&\delta_m=\left(-1\right)^m-1,\ \delta_n=\left(-1\right)^n-1,\ t_{14}=-a_{13}\alpha^2-a_{32}\beta^2,\ t_{15}=-\left(a_{14}+a_{33}\right)\alpha\beta,\\
&t_{16}=\frac{4\delta_m\delta_n}{9mn\pi^2}\left[2\left(-a_{11}\alpha^3-a_{31}\alpha\beta^2\right)+\left(a_{12}+a_{31}\right)\alpha\beta^2\right],\ t_{21}=-\left(a_{31}+a_{21}\right)\alpha\beta,\\
&t_{22}=-a_{31}\alpha^2-a_{22}\beta^2,\ t_{23}=\left(-a_{34}-a_{25}\right)\alpha^2\beta-\frac{a_{22}\beta}{R}-a_{26}\beta^3,\ t_{24}=-\left(a_{32}+a_{23}\right)\alpha\beta,\\
&t_{25}=-a_{33}\alpha^2-a_{24}\beta^2,\ t_{26}=\frac{4\delta_m\delta_n}{9mn\pi^2}\left[2\left(-a_{31}\alpha^2\beta-a_{22}\beta^3\right)+\left(a_{31}+a_{21}\right)\alpha^2\beta\right],\\
&t_{31}=b_{11}\alpha^3+\left(2b_{31}+b_{21}\right)\alpha\beta^2-\frac{a_{21}\alpha}{R},\ t_{32}=\left(b_{12}+2b_{31}\right)\alpha^2\beta+b_{22}\beta^3-\frac{a_{22}\beta}{R},.\\
&t_{33}=\frac{-a_{22}}{R^2}-K_1-\left(\frac{a_{25}}{R}-\frac{b_{12}}{R}+K_2\right)\alpha^2-\left(\frac{a_{26}}{R}-\frac{b_{22}}{R}+K_2\right)\beta^2+b_{15}\alpha^4
\end{aligned}\tag{D1}$$

$$+\left(b_{16}+2b_{34}+b_{25}\right)\alpha^2\beta^2 \ +b_{26}\beta^4,$$

$$t_{34}=\frac{-a_{23}\alpha}{R}+b_{13}\alpha^3+\left(2b_{32}+b_{23}\right)\alpha\beta^2,\qquad t_{35}=\frac{-a_{24}\beta}{R}+\left(b_{14}+2b_{33}\right)\alpha^2\beta+b_{24}\beta^3,$$

$$t_{36}=\left\{\left(b_{11}+a_{15}\right)\alpha^4+\left(2b_{31}+a_{16}+a_{25}\right)\alpha^2\beta^2\ +\left(b_{22}+a_{26}\right)\beta^4\ +\frac{a_{12}\alpha^2}{R}+\frac{a_{22}\beta^2}{R}\right.$$

$$+\frac{1}{2}\left[-b_{11}\alpha^4-\left(2b_{31}+b_{21}\right)\alpha^2\beta^2+\frac{a_{21}\alpha^2}{2R}\right]+\frac{\alpha^2\beta^2}{4}\left(b_{12}+2b_{31}+b_{21}+2a_{34}\right)$$

$$\left.+\frac{1}{2}\left[-\left(b_{12}+2b_{31}\right)\alpha^2\beta^2-b_{22}\beta^4\ +\frac{a_{22}\beta^2}{2R}\right]\right\}\frac{16\delta_m\delta_n}{9mn\pi^2},$$

$$t_{37}=\left(a_{11}\alpha^3+a_{21}\alpha\beta^2+\frac{1}{2}a_{31}\alpha\beta^2\right)\frac{16\delta_m\delta_n}{9mn\pi^2},\ \ t_{38}=\left(a_{12}\alpha^2\beta+a_{22}\beta^3+\frac{1}{2}a_{31}\alpha^2\beta\right)\frac{16\delta_m\delta_n}{9mn\pi^2},$$

