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An Analytical Time Domain Solution for the Forced Vibration Analysis of Thick-Walled Cylinders

Abstract

In this paper, we propose a time domain analytical solution for the forced vibration analysis of thick-walled hollow cylinders in presence of polar orthotropy. In this regard, solution of the governing equation is decomposed into two parts. The role of the first one is to satisfy boundary conditions utilizing the method of separation of variables besides of Fourier series expansion of the non-homogenous boundary conditions. The second part has been also expressed as the series of orthogonal characteristic functions with the aim of satisfaction of initial conditions. The proposed analytical solution has been implemented to evaluate the dynamic response of the cylinder in solution of some sample problems which are chosen from previous studies.

Keywords

Forced vibration, Thick-walled cylinders, Analytical time domain solution, Polar orthotropy

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1 INTRODUCTION

The increasing application of thick-walled cylinders subjected to dynamic inner pressure in diverse fields such as aerospace engineering, civil engineering and submarine structures has made these members of paramount importance. In this regards, so many studies have been done to compute time dependent responses of the both isotropic and anisotropic cylinders (Huang, 1969; Keles and Tutuncu, 2009; Shakeri et al. 2006; Baba and Keles, 2015; Ghannad and Gharooni, 2015). In most of these researches, the time dependency of the governing equation has been eliminated utilizing the Laplace transform (Huang, 1969; Keles and Tutuncu, 2009; Baba and Keles, 2015).

Recently, Baba and Keles (2015) proposed an analytical solution for the anisotropic hollow cylinders under the internal dynamic pressure in Laplace domain. They also employed the modified Durbin's numerical inversion to obtain solution in time. Since the transformation of the dynamic response from the Laplace domain to the time domain problems is associated with some difficulties, here the new idea is employed from previous studies (Shamsaei and Boroomand, 2011; Movahedian and Boroomand, 2014; Movahedian et al. 2013) to propose an analytical time domain solution for the governing differential equation of the mentioned problem. This solution enables us to estimate the dynamic responses of the cylinder, i.e. the radial and hoop stresses or radial displacement, with desirable accuracy directly in time.

The layout of the paper is as follows, in the next section, the model used for the dynamic analysis of orthotropic hollow cylinders is described and the governing equations are derived. In section 3, the superposition principle is employed to express the solution in terms of two parts. In section 4, the proposed solution is applied to cases which were studied by Baba and Keles (2015) in order to validate the study. In addition, a problem with non-homogenous initial conditions is included in this part. Finally, in section 5, the summary of the conclusions made throughout the paper are provided.

2 PROBLEM STATEMENT

In this section, the governing differential equation of the vibration of thick-walled hollow cylinder in presence of polar orthotropy is derived. Utilizing the axisymmetric conditions, the radial strain, ε_r, and tangential strain, ε_θ are related to radial displacement u, as

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r}. \tag{1}$$

Considering the polar orthotropy of the cylinder, the stress-strain relation can be expressed as

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix} \tag{2}$$

where C_{11} and C_{22} are the stiffness modules in the radial and circumferential directions and C_{12} is the material parameter that includes the Poisson's effect.

Consider an element on the thick walled hollow cylinder bounded by lines (r,θ) and $(r + dr, \theta + d\theta)$. Due to symmetry, the radial and hoop stresses remain constant along angular coordinate, i.e., $[\partial\sigma_r / \partial\theta] = 0$ and $[\partial\sigma_\theta / \partial\theta] = 0$, and the shear stress component, $\tau_{r\theta}$, must be zero. In this way, equilibrium equation in the radial direction gives,

$$\frac{\partial(h\sigma_r)}{\partial r} + \frac{h(\sigma_r - \sigma_\theta)}{r} = \rho h \frac{\partial^2 u(r,t)}{\partial t^2} \tag{3}$$

where u is the displacement component in radial direction that must be found in polar coordinate r and time t. Also ρ and h are the material density and element's thickness. Figure 1 shows the geometry of thick walled hallow cylinder as well as stress component on the specified element in polar coordinate.

