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Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review

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Abstract

This review features a survey of some recent developments in asymptotic techniques and new developments, which are valid not only for weakly nonlinear equations, but also for strongly ones. Further, the achieved approximate analytical solutions are valid for the whole solution domain. The limitations of traditional perturbation methods are illustrated, various modified perturbation techniques are proposed, and some mathematical tools such as variational theory, homotopy technology, and iteration technique are introduced to over-come the shortcomings.In this review we have applied different powerful analytical methods to solve high nonlinear problems in engineering vibrations. Some patterns are given to illustrate the effectiveness and convenience of the methodologies.

Keywords

Nonlinear Vibration; Nonlinear Response; Analytical Methods; Parameter Perturbation Method (PPM); Variational Iteration Method(VIM); Homotopy Perturbation Method (HPM); Iteration Perturbation Method (IPM); Energy Balance Method (EBM); Parameter-Expansion Method (PEM); Variational Approach (VA); Improved Amplitude Frequency Formulation (IAFF); Max-Min Approach (MMA); Hamiltonian Approach (HA); Homotopy Analysis Method (HAM); Review

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1 INTRODUCTION

Most of engineering problems, especially some oscillation equations are nonlinear, and in most cases it is difficult to solve such equations, especially analytically. Recently, nonlinear oscillator models have been widely considered in physics and engineering. It is obvious that there are many nonlinear equations in the study of different branches of science which do not have analytical solutions. Due to the limitation of existing exact solutions, many analytical and numerical approaches have been investigated. Therefore, these nonlinear equations must be solved using other methods. Many researchers have been working on various analytical methods for solving nonlinear oscillation systems in the last decades. Perturbation technique is one the well- known methods [3, 11, 34, 37, 39, 85], the traditional perturbation method contains many shortcomings. They are not useful for strongly nonlinear equations, so for overcoming the shortcomings, many new techniques have been appeared in open literatures.

It should be mentioned that several books appeared on the subject of mathematical methods in engineering problem during the past decade [10, 48, 54, 77, 113, 125, 133, 137, 144–146, 180, 186, 187].

The aim of this article is to review the recent research on the approximate analytical methods for nonlinear vibrations. The applications of these methods have been appeared in open literatures in the last three years. There are hundreds of published papers too numerous to refer to all of them, but for the purpose of filling the gaps in the present summary, Refs[14, 15, 28, 30, 36, 40, 47, 55, 66, 75, 76, 83, 88, 89, 94, 142, 166, 170, 173, 178, 192, 193, 210, 217]may offer good help in overcoming the inevitable shortcomings in a condensed presentation. To show the efficiency and accuracy of the methods some comparisons have done with the results obtained by those methods and numerical methods and they are valid for whole domain. Some of the ideas first appeared in this review article, and most cited references were published in the last three years, revealing the most emerging research fronts. In this review, the basic idea of each method is presented then some examples are illustrated and discussed to show the application of these methods.

2 PARAMETERIZED PERTURBATION METHOD (PPM)

Recently, nonlinear oscillator models have been widely considered in physics and engineering. Study of nonlinear problems which are arisen in many areas of physics and also engineering is very significant for scientists. Surveys of the literature with numerous references have been given by many authors utilizing various analytical methods for solving nonlinear oscillation systems. Non-linear problems continue to be as a challenge, and heed has mainly concentrated on qualitative changes of systems bifurcations and instability. Parameterized Perturbation Method (PPM) is one of the well-known methods for solving nonlinear vibration equations. The method was proposed in by He in 1999 [80].It was rarely used recently, but this method is a kind of powerful tool for treating weakly nonlinear problems, but they are less effective for analyzing strongly nonlinear problems [37, 50, 86, 92, 115, 160].

2.1 Basic idea of Parameterized Perturbation Method

For the nonlinear equation $L(u) + N(u) = 0$, where L and N are general linear and nonlinear differential operators respectively, a linear transformation can be introduced as:

$$u = \varepsilon\nu \tag{1}$$

We can assume that ν can be written as a power series in ε,as following

$$\nu = \nu_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + ..., \tag{2}$$

And

$$\nu = \lim_{\varepsilon \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \nu_3 + \ldots \tag{3}$$

2.2 Application of Parameterized Perturbation Method

Two examples have considered showing the applicability of this method.

Example 1

Consider the following Duffing equation:

$$\ddot{u} + \alpha u + \beta u^3 = 0, \qquad u(0) = A,\ \dot{u}(0) = 0 \tag{4}$$

We let $u = \varepsilon\nu$ in Eq. (4) and obtain

$$\ddot{\nu} + \alpha\nu + \varepsilon^2\beta\nu^3 = 0, \quad \nu(0) = A/\varepsilon\,,\ \dot{\nu}(0) = 0 \tag{5}$$

Supposing that the solution of Eq. (5)and ω^2can be expressed in the form

$$\nu = \nu_0 + \varepsilon^2\nu_1 + \varepsilon^4\nu_2 + \varepsilon^6\nu_3 \tag{6}$$

$$\alpha = \omega^2 + \varepsilon^2\omega_1 + \varepsilon^4\omega_2 + \varepsilon^6\omega_3 \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5) and equating coefficients of like powers of ε yields the following equations

$$\ddot{\nu}_0 + \omega^2\nu_0 = 0, \qquad \nu_0(0) = A/\varepsilon \quad,\ \dot{\nu}_0(0) = 0, \tag{8}$$

$$\ddot{\nu}_1 + \omega^2\nu_1 + \omega_1\nu_0 + \beta\nu_0^3 = 0\ , \qquad \nu_1(0) = 0\ ,\ \dot{\nu}_1(0) = 0 \tag{9}$$