$$t_{39}=\left(a_{13}\alpha^3+a_{23}\alpha\beta^2+\frac{1}{2}a_{32}\alpha\beta^2\right)\frac{16\delta_m\delta_n}{9mn\pi^2},\ \ t_{310}=\left(a_{14}\alpha^2\beta+a_{24}\beta^3+\frac{1}{2}a_{33}\alpha^2\beta\right)\frac{16\delta_m\delta_n}{9mn\pi^2},$$

$$t_{311}=\frac{3}{32}\left(-a_{11}\alpha^4-a_{21}\alpha^2\beta^2-a_{12}\alpha^2\beta^2-a_{22}\beta^4\ +\frac{4}{3}a_{31}\alpha^2\beta^2\right),\qquad t_{312}=\frac{4\delta_m\delta_n}{mn\pi^2},$$

$$t_{41}=\left(-b_{11}+\lambda c_{11}\right)\alpha^2-\left(b_{31}-\lambda c_{31}\right)\beta^2,\quad t_{42}=-\left(b_{12}+b_{31}-\lambda c_{12}-\lambda c_{31}\right)\alpha\beta,$$

$$t_{43}=\left[\frac{-b_{12}}{R}-d_{11}-d_{13}+3\lambda\left(e_{11}+e_{13}\right)+\frac{\lambda c_{12}}{R}\right]\alpha\ -\left(b_{15}-\lambda c_{15}\right)\alpha^3$$

$$-\left(b_{16}+b_{34}-\lambda c_{16}-\lambda c_{34}\right)\alpha\beta^2,$$

$$t_{44}=-\left(b_{13}-\lambda c_{13}\right)\alpha^2-\left(b_{32}-\lambda c_{32}\right)\beta^2\ -d_{11}-d_{12}+3\lambda\left(e_{11}+e_{12}\right),$$

$$t_{45}=-\left(b_{14}+b_{33}-\lambda c_{14}-\lambda c_{33}\right)\alpha\beta,$$

$$t_{46}=\frac{-4\delta_m\delta_n}{9mn\pi^2}\left[2\left(b_{11}-\lambda c_{11}\right)\alpha^3+2\left(b_{31}-\lambda c_{31}\right)\alpha\beta^2\ -\left(b_{12}+b_{31}-\lambda c_{12}-\lambda c_{31}\right)\alpha\beta^2\right],$$

$$t_{51}=\left(-b_{31}-b_{21}+\lambda c_{31}+\lambda c_{21}\right)\alpha\beta,\quad t_{52}=-\left(b_{31}-\lambda c_{31}\right)\alpha^2-\left(b_{22}-\lambda c_{22}\right)\beta^2,$$

$$t_{53}=-\left(b_{34}+b_{25}-\lambda c_{34}-\lambda c_{25}\right)\alpha^2\beta\ +\left[\frac{-b_{22}}{R}-d_{21}-d_{23}+3\lambda\left(e_{21}+e_{23}\right)+\frac{\lambda c_{22}}{R}\right]\beta$$

$$t_{54}=-\left(b_{32}+b_{23}-\lambda c_{32}-\lambda c_{23}\right)\alpha\beta,\ t_{55}=-\left(b_{33}-\lambda c_{33}\right)\alpha^2-\left(b_{24}-\lambda c_{24}\right)\beta^2$$

$$-d_{21}-d_{22}+3\lambda\left(e_{21}+e_{22}\right),$$

$$t_{56}=\frac{-4\delta_m\delta_n}{9mn\pi^2}\left[2\alpha^2\beta\left(b_{31}-\lambda c_{31}\right)+2\beta^3\left(b_{22}-\lambda c_{22}\right)-\alpha^2\beta\left(b_{31}+b_{21}-\lambda c_{31}-\lambda c_{21}\right)\right]$$