Substituting definitions of (1) and (2), in the above relation leads to the following equation, (the element's thickness has been removed from both sides of (3)),

$$\frac{\partial(h\sigma_r)}{\partial r} + \frac{h(\sigma_r - \sigma_\theta)}{r} = \rho h \frac{\partial^2 u(r,t)}{\partial t^2} \tag{4}$$

in which, $c = \sqrt{C_{11}/\rho}$ and $n = \sqrt{C_{22}/C_{11}}$ is a non-dimensional parameter that indicates degree of anisotropy of material. If the variation of internal and external pressures are respectively expressed by functions $P_I(t)$ and $P_E(t)$, the governing boundary conditions at $r = a$ and $r = ka$ can be stated as follow

$$\sigma_r \Big|_{r=a} = -P_I(t), \qquad \sigma_r \Big|_{r=ka} = P_E(t). \tag{5}$$

Aforementioned conditions can be expressed in terms of radial displacement, u, as follow

$$\left[C_{11}\left(\frac{\partial u}{\partial r} + \frac{1}{\alpha}\frac{u}{r}\right)\right]_{r=a} = -P_I(t), \qquad \left[C_{11}\left(\frac{\partial u}{\partial r} + \frac{1}{\alpha}\frac{u}{r}\right)\right]_{r=ka} = P_E(t), \tag{6}$$

where $\alpha = C_{11}/C_{12}$. The general form of the initial displacement and initial velocity conditions of can be satisfied by the following equation

$$\left[u(r,t)\right]_{t=0} = u_0(r), \qquad \left[\dot{u}(r,t)\right]_{t=0} = \dot{u}_0(r). \tag{7}$$

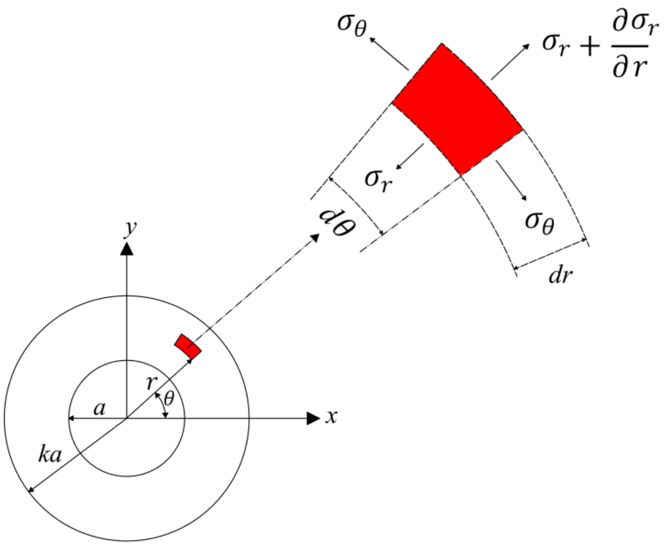


Figure 1: The geometry of thick walled hallow cylinder in polar coordinate.

3 THE SOLUTION METHOD

The aim here is to find the time domain analytical solution of the Equation (4) through employing Fourier's series expansion of boundary conditions as well as defining a suitable characteristic problem to satisfy initial conditions. In this regard, we split the solution into two parts as follows:

$$u(r,t) = u_1(r,t) + u_2(r,t) \tag{8}$$

In the above relation the role of $u_1(r,t)$ is to fully satisfy the boundary conditions stated in (6). After determining $u_1(r,t)$, obviously, the task of satisfying the actual initial conditions remains for $u_2(r,t)$ which will be explained later. Prior to construction of $u_1(r,t)$, one should expand the right hand sides of conditions in (6) in terms of Fourier series in time; i.e. Fourier sine series as:

$$P_I(t) = \sum_{i=1}^{N} A_i \sin \omega_i t, \qquad A_i = \frac{2}{T}\int_0^T P_I(t)\sin \omega_i t dt, \tag{9}$$

$$P_E(t) = \sum_{i=1}^{N} B_i \sin \omega_i t, \qquad B_i = \frac{2}{T}\int_0^T P_I(t)\sin \omega_i t dt, \tag{10}$$

where N is the number of the basis functions to be used, $\omega_i = i\pi/T$ and T is the length of finite time interval ($t \in [0,T]$ instead of $t \in [0,\infty)$). The magnitude of T may be determined by inspection, i.e. in successive solutions, one can enlarge T until the final solution to u converges to a solution for smaller time interval, $t \in [0,T_1]$, $T_1 < T$. Utilizing the method of separation of variables leads to express $u_1(r,t)$ in the following form:

$$u_1(r,t) = \sum_{i=1}^{N} \overline{u}_{1,i} \sin \omega_i t \tag{11}$$

where $\overline{u}_{1,i}(r)$ is the solution to the following ordinary differential equation which comes from substituting (11) in Equation (4).