Solving Eq. (8) results in

$$\nu_0 = \frac{A}{\varepsilon}\cos\omega\, t \tag{10}$$

Equation (9), therefore, can be re-written down as

$$\ddot{\nu}_1 + \omega^2\nu_1 + \left(\omega_1 + \frac{3\beta A^2}{4\varepsilon^2}\right)\frac{A}{\varepsilon}\cos\left(\omega\, t\right) + \frac{\beta A^3}{4\varepsilon^3}\cos\left(3\omega\, t\right) = 0. \tag{11}$$

Avoiding the presence of a secular terms needs:

$$\omega_1 = -\frac{3\beta A^2}{4\varepsilon^2} \tag{12}$$

Substituting Eq. (12) into Eq. (7)

$$\omega_{PPM} = \sqrt{\alpha + \frac{3}{4}\beta A^2} \tag{13}$$

Solving Eq. (11), gives:

$$\nu_1 = -\frac{A^3\beta}{32\omega^2\varepsilon^3}\left(\cos(\omega t) - \cos(3\omega t)\right) \tag{14}$$

Its first-order approximation is sufficient, and then we have:

$$u = \varepsilon\nu = \varepsilon(\nu_0 + \varepsilon^2\nu_1) = A\cos(\omega t) - \frac{A^3\beta}{32\omega^2\varepsilon^3}\left[\cos(\omega t) - \cos(3\omega t)\right] \tag{15}$$

The exact frequency of this problem is:

$$\omega_{Exact} = 2\pi \Big/ 4\sqrt{2}\int_0^{\pi/2} \frac{dt}{\sqrt{\beta \mathrm{A}^2\cos^2(\mathrm{t}) + \beta\mathrm{A}^2 + 2\alpha}} \tag{16}$$

Table 1 Comparison of the approximate frequencies with the exact period.

A	α	β	Present Study (PPM)	Exact Solution	Error % $(\omega_{PPM} - \omega_{ex})/\omega_{ex}$
0.1	0.5	0.1	0.7076	0.7076	0.0000
0.5	0.1	2	0.6892	0.6800	1.3501
1	2	0.5	1.5411	1.5403	0.0520
2	5	2	3.3166	3.2958	0.6313
5	2	5	9.7852	9.5818	2.1228
10	1	0.5	6.2048	6.0772	2.0994
15	0.5	2	18.3848	17.9866	2.2135
20	5	1	17.4642	17.0977	2.1436

The maximum relative error is less than 2.2135% for this example.

Example 2

We consider the following nonlinear oscillator [89];

$$\left(1 + u^2\right)\ddot{u} + u = 0\ ,\ \ u(0) = A\ \ ,\ \dot{u}(0) = 0. \tag{17}$$

We let $u = \varepsilon\nu$in Eq. (17) and obtain

$$\ddot{\nu} + 1.\nu + \varepsilon^2\nu^2\ddot{\nu} = 0, \quad \nu(0) = \frac{A}{\varepsilon},\ \dot{\nu}(0) = 0. \tag{18}$$

Supposing that the solution of Eq. (18)and ω^2can be expressed in the form

$$\nu = \nu_0 + \varepsilon^2\nu_1 + \varepsilon^4\nu_2 + \ldots \tag{19}$$

$$1 = \omega^2 + \varepsilon^2\omega_1 + \varepsilon^4\omega_2 + ... \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (18) and equating coefficients of like powers of ε yields the following equations

$$\ddot{\nu}_0 + \omega^2\nu_0 = 0, \qquad \nu_0(0) = \frac{A}{\varepsilon} \ , \ \dot{\nu}_0(0) = 0, \tag{21}$$

$$\ddot{\nu}_1 + \omega^2\nu_1 + \omega_1\nu_0 + \nu_0^2\ddot{\nu}_0 = 0\ , \quad \nu_1(0) = 0\ , \dot{\nu}_1(0) = 0. \tag{22}$$

Solving Eq. (21) results in

$$\nu_0 = \frac{A}{\varepsilon}\cos\omega t \tag{23}$$

Equation (22), therefore, can be re-written down as

$$\nu_1'' + \omega^2\nu_1 + \frac{\omega_1 A}{\varepsilon}\cos\omega t - \frac{\omega^2A^3}{\varepsilon^3}\cos^3\omega t = 0 \tag{24}$$

Or

$$\nu_1'' + \omega^2\nu_1 + (\frac{\omega_1 A}{\varepsilon} - \frac{3\omega^2A^3}{4\varepsilon^3})\cos 3\omega t = 0. \tag{25}$$

We let

$$\omega_1 = \frac{3\omega^2A^2}{4\varepsilon^2} \tag{26}$$

In Eq. (25) so that the secular term can be eliminated. Solving Eq. (25) yields;

$$\nu_1 = \frac{A^3}{32\varepsilon^3}(\cos\omega t - \cos 3\omega t) \tag{27}$$

Thus we obtain the first-order approximate solution of the original Eq. (17), which reads

$$u = \varepsilon(\nu_0 + \varepsilon^2\nu_1) = A\cos\omega t - \frac{A^3}{32}(\cos\omega t - \cos 3\omega t) \tag{28}$$

Substituting Eq. (26) into Eq. (20) results in

$$1 = \omega^2 + \varepsilon^2\omega_1 = \omega^2 + \frac{3\omega^2A^2}{4} \tag{29}$$

Then we have;

$$\omega_{PPM} = \frac{1}{\sqrt{1 + \frac{3}{4}A^2}} \tag{30}$$

Eq. (30) gives the same frequency as that resulting from the artificial parameter Linstedt–Poincare method [89].