The coefficients $g_i\left(i=\overline{1,5}\right)$ Eq. (37) are given as

$$
\begin{aligned}
& l_1 = \frac{-t_{22}t_{14} + t_{12}t_{24}}{t_{11}t_{22} - t_{12}t_{21}},\ l_2 = \frac{-t_{22}t_{15} + t_{12}t_{25}}{t_{11}t_{22} - t_{12}t_{21}},\ l_3 = \frac{-t_{22}t_{13} + t_{12}t_{23}}{t_{11}t_{22} - t_{12}t_{21}},\ l_4 = \frac{-t_{22}t_{16} + t_{12}t_{26}}{t_{11}t_{22} - t_{12}t_{21}}, \\
& l_5 = \frac{-t_{11}t_{24} + t_{21}t_{14}}{t_{11}t_{22} - t_{12}t_{21}},\ l_6 = \frac{-t_{11}t_{25} + t_{21}t_{15}}{t_{11}t_{22} - t_{12}t_{21}},\ l_7 = \frac{-t_{11}t_{23} + t_{21}t_{13}}{t_{11}t_{22} - t_{12}t_{21}}, l_8 = \frac{-t_{11}t_{26} + t_{21}t_{16}}{t_{11}t_{22} - t_{12}t_{21}} \\
& l_9 = \left[-\left(t_{51}l_2 + t_{52}l_6 + t_{55}\right)\left(t_{41}l_3 + t_{42}l_7 + t_{43}\right) + \left(t_{41}l_2 + t_{42}l_6 + t_{45}\right)\left(t_{51}l_3 + t_{52}l_7 + t_{53}\right)\right]\frac{1}{g_1} \\
& l_{10} = \left[-\left(t_{51}l_2 + t_{52}l_6 + t_{55}\right)\left(t_{41}l_4 + t_{42}l_8 + t_{46}\right) + \left(t_{41}l_2 + t_{42}l_6 + t_{45}\right)\left(t_{51}l_4 + t_{52}l_8 + t_{56}\right)\right]\frac{1}{g_1} \\
& l_{11} = \left[-\left(t_{41}l_1 + t_{42}l_5 + t_{44}\right)\left(t_{51}l_3 + t_{52}l_7 + t_{53}\right) + \left(t_{51}l_1 + t_{52}l_2 + t_{54}\right)\left(t_{41}l_3 + t_{42}l_7 + t_{43}\right)\right]\frac{1}{g_1} \\
& l_{12} = \left[-\left(t_{41}l_1 + t_{42}l_5 + t_{44}\right)\left(t_{51}l_4 + t_{52}l_8 + t_{56}\right) + \left(t_{51}l_1 + t_{52}l_2 + t_{54}\right)\left(t_{41}l_4 + t_{42}l_8 + t_{46}\right)\right]\frac{1}{g_1} \\
& g_1 = t_{31}\left(l_1 + l_2l_9 + l_3l_{11}\right) + t_{32}\left(l_6l_9 + l_7l_{11} + l_5\right) + t_{34}l_9 + t_{35}l_{11} + t_{33}, \\
& g_2 = t_{31}\left(l_2l_{10} + l_3l_{12} + l_4\right) + t_{32}\left(l_6l_{10} + l_7l_{12} + l_8\right) + t_{34}l_{10} + t_{35}l_{12}, \\
& g_3 = t_{38}\left(l_1 + l_2l_9 + l_3l_{11}\right) + t_{39}\left(l_6l_9 + l_7l_{11} + l_5\right) + t_{310}l_9 + t_{311}l_{11}, \\
& g_4 = t_{38}\left(l_2l_{10} + l_3l_{12} + l_4\right) + t_{39}\left(l_6l_{10} + l_7l_{12} + l_8\right) + t_{310}l_{10} + t_{311}l_{12} + t_{312}, \\
& g_5 = I_0 + \lambda^2 I_6\left(\alpha^2 + \beta^2\right) - \frac{1}{I_0L_2 - J_1^2}\left[\left(\alpha^2\lambda^2 I_3 + \beta^2\lambda^2 I_3\right)\left(I_3L_2 - J_1J_4\right) + \right. \\
& \left.\left(\alpha^2\lambda^2 J_4 + \beta^2\lambda^2 J_4\right)\left(I_0J_4 - I_3J_1\right)\right],\ g_6 = \alpha^2 N_x^0 - g_1 - \Phi_{1T}.
\end{aligned}
\tag{D2}
$$