$$r^2 \frac{d^2\overline{u}_{1,i}}{dr^2} + r\frac{d\overline{u}_{1,i}}{dr} + (\frac{r^2\omega_i^2}{c^2} - n^2)\overline{u}_{1,i} = 0. \tag{12}$$

The aforementioned equation is known as the Bessel differential equation of order n. The solution of which can be expressed as the combination of the Bessel function of the first kind, $J_n(\omega_i r/c)$, and the second kind, $Y_n(\omega_i r/c)$. In other words, the solution can be stated as

$$\overline{u}_{1,i}(r) = c_{1,i}J_n(\omega_i r/c) + c_{2,i}Y_n(\omega_i r/c) \tag{13}$$

The constant coefficients $c_{1,i}$ and $c_{2,i}$ in the above relation are determined by satisfaction of the radial stress boundary conditions at $r = a$ and $r = ka$ in (6) as

$$\left[C_{11}\left(\frac{d\overline{u}_{1,i}}{dr} + \frac{1}{\alpha}\frac{\overline{u}_{1,i}}{r}\right)\right]_{r=a} = -A_i, \tag{14}$$

$$\left[C_{11} \left(\frac{d\overline{u}_{1,i}}{dr} + \frac{1}{\alpha} \frac{\overline{u}_{1,i}}{r} \right) \right]_{r=ka} = B_i, \tag{15}$$

Finally, after doing some simplifications, $u_1(r,t)$ is therefore written as

$$u_1(r,t) = \sum_{i=1}^{N} \frac{ca\alpha}{C_{11}} \frac{Y_n(\omega_i r/c)\varphi_{1,i} + J_n(\omega_i r/c)\varphi_{2,i}}{\varphi_{3,i} + \varphi_{4,i}} \sin \omega_i t \tag{16}$$

where

$$\begin{aligned} \varphi_{1,i} &= ck(n\alpha - 1)B_i J_n(\omega_i a/c) + c(n\alpha - 1)A_i J_n(\omega_i ka/c) \\ &\quad - ka\,\alpha\,\omega_i [B_i J_{n-1}(\omega_i a/c) + A_i J_{n-1}(\omega_i ka/c)] \end{aligned} \tag{17}$$

$$\begin{aligned} \varphi_{2,i} &= -c(n\alpha - 1)[kB_i Y_n(\omega_i a/c) + A_i Y_n(\omega_i ka/c)] \\ &\quad + ka\,\alpha\,\omega_i [B_i Y_{n-1}(\omega_i a/c) + A_i J_{n-1}(\omega_i ka/c)] \end{aligned} \tag{18}$$

$$\varphi_{3,i} = c^3(n\alpha - 1)^2 [J_n(\omega_i ka/c) Y_n(\omega_i a/c) - J_n(\omega_i a/c) Y_n(\omega_i ka/c)] \tag{19}$$

$$\begin{aligned} \varphi_{4,i} &= +a\alpha\omega_i \{c(n\alpha - 1)[-J_n(\omega_i ka/c) Y_{n-1}(\omega_i a/c) + kJ_n(\omega_i a/c) Y_{n-1}(\omega_i ka/c) \\ &\quad - kJ_{n-1}(\omega_i ka/c) Y_n(\omega_i a/c) + J_{n-1}(\omega_i a/c) Y_n(\omega_i ka/c)] \\ &\quad + ka\alpha\omega_i [J_{n-1}(\omega_i ka/c) Y_{n-1}(\omega_i a/c) - J_{n-1}(\omega_i ka/c) Y_{n-1}(\omega_i ka/c)]\} \end{aligned} \tag{20}$$