3 VARIATIONAL ITERATION METHOD (VIM)

Nonlinear phenomena play a crucial role in applied mechanics and physics. By solving nonlinear equations we can guide authors to know the described process deeply. But it is difficult for us to obtain the exact solution for these problems. In recent decades, there has been great development in the numerical analysis and exact solution for nonlinear partial equations. There are many standard methods for solving nonlinear partial differential equations. The variational iteration method was first proposed by He [82]used to obtain an approximate analytical solutions for nonlinear problems.In VIM in most cases only one iteration leads to high accuracy of the solution and it doesn't need any linearization or discretization, and large computational work. The VIM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations [35, 57, 62, 99, 104, 117, 122, 136, 139, 153, 167, 177, 179, 184, 191, 202, 206].We have considered three examples to show the implement of the VIM.

3.1 Basic idea of Variational Iteration Method

To illustrate its basic concepts of the new technique, we consider following general differential equation[82]:

$$Lu + Nu = g\left(x \right) \tag{31}$$

Where, L is a linear operator, and N a nonlinear operator, g(x) an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correct functional as follows:

$$u_{(n+1)}(t) = u_n(t) + \int_0^t \lambda\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}d\tau \tag{32}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\tilde{} = 0\ n\ \delta u.$

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

3.2 Application of Variational Iteration Method

Example 1

The equation of motion of a mass attached to the center of a stretched elastic wire in dimensionless is[181]:

$$\ddot{u} + u - \frac{\eta u}{\sqrt{1+u^2}} = 0\ , \quad 0 < \lambda \le 1 \tag{33}$$

With initial conditions

$$u(0) = A \quad , \dot{u}(0) = 0 \tag{34}$$

Assume that the angular frequency of the system (33) is ω, we have the following linearized equation:

$$\ddot{u} + \omega^2 u = 0 \tag{35}$$

So we can rewrite Eq. (33) in the form

$$\ddot{u} + \omega^2 u + g(u) = 0 \tag{36}$$

Where $g(u) = (1-\omega^2)u - \frac{\eta u}{\sqrt{1+u^2}}$

Applying the variational iteration method, we can construct the following functional equation:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\ddot{u}(\tau) + \omega^2 u_n(\tau) - g(\tau))d\tau \tag{37}$$

Where $\tilde{g}$ is considered as a restricted variation, i.e.,$\delta\tilde{g} = 0$.

Calculating variation with the respect to u_n and nothing that$\delta\tilde{g}(u_n) = 0$. We have the following stationary conditions:

$$\begin{aligned} &\lambda'' + \omega^2\lambda(\tau) = 0, \\ &\lambda(\tau)\,|_{\tau=t} = 0, \\ &1 - \lambda'(\tau)\,|_{\tau=t} = 0. \end{aligned} \tag{38}$$

The Lagrange multiplier, therefore, can be identified as;

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \tag{39}$$

Substituting the identified multiplier into Eq.(37) results in the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t) \times \left(\ddot{u}(\tau) + u(\tau) - \frac{\eta u(\tau)}{\sqrt{1+u^2(\tau)}}\right)d\tau \tag{40}$$

Assuming its initial approximate solution has the form

$$u_0 = A\cos(\omega\ t) \tag{41}$$

And substituting Eq. (41) into Eq. (33) leads to the following residual:

$$R_0(t) = -A\omega^2\cos(\omega t) + A\cos(\omega t) - \left(\frac{A\eta}{\sqrt{1+A^2}} + \frac{1}{2}\frac{A^3\eta\omega^2 t^2}{(1+A^2)} + O(t^3)\right)\cos(\omega t). \tag{42}$$

By the formulation (40), we can obtain

$$u_1(t) = A\cos(\omega t) + \int_0^t \frac{1}{\omega}\sin\omega(\tau - t)\,R_0(\tau)d\tau., \tag{43}$$

In order to ensure that no secular terms appear in u_1, resonance must be avoided. To do so, the coefficient of $\cos(\omega\,t)$in Eq. (42) requires being zero, i.e.,

$$\omega_{VIM} = \frac{\sqrt{1 + A^2 - \sqrt{1 + A^2}\eta}}{\sqrt{1 + A^2}} \tag{44}$$

And period of oscillation for this system by variational iteration method is;

$$T_{VIM} = \frac{2\pi\sqrt{1 + A^2}}{\sqrt{1 + A^2 - \sqrt{1 + A^2}\eta}} \tag{45}$$

Table 2 Comparison of the approximate periods with the exact period[1].

A	η	T $_{VIM}$	T$_{exact}$[181]	Error %
0.1	0.1	6.621237	6.62168	0.00669
1	0.1	6.517854	6.537508	0.300634
10	0.1	6.314678	6.322938	0.130635
0.1	0.5	8.863794	8.869257	0.061595
1	0.5	7.814722	7.992133	2.21982
10	0.5	6.445572	6.490208	0.687744
0.1	0.75	12.47385	12.49673	0.183088
1	0.75	9.168186	9.625404	4.750118
10	0.75	6.531632	6.602092	1.067237

Table 2 shows an excellent agreement of the VIM with the exact one.

Example 2

For the second example, we consider Duffing equation:

$$\ddot{u} + u + \varepsilon u^3 = 0 \tag{46}$$

With initial conditions

$$u(0) = A \quad , \dot{u}(0) = 0 \tag{47}$$

Assume that the angular frequency of the Eq.(46) is ω, we have the following linearized equation:

$$\ddot{u} + \omega^2 u = 0 \tag{48}$$

So we can rewrite Eq. (46) in the form

$$\ddot{u} + \omega^2 u + g(u) = 0 \tag{49}$$

Where $g(u) = u + \varepsilon u^3 - \omega^2 u$.

Applying the variational iteration method, we can construct the following functional equation:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\ddot{u}(\tau) + \omega^2 u_n(\tau) - g(\tau))d\tau \tag{50}$$

Where $\tilde{g}$ is considered as a restricted variation, i.e.,$\delta\tilde{g} = 0$.

Calculating variation with the respect to u_n and nothing that $\delta\tilde{g}(u_n) = 0$. We have the following stationary conditions:

$$\begin{aligned} &\lambda'' + \omega^2\lambda(\tau) = 0, \\ &\lambda(\tau)\,|_{\tau=t} = 0, \\ &1 - \lambda'(\tau)\,|_{\tau=t} = 0. \end{aligned} \tag{51}$$

The Lagrange multiplier, therefore, can be identified as;

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \tag{52}$$

Substituting the identified multiplier into Eq.(50) results in the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t) \times \left(\ddot{u}_n(\tau) + u_n(\tau) + \varepsilon u_n^3(\tau)\right) d\tau \tag{53}$$

Assuming its initial approximate solution has the form

$$u_0 = A\cos(\omega\, t) \tag{54}$$

And substituting Eq. (54) into Eq. (46) leads to the following residual:

$$R_0(t) = \left(1 - \omega^2 + \frac{3}{4}\varepsilon A^2\right) A\cos(\omega t) + \frac{1}{4}\,\varepsilon A^3\cos(3\omega t)\,. \tag{55}$$

By the formulation (53), we can obtain

$$u_1(t) = A\cos(\omega t) + \int_0^t \frac{1}{\omega}\sin\omega(\tau - t)\,R_0(\tau)d\tau., \tag{56}$$

To avoid secular terms appear in u_1,the coefficient of $\cos(\omega\, t)$ in Eq. (55)requires being zero, i.e.

$$\omega_{VIM} = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{57}$$

And period of this system is:

$$T_{VIM} = \frac{2\pi}{\sqrt{1 + (3/4)\,\varepsilon A^2}} \tag{58}$$

The exact solution is [89]:

$$T_{Exact} = \frac{4}{\sqrt{1+\varepsilon A^2}} \int_0^{\pi/2} \frac{dt}{\sqrt{1-k\sin^2 t}} \tag{59}$$

Where $k = 0.5\varepsilon A^2 \big/ \left(1+\varepsilon A^2\right)$.

Example 3

The governing equation of Mathieu-Duffing system which is considered in this study is described by the following high-order nonlinear differential equation[45];

$$\ddot{u} + [\delta + 2\varepsilon cos(2t)]u - \phi u^3 = 0 \tag{60}$$

Where dots indicate differentiation with respect to the time (t), ε<<1 is a small parameter,ϕ is the Parameter of nonlinearity and δ is the transient curve and can be defined as [45];

$$\delta = \phi u_0^2 (1 - \frac{2\varepsilon}{2+\phi u_0^2}). \tag{61}$$

The initial condition considered in this study is defined by [45];

$$u(0) = 0.1\ ,\dot{u}(0) = 0 \tag{62}$$

According to the VIM, we can construct the correction functional of Eq. (60) as follows

$$u_{(n+1)}(t) = u_n(t) + \int_0^{\tau} \lambda\{\ddot{u}_n + [\delta + 2\varepsilon cos(2\tau)]u_n - \phi u_n^3\}d\tau \tag{63}$$

Where λ is General Lagrange multiplier.

Making the above correction functional stationary, we can obtain following stationary conditions

$$\begin{aligned} \lambda''(\tau) &= 0, \\ \lambda(\tau)_{\tau=t} &= 0, \\ 1 - \lambda'(\tau)\,|_{\tau=t} &= 0, \end{aligned} \tag{64}$$

The Lagrange multiplier, can be identified as:

$$\lambda = \tau - t \tag{65}$$

Leading to the following iteration formula

$$u_{(n+1)}(t) = u_n(t) + \int_0^{t} (\tau - t)\{\ddot{u}_n + [\delta + 2\varepsilon cos(2t)]u_n - \phi u_n^3\}d\tau \tag{66}$$

If, for example, the initial conditions are $u(0) = 0.1$ and$\dot{u}(0) = 0$, we began with$u_0(t) = 0.1$, by the above iteration formula (63) we have the following approximate solutions

$$u_1(t) = 0.1 - 0.05\varepsilon - 0.05\delta t^2 + 0.05\varepsilon cos(2t) + 0.0005\phi t^2 \tag{67}$$

In the same way, we obtain as u_2 (t)follows:

$$
\begin{aligned}
u_2(t) &= 0.1 - 0.05\varepsilon - 0.05\delta t^2 + 0.05\varepsilon cos(2t) + 0.0005\phi t^2 + 0.1875\varepsilon^2 - 0.328125 \times 10^{-3}\phi\varepsilon^2 \\
&+0.2724609375 \times 10^{-5}\phi^2\varepsilon^2 - 0.5625 \times 10^{-5}\varepsilon\phi^2 + 0.9461805556 \times 10^{-4}\varepsilon\phi^3 \\
&+6.696428 \times 10^{-8}\delta^2\phi^2 t^8 + 1.171875 \times 10^{-7}\varepsilon^2\phi^2 t^2 + 0.125 \times 10^{-5}\phi^2 t^4 + 3.75 \times 10^{-8} t^3\varepsilon\phi^3 sin(2t) \\
&+0.140625 \times 10^{-4}\varepsilon\delta\phi^2 cos(2t) + 8.4375 \times 10^{-8} t^2\varepsilon\phi^3 cos(2t) + 0.1875 \times 10^{-5} t^2\varepsilon^2\phi^2 cos(2t) \\
&-0.375 \times 10^{-5} t\varepsilon^2\phi^2 sin(2t) + 0.00025 t^2\phi\varepsilon cos(2t) - 0.375 \times 10^{-5}\varepsilon\phi^2 cos(2t) t^2 \\
&-2.34375 \times 10^{-7}\varepsilon^2\phi^2 cos^2(2t) t^2 - 0.87890625 \times 10^{-5}\phi\varepsilon^2\delta cos(2t)^2 - 0.025 t^2\delta\varepsilon cos(2t) \\
&+0.00028125\phi\varepsilon^2\delta cos(2t) - 0.000703125\phi\varepsilon\delta^2 cos(2t) - 0.0005625\phi\varepsilon\delta cos(2t) \\
&-0.140625 \times 10^{-4}\varepsilon\delta\phi^2 + 0.05 t\delta\varepsilon sin(2t) - 0.5 \times 10^{-3} t\phi\varepsilon sin(2t) + 0.75 \times 10^{-5}\varepsilon\phi^2 sin(2t) t \\
&-1.125 \times 10^{-7} t\varepsilon\phi^3 sin(2t) - 9.375 \times 10^{-9} t^4\varepsilon\phi^3 cos(2t) + 0.5625 \times 10^{-3}\phi\varepsilon\delta + 0.70312 \times 10^{-3}\phi\varepsilon\delta^2 \\
&-0.2724609375 \times 10^{-3}\phi\varepsilon^2\delta - 0.05\delta\varepsilon + 0.00075\phi\varepsilon + 7.03125 \times 10^{-8}\varepsilon\phi^3 + 4.6875 \times 10^{-7}\varepsilon^2\phi^2 t^4 \\
&+0.9461805556 \times 10^{-4}\phi\varepsilon^3 - 0.5625 \times 10^{-5}\varepsilon\phi^2 - 0.46875 \times 10^{-4}\phi\varepsilon^2\delta t^4 + 0.000125\phi\varepsilon\delta t^4 \\
&-0.125 \times 10^{-4} t^6\phi\delta^2\varepsilon + 2.5 \times 10^{-9}\phi^3 t^6 + 0.2724609375 \times 10^{-5}\varepsilon^2\phi^2 - 0.328125 \times 10^{-3}\phi\varepsilon^2 \\
&+0.1875 \times 10^{-5} t^4\delta\varepsilon\phi^2 cos(2t) + 2.23214285710^{-12}\phi^4 t^8 - 7.03125 \times 10^{-8}\varepsilon\phi^3 cos(2t) \\
&-0.28125 \times 10^{-5}\varepsilon^2\phi^2 cos(2t) - 0.75 \times 10^{-3}\phi\varepsilon cos(2t) + 0.375 \times 10^{-3}\phi\varepsilon^2 cos(2t) \\
&-0.46875 \times 10^{-4}\phi\varepsilon^2 cos^2(2t) - 0.34722222 \times 10^{-5}\phi\varepsilon^3 cos^3(2t) + 0.5625 \times 10^{-5}\varepsilon\phi^2 cos(2t) + \ldots
\end{aligned}
\tag{68}
$$

And so on. In the same manner, the rest of the components of the iteration formula can be obtained.

Figures 1 to 3 indicate that the VIM experiences a high accuracy. The figures illustrate the time history diagram of the displacement, velocity and phase plan, respectively.

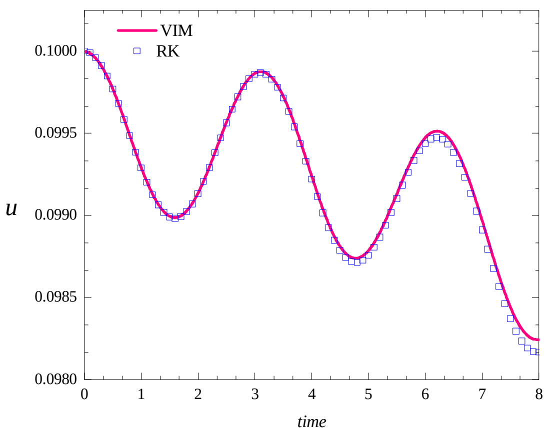

Figure 1 Comparison of time history diagram of displacements between VIM and RK solutions at $\varphi = 2, \varepsilon = 0.01, u(0) = 0.1, \dot{u}(0) = 0$.

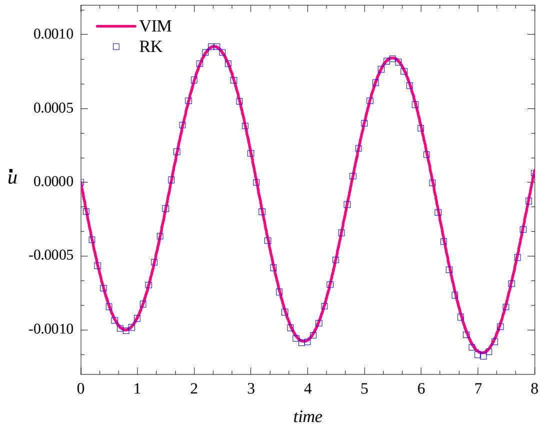

Figure 2 Comparison of time history diagram of velocity between VIM and RK solutions at$\varphi = 2$,$\varepsilon = 0.01$,$u(0) = 0.1$,$\dot{u}(0) = 0$.