At this point, the second part of relation (8) must be determined. In this regard, the method of separation of variables is applied by substituting

$$u_2(r,t) = \overline{u}_2(r) T(t) \tag{21}$$

in (4) which yields:

$$\frac{r^2 \dfrac{d^2\overline{u}_2(r)}{dr^2} + r \dfrac{d\overline{u}_2(r)}{dr} - \dfrac{n^2}{r^2} \overline{u}_2(r)}{\overline{u}_2(r)} = \frac{1}{c^2 T(t)} \frac{d^2 T(t)}{dt^2} = -\beta^2, \tag{22}$$

where $-\beta^2$ is the separation constant. The solutions of the two separated ordinary differential equations in (22) for $\overline{u}_2(r)$ and $T(t)$ are respectively expressed as:

$$\overline{u}_2(r) = c_3 J_n(r\beta) + c_4 Y_n(r\beta) \tag{23}$$

and

$$T(t) = a \sin c\beta t + b \cos c\beta t \tag{24}$$

In order to determine the unknown coefficients of c_3 , c_4 and β, a characteristic problem must be formed by substituting (21) in the homogenous form of stress boundary conditions, i.e., $P_I(t) = P_E(t) = 0$ in (6), which results in following system of algebraic equations,

$$\mathbf{A}_\beta \mathbf{C} = \mathbf{0} \tag{25}$$

In the above relation, $\mathbf{A}_\beta$ is a 2×2 matrix depending on β, and $\mathbf{C}$ is a 2×1 vector containing the coefficients of c_3 and c_4. The components of $\mathbf{A}_\beta$ are:

$$\mathbf{A}_\beta = \begin{bmatrix} \dfrac{a\alpha\beta J_{n-1}(a\beta) + (1-n\alpha)J_n(a\beta)}{a\alpha} & \dfrac{a\alpha\beta Y_{n-1}(a\beta) + (1-n\alpha)Y_n(a\beta)}{a\alpha} \\ \dfrac{ka\alpha\beta J_{n-1}(ka\beta) + (1-n\alpha)J_n(ka\beta)}{ka\alpha} & \dfrac{ka\alpha\beta Y_{n-1}(ka\beta) + (1-n\alpha)Y_n(ka\beta)}{ka\alpha} \end{bmatrix}. \tag{26}$$

To have non-trivial solution to (25), the determinant of $\mathbf{A}_\beta$ is set zero.

$$\left|\mathbf{A}_\beta\right| = 0. \tag{27}$$

The above mentioned issue a non-standard eigenvalue problem which should be solved for β. Moreover, by substituting the roots of (27), i.e. β_j , $j = 1,2,\ldots$ in (25), the corresponding null space $\mathbf{C}_j$, $j = 1,2,\ldots$ with components $c_{3,j}$ and $c_{4,j}$, have been computed. In this way, the second part of the $u(r,t)$ is therefore written as

$$u_2(r,t) = \sum_{j=1}^{M} \overline{u}_{2,j}(r) \times \left[a_j \sin c\beta_j\, t + b_j \cos\, c\beta_j\, t\right], \tag{28}$$

in which $\overline{u}_{2,j}(r) = c_{3,j} J_n(r\,\beta_j) + c_{4,j} Y_n(r\beta_j)$ and (a_j, b_j) denote a set of new unknown coefficients to be determined by satisfying the initial displacement and velocity conditions in (7). Utilizing the orthogonality of the set $\overline{u}_{2,j}(r)$ with respect to weight function $w(r) = r$, (See Hildebrand (1976), for more details on the properties of Strum-Liouville problems), these coefficients have been determined by inserting (28) and (11) in (7) as follows

$$a_j = \frac{\displaystyle\int_a^{ka} r\overline{u}_{2,j}(r)\left[\dot{u}_0(r) - \sum_{i=1}^{N}\overline{u}_{1,i}(r)\omega_i\right]dr}{\displaystyle\int_a^{ka} c\beta_j r\left[\overline{u}_{2,j}(r)\right]^2 dr}, \tag{29}$$

$$b_j = \frac{\int_a^{ka} r\bar{u}_{2,j}(r)u_0(r)dr}{\int_a^{ka} r\left[\bar{u}_{2,j}(r)\right]^2 dr}. \tag{30}$$

In this way, the radial displacement of a thick-walled cylinder can now be written as:

$$\begin{aligned} u(r,t) = {} & \sum_{i=1}^{N} \frac{ca\alpha}{C_{11}} \frac{Y_n(\omega_i r / c)\varphi_{1,i} + J_n(\omega_i r / c)\varphi_{2,i}}{\varphi_{3,i} + \varphi_{4,i}} \sin(\omega_i t) + \\ & \sum_{j=1}^{M} [c_{3,j} J_n(r\beta_j) + c_{4,j} Y_n(r\beta_j)] \times \left[a_j \sin(c\beta_j t) + b_j \cos(c\beta_j t)\right]. \end{aligned} \tag{31}$$

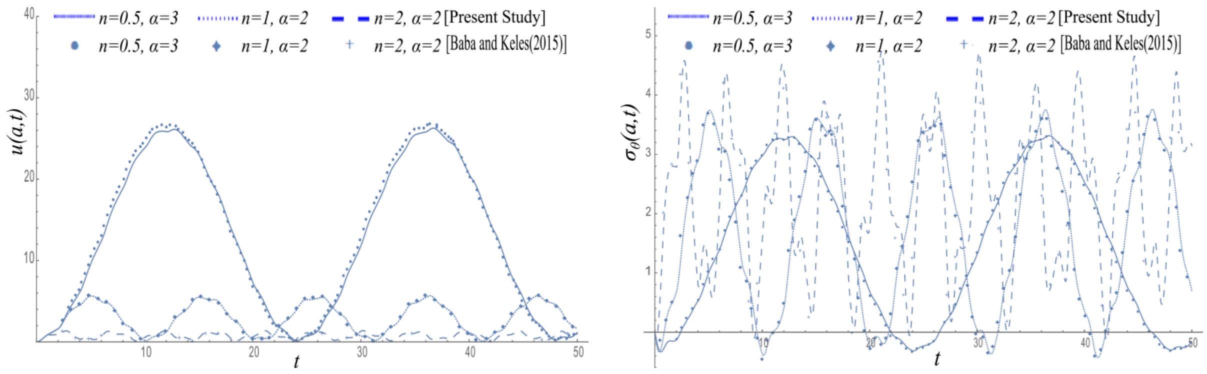
4 RESULTS AND DISCUSSIONS

The proposed method has been used for both homogenous and nonhomogeneous initial displacement conditions. In the both cases the specifications of orthotropic hollow cylinder are taken form Baba and Keles (2015), as $C_{11} = 1$, $\rho = 1$, $a = 1$ and $ka = 2$. Moreover, the analytical solution in (31) has been computed using the first 100 terms of the series of $u_1(r,t)$ with $T = 80$ for the Fourier sin series expansion of $P_I(t)$ in (9), and the first 100 terms of the series of $u_2(r,t)$, i.e. $M = 100$. In Table 1, the first 15 sets of eigenvalues, β_j, and ratio of the components of the related eigenvector, $\mathbf{C}_j$, have been provided for three types of material with different degrees of anisotropy.

	$n = 0.5$, $\alpha = 3$		$n = 2$, $\alpha = 2$		$n = 1$, $\alpha = 2$	
j	β_j	$c_{3,j} / c_{4,j}$	β_j	$c_{3,j} / c_{4,j}$	β_j	$c_{3,j} / c_{4,j}$
1	0.25992	-0.91437	1.34969	-0.46413	0.61059	-0.92914
2	3.16786	-12.65974	3.45185	0.11203	3.20507	1.46431
3	6.29641	-25.17978	6.43419	-0.35032	6.31359	1.19987
4	9.43361	-37.73051	9.52479	-0.52919	9.44485	1.12786
5	12.573	-50.28905	12.6412	-0.62901	12.58137	1.0941
6	15.71327	-62.85071	15.76776	-0.69341	15.71994	1.07447
7	18.85398	-75.41394	18.89936	-0.73859	18.85953	1.06163
8	21.99494	-87.97806	22.03382	-0.77208	21.99969	1.05257
9	25.13606	-100.54275	25.17007	-0.79793	25.14021	1.04583
10	28.27728	-113.10781	28.30751	-0.81849	28.28097	1.04063
11	31.41858	-125.67314	31.44578	-0.83525	31.4219	1.03649
12	34.55993	-138.23865	34.58466	-0.84916	34.56295	1.03311
13	37.70132	-150.80431	37.72399	-0.8609	37.70409	1.03031
14	40.84274	-163.37007	40.86367	-0.87095	40.8453	1.02794
15	43.98419	-175.93592	44.00362	-0.87964	43.98656	1.02592