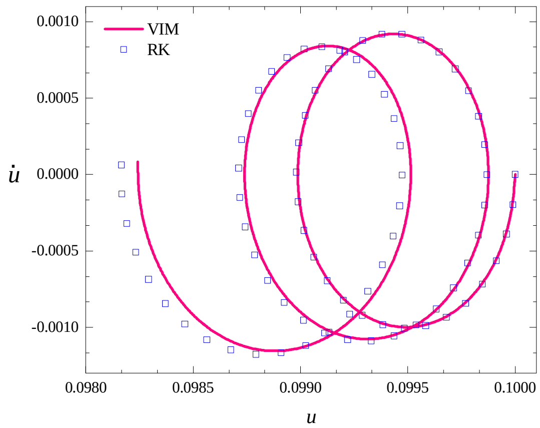

Figure 3 Comparison of VIM with RK ,$\dot{u}$versus $u$atϕ= 2, $\varepsilon = 0.01$, $\delta = 0.02$

4 HOMOTOPY PERTURBATION METHOD (HPM)

Until recently, the application of the homotopy perturbation method in nonlinear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem easy to solve into the difficult problem under study. The homotopy perturbation method proposed by He in 1999[81]. Elementary introduction and interpretation of the method are given in the following publications [5, 9, 24, 27–33, 59, 63, 64, 68, 84, 91, 93, 95, 96, 98, 101, 102, 123, 148, 168, 174, 176, 208, 218]. HPM can solve a large class of nonlinear problems with approximations converging rapidly to accurate solutions. This method is the most effective and convenient one for both weakly and strongly nonlinear equations.

4.1 Basic idea of Homotopy Perturbation Method

To explain the basic idea of the HPM for solving nonlinear differential equations, one may consider the following nonlinear differential equation[81]:

$$A(u) - f(r) = 0 \; r \in \Omega \tag{69}$$

That is subjected to the following boundary condition:

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0 \; r \in \Gamma \tag{70}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the solution domain(Ω), and $\partial u/\partial t$ denotes differentiation along the outwards normal to Γ. Generally, the operator A may be divided into two parts: a linear part L and a nonlinear part N. Therefore, Eq. (69) may be rewritten as follows:

$$L(x) + N(x) - f(r) = 0 \; r \in \Omega \tag{71}$$

In cases where the nonlinear Eq. (69) includes no small parameter, one may construct the following homotopy equation

$$H(\nu, p) = (1-p)\; [L(\nu) - L(u_0)] + p[A(\nu) - f(r)\;] = 0 \tag{72}$$

Where

$$\nu(r,p):\;\; \Omega \times [0,1] \to R \tag{73}$$

In Eq. (72), $p \in [0\;,\;1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. One may assume that solution of Eq. (72) may be written as a power series in p, as the following:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots \tag{74}$$

The homotopy parameter p is also used to expand the square of the unknown angular frequency u as follows:

$$\omega_0 = \omega^2 - p\omega_1 - p^2\omega_2 - \ldots \tag{75}$$

Or

$$\omega^2 = \omega_0 + p\omega_1 + p^2\omega_2 + \ldots \tag{76}$$

Where ω_0 is the coefficient of $u(r)$ in Eq. (69) and should be substituted by the right hand side of Eq. (76). Besides, $\omega_i\,(i = 1, 2, \ldots)$ are arbitrary parameters that have to be determined.

The best approximations for the solution and the angular frequency ω are

$$u = lim_{p\to 1}\nu = \nu_0 + \nu_1 + \nu_2 + \cdots \tag{77}$$

$$\omega^2 = \omega_0 + \omega_1 + \omega_2 + \ldots \tag{78}$$

When Eq. (72) corresponds to Eq. (69) and Eq. (77) becomes the approximate solution of Eq. (69)

4.2 Application of Homotopy Perturbation Method

Example 1

We consider the mathematical pendulum. When friction is neglected; the differential equation governing the free oscillation of the mathematical pendulum is given by[82];

$$\ddot{\theta} + \Omega^2 \sin\theta = 0, \qquad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{79}$$

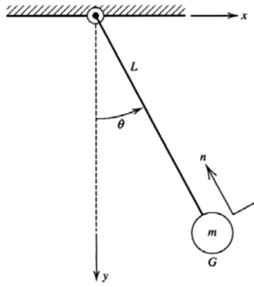

Figure 4 The simple pendulum

When θ designates the deviation angle from the vertical equilibrium position, $\Omega^2 = \frac{g}{l}$where g is the gravitational acceleration, l the length of the pendulum[82].

In order to apply the homotopy perturbation method to solve the above problem, the approximation $\sin\theta \approx \theta - (1/6)\,\theta^3 + (1/120)\,\theta^5$is used

Now we apply homotopy perturbation to Eq. (79). We construct a homotopy in the following form:

$$H(\theta, p) = (1-p)\ \left[\ddot{\theta} + \Omega^2\theta\right] + p\left[\ddot{\theta} + \Omega^2\left(\theta - (1/6)\,\theta^3 + (1/120)\,\theta^5\right)\right] = 0 \tag{80}$$

According to HPM, we assume that the solution of Eq. (80) can be expressed in a series of p;

$$\theta(t) = \theta_0(t) + p\theta_1(t) + p^2\theta_2(t) + \ldots.. \tag{81}$$

Just the coefficient of θ,(Ω^2) expanded into a series in p in a similar way:

$$\Omega^2 = \omega^2 - p\omega_1 - p^2\omega_2 + \ldots \tag{82}$$

Substituting Eq.(81) and Eq. (82) into Eq. (80) after some simplification and substitution and rearranging based on powers of p-terms, we have:

$$p^0: \ \ddot{\theta}_0 + \omega^2\theta_0 = 0, \qquad \theta_0(0) = A, \quad \dot{\theta}_0(0) = 0 \tag{83}$$

$$p^1: \ \ddot{\theta}_1 + \omega^2\theta_1 = \omega_1\theta + (\frac{\Omega^2}{6})\omega^2\theta^3 - (\frac{\Omega^2}{120})\omega^2\theta^5, \qquad \theta_1(0) = 0, \quad \dot{\theta}_1(0) = 0 \tag{84}$$

$$\vdots$$

Considering the initial conditions $\theta_0(0) = A$and $\dot{\theta}_0(0) = 0$ the solution of Eq. (83) is $\theta_0 = A\cos\omega t$ Substituting the result into Eq. (84), we have:

$$p^1: \ \ddot{\theta}_1 + \omega^2\theta_1 = \omega_1 A\cos(\omega t) + \frac{1}{6}\omega^2 A^3\cos^3(\omega t) - \frac{1}{120}\omega^2 A^5\cos^5(\omega t) \tag{85}$$