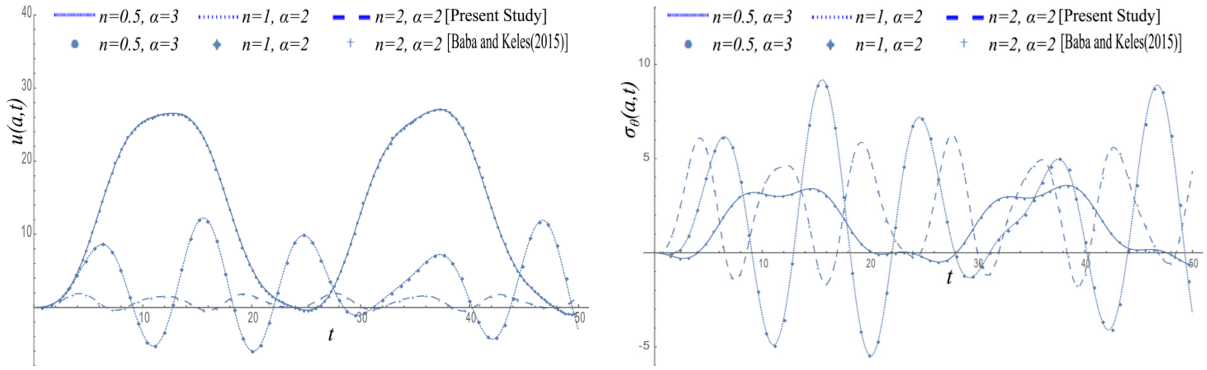
Table 1: Results of the first 15 sets of the non-standard eigenvalue problem in (25) for three anisotropy's types of the material.

In the case of homogenous initial conditions, three different dynamic inner pressure functions were considered as $P_I(t)=1$, $P_I(t)=1-\exp(0.8t)$ and $P_I(t)=1-\cos(0.8t)$ to validate results with those stated in Baba and Keles (2015) (The variations of outer pressure was not considered in the mentioned reference, i.e. $P_E(t)=0$). In this regards, the variation of radial displacement, $u(r,t)$, and hoop stress, $\sigma_\theta(r,t)$ at $r = a$ are illustrated in Figures 2 to 4. As can be seen, the results of in Baba and Keles (2015) and presented method follow similar trends in evaluation of both u_r and σ_θ.



(a) (b)

Figure 2: Variation of (a) $u(a,t)$ and (b) $\sigma_\theta(a,t)$ due to $P_I(t)=1$ for three different degrees of anisotropy.



(a) (b)

Figure 3: Variation of (a) $u(a,t)$ and (b) $\sigma_\theta(a,t)$ due to $P_I(t)=1-\exp(0.8t)$

for three different degrees of anisotropy.

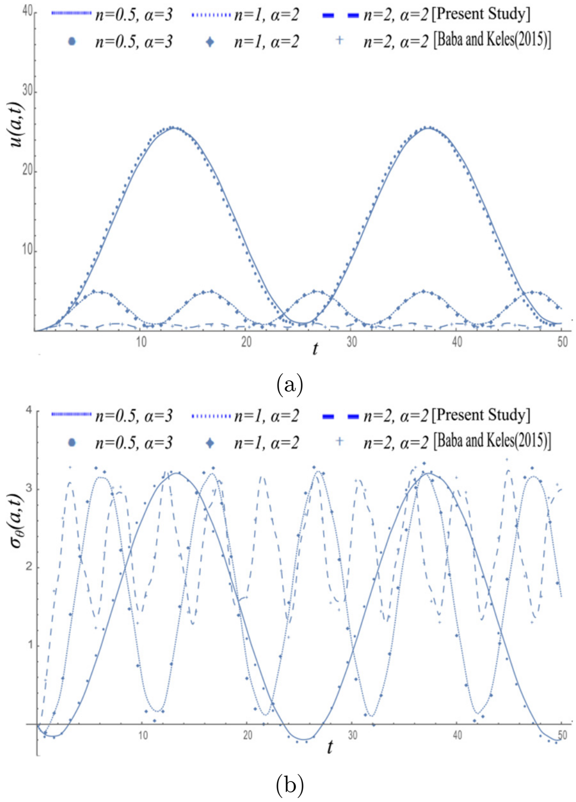


Figure 4: Variation of (a) $u(a,t)$ and (b) $\sigma_\theta(a,t)$ due to $P_I(t)=1-\cos(0.8t)$ for three different degrees of anisotropy.

As mentioned previously, the presented analytical solution is able to predict dynamic response of the hollow cylinder even in presence of nonhomogeneous initial and external pressure boundary conditions. In this regards, the second sample problem has been chosen to investigate the forced vibration of the orthotropic cylinder due to following variations of the inner and outer pressures,

$$P_I(t) = 1 - \cos(0.8t), \qquad P_E(t) = 1 \tag{32}$$

The initial displacement and initial velocity conditions are also considered as follows:

$$u_0(r) = (r-a)^2(r-k\,a)^2, \qquad \dot{u}_0(r) = 0 \tag{33}$$

Figure 5 depicts the variations of $\sigma_r(r,t)$ and $\sigma_\theta(r,t)$ within the thickness of the cylinder with $n = 0.5$ and $\alpha = 3$ at two time steps $t = 2$ and $t = 5$.

In order to provide design criteria from the standpoint of fatigue of orthotropic cylinders, the variations of $\sigma_\theta(a,t)$ and $\sigma_\theta(ka,t)$ are illustrated in Figure 6 for three different degrees of anisotropy. As can be seen, increasing degrees of anisotropy, n, will result in an increase in the frequency of cylinder's response