For achieving the secular term, we use Fourier expansion series as follows:

$$\begin{aligned}
&\Phi(\omega, t) = \left(-\tfrac{1}{8}\omega^2 A^3 + \tfrac{1}{192}\omega^2 A^5 - \omega_1 A\right) cos(\omega t) - \tfrac{1}{24}\,\omega^2 A^3 cos(3\omega t) \\
&+\tfrac{1}{1920}\omega^2 A^5 cos(5\omega t) + \tfrac{1}{384}\,\omega^2 A^5 cos(3\omega t) \\
&= \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right] \\
&= b_1\cos(\omega t) + b_3\cos(3\omega t) + ...
\end{aligned} \tag{86}$$

Substituting Eq. (86) into right hand of Eq. (85) yields:

$$p^1: \ \ddot{\theta}_1 + \omega^2\theta_1 = \left[-(1/8)\omega^2 A^3 + (1/192)\omega^2 A^5 - \omega_1 A\right]\cos(\omega t) + \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right] \tag{87}$$

Avoiding secular term, gives:

$$\omega_1 = -\frac{1}{192}\omega^2 A^2(-24 + A^2) \tag{88}$$

From Eq. (82) and setting$p = 1$, we have:

$$\Omega^2 = \omega^2 - \omega_1 \tag{89}$$

Comparing Eqs. (88) and (89), we can obtain:

$$\omega = \Omega\sqrt{1 - \frac{1}{8}A^2 + \frac{1}{192}A^4} \tag{90}$$

The exact frequency of this problem is:

$$\omega_{Exact} = 2\pi \bigg/ 2\sqrt{2}\int_0^{\pi/2} \frac{A\sin^2(t)\,dt}{\Omega\sqrt{\cos(A\cos(t)) - \cos(A)}} \tag{91}$$

Table 3 Comparison of the approximate frequencies with the exact period.

A	Ω	Present Study (HPM)	Exact Solution	Error % $(\omega_{HPM} - \omega_{ex})/\omega_{ex}$
0.1	2	1.99875	1.99875	0.0000
0.2	3	2.992503	2.992502	0.0001
0.5	4	3.937665	3.937579	0.0022
0.8	2	1.920555	1.92025	0.0159
1	1	0.938194	0.937792	0.0429
1.2	2	1.822965	1.821145	0.0999
1.5	1	0.863202	0.860608	0.3013
1.8	0.5	0.403012	0.399787	0.8066
2	1	0.763763	0.7525	1.4968

Example 2

The motion of a particle on a rotating parabola is considered for second example. The governing equation of motion and can be expressed as;

$$\ddot{u} + a\,u\dot{u}^2 + a\,u\ddot{u} + \alpha_1 u + \alpha_2 u^3 + \alpha_3 u^5 = 0, \qquad u(0) = A, \qquad \dot{u}(0) = 0 \tag{92}$$

Now we apply homotopy-perturbation to Eq(92).We construct a homotopy in the following form:

$$H(u,p) = (1-p)\ [\ddot{u} + \alpha_1 u] + p\left[\ddot{u} + a\,u\dot{u}^2 + au\ddot{u} + \alpha_1 u + \alpha_2 u^3 + \alpha_3 u^5\right] = 0 \tag{93}$$

According to HPM, we assume that the solution of (93) can be expressed in a series of p

$$u(t) = u_0(t) + pu_1(t) + p^2 u_2(t) + \ldots.. \tag{94}$$

The coefficient α_1 expanded into a series in p in a similar way.

$$\alpha_1 = \omega^2 - p\omega_1 - p^2\omega_2 + \ldots \tag{95}$$

Substituting (94) and (95) into (93) after some simplification and substitution and rearranging based on powers of p-terms, we have:

$$p^0 = \ddot{u}_0 + \omega^2 u_0 = 0, \qquad u_0(0) = A, \qquad \dot{u}_0(0) = 0 \tag{96}$$

And,

$$p^1 = \ddot{u}_1 + \omega^2 u_1 = \omega_1 u_0 - au_0\dot{u}_0^2 - au_0\ddot{u}_0 - \alpha_2 u_0^3 - \alpha_3 u_0^5, \qquad u_1(0) = 0, \qquad \dot{u}\,(0) = 0 \tag{97}$$

Considering the initial conditions $u_0(0) = A$ and $\dot{u}_0(0) = 0$ the solution of Eq. (96) is $u_0 = A\cos(\omega t)$ Substituting the result into Eq. (97), we have:

Table 4 Comparison of HPM solution and Runge-Kutta algorithm.