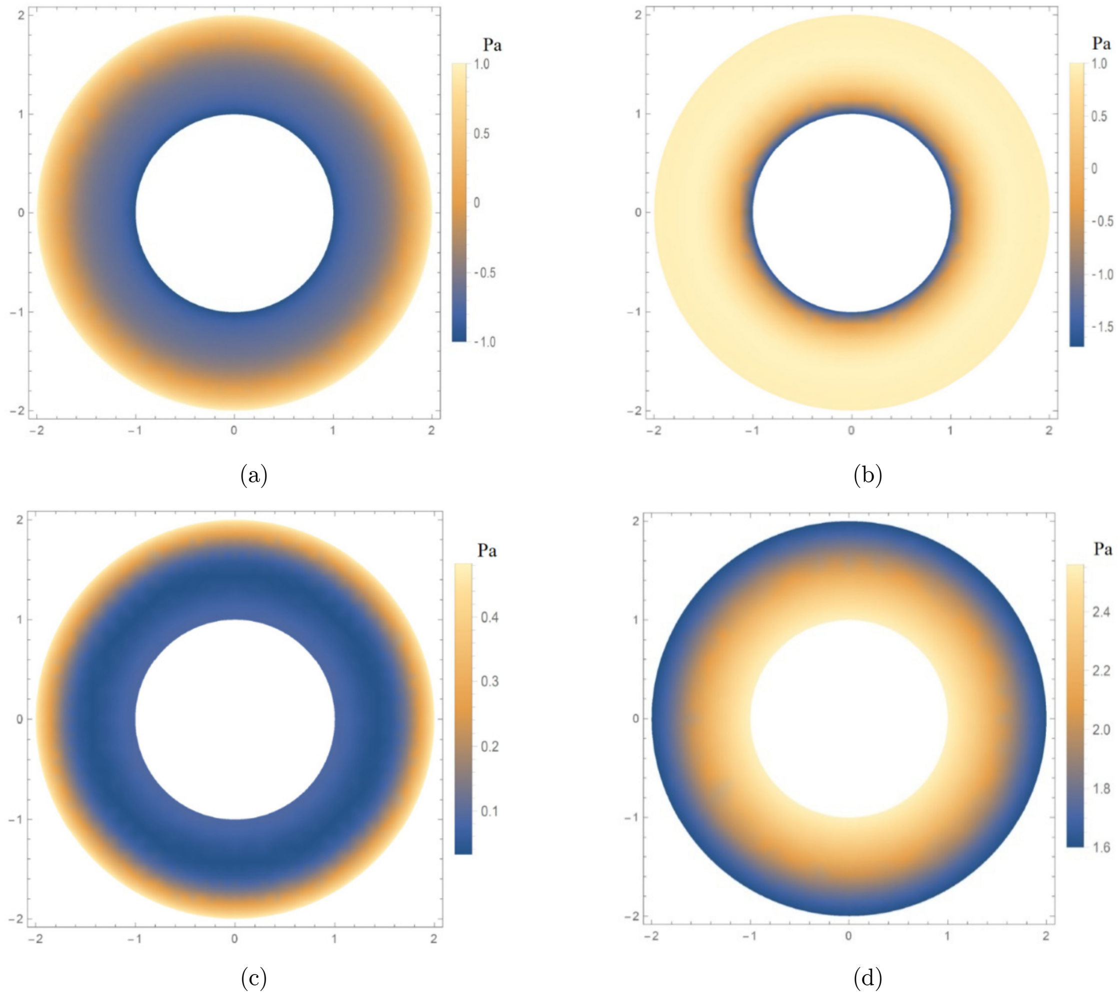


Figure 5: Variation of (a) $\sigma_r(r,2)$, (b) $\sigma_r(r,5)$, (c) $\sigma_\theta(r,2)$ and (d) $\sigma_\theta(r,5)$ within the cylinder's thickness.

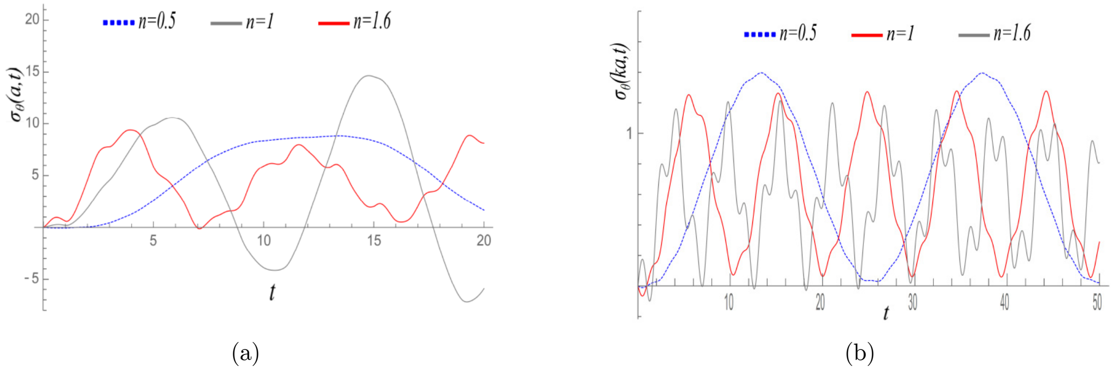


Figure 6: Variation of (a) $\sigma_\theta(a,t)$, (b) $\sigma_\theta(ka,t)$ due to $P_I(t)=1-\cos(0.8t)$

and $P_E(t)=1$ for three different degrees of anisotropy.

5 CONCLUSIONS

In the present study, a semi analytical time domain solution has been proposed for the governing equation to the vibration of thick-walled hollow cylinder in the presence of polar orthotropy. The effects of different material properties and internal pressure variations on the dynamic responses of hollow cylinder have been investigated. The sufficient accuracy of the presented method has been also illustrated in comparison of the obtained results with those reported in Baba and Keles (2015). Finally, the superiorities of the presented solution can be listed as follows:

- Employing the analytical solution, the dynamic response of the cylinder can be evaluated directly in time with no need to use any transformation such as inverse Laplace transform.
- The proposed scheme can be used to evaluate dynamic response of polar orthotropic cylinders in presence of exterior pressure or non-homogenous initial conditions, which may be useful for designing purposes.
- The presented method can be extended to evaluate transient response of the pipe conveying fluid due to internal and external temperature variations.

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