Case1: $A=0.5$, $a=0.2$, $\alpha_1=2$, $\alpha_3=0.5$			Case2: $A=1$, $a=0.5$, $\alpha_1=1$, $\alpha_2=0.5$, $\alpha_3=0.2$		
t	HPM *u(t)*	Runge -Kutta *u(t)*	t	HPM *u(t)*	Runge -Kutta *u(t)*
0	0.4	0.4	0	1	1
0.5	0.299437	0.299766	0.5	0.860691	0.853713
1	0.049196	0.049299	1	0.469752	0.457651
1.5	-0.225475	-0.225875	1.5	-0.080103	-0.072308
2	-0.387780	-0.387848	2	-0.600724	-0.581112
2.5	-0.355266	-0.355443	2.5	-0.927124	-0.919896
3	-0.144634	-0.144902	3	-0.988938	-0.989543
3.5	0.137916	0.138260	3.5	-0.774269	-0.769674
4	0.351899	0.352131	4	-0.325557	-0.324619
4.5	0.389486	0.389524	4.5	0.237755	0.215412
5	0.231375	0.231700	5	0.715872	0.692419
5.5	-0.042066	-0.042245	5.5	0.972568	0.966842
6	-0.295018	-0.294619	6	0.955930	0.958391

$$\begin{aligned} p^1 = \ddot{u}_1 + \omega^2 u_1 = \omega_1 A\cos(\omega t) - a\omega^2 A^3\cos(\omega t)\sin^2(\omega t) - a\omega^2 A^3\cos^3(\omega t) \\ - \alpha_2 A^3\cos^3(\omega t) - \alpha_3 A^5\cos^5(\omega t) \end{aligned} \tag{98}$$

No secular term in p^1 requires that

$$\omega_1 = -\frac{1}{8}A^2\left(-4a\omega^2 + 6\alpha_2 + 5\alpha_3 A^2\right) \tag{99}$$

Substituting (99) in to Eq (95) and setting $p=1$, we can obtain the frequency of the nonlinear oscillator as follows:

$$\omega_{HPM} = \frac{1}{2}\frac{\sqrt{(2+A^2a)\left(8\alpha_1 + 5A^4\alpha_3 + 6\alpha_2 A^2\right)}}{(2+A^2a)} \tag{100}$$

Table 4 shows the high accuracy of the Homotopy Perturbation Method with the Runge-Kutta Method.

Example 3

In this section, we will consider the system with linear and nonlinear springs in series as it is shown in Fig. 5.

In this figure, k_1 is the stiffness coefficient of the first linear spring , the coefficients associated with the linear and nonlinear portions of spring force in the second spring with cubic nonlinear characteristic are described by k_2 and k_3, respectively. Let ε be defined as:

$$\varepsilon = k_2/k_3 \tag{101}$$

The case of $k_3 > 0$ corresponds to a hardening spring while $k_3 < 0$ indicates a softening one.

Let x and y denote the absolute displacements of the connection point of two springs, and the mass m, respectively. By introducing two new variables

$$u = y - x, r = x. \tag{102}$$

Telli and Kopmaz [185] obtained the following governing equation for v and r:

$$(1 + 3\varepsilon\eta u^2)\ddot{u} + 6\varepsilon\eta u\dot{u}^2 + \omega_e^2 u + \varepsilon\omega_e^2 u^3 = 0, \tag{103}$$

$$r = x = \xi(1 + \varepsilon u^2)u, y = (1 + \xi + \xi\varepsilon u^2)u, \tag{104}$$

Where a prime denotes differentiation with respect to time and

$$\xi = k_2/k_1, \quad \eta = \frac{\xi}{1+\xi}, \quad \omega_0^2 = \frac{k_2}{m(1+\xi)}. \tag{105}$$

Eq. (103) is an ordinary differential equation in u. For Eq. (103), we consider the following initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{106}$$

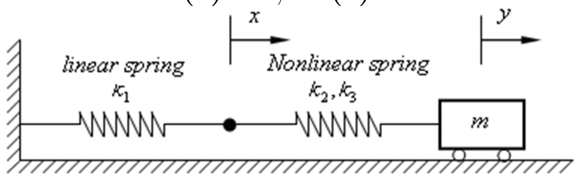

Figure 5 Nonlinear free vibration of a system of mass with serial linear and Nonlinear stiffness on a frictionless contact surface[185]

Eq. (103) can be rewritten as the following form:

$$\ddot{u} + 1.u = p.\left[-3\ddot{u}\varepsilon\eta u^2 - 6\varepsilon\eta u\dot{u}^2 - \omega_0^2\varepsilon u^3 - \omega_0^2 u + u\right] = 0, \quad p \in [0,1]. \tag{107}$$

Substituting Eqs. (74)and (75) into Eq. (107) and expanding, we can write the first two linear equations as follows:

$$p^0: \quad \ddot{u}_0 + \omega^2 u_0 = 0, \qquad u_0(0) = A, \quad \dot{u}_0(0) = 0 \tag{108}$$

$$p^1: \ddot{u}_1 + \omega^2 u_1 = -3u_0^{''}\eta\varepsilon u_0^2 - 6\eta\varepsilon u_0 u_0^{'2}\omega_0^2\varepsilon u_0^3 + \left(1 + \gamma_1 - \omega_0^2\right)u_0, \tag{109}$$

$$\vdots$$

Solving Eq. (108) gives: $u_0 = A\cos\omega t$. Substituting u_0 into Eq. (109) , yield:

$$\begin{aligned} p^1\, u_1 + \omega^2 u_1 = {} & 9A^3\eta\varepsilon\omega^2\cos^3\omega t - 6\eta\varepsilon\omega^2 A^3\cos\omega t \\ & + \left(1 + \gamma_1 - \omega_0^2\right)A\cos\omega t - \omega_0^2\varepsilon A^3\cos^3\omega t, \end{aligned} \tag{110}$$

$$\vdots$$

For achieving the secular term, we use Fourier expansion series as follows:

$$\begin{aligned}
&9A^3\eta\,\varepsilon\,\omega^2\cos^3\omega\,t - 6\,\eta\,\varepsilon\,\omega^2 A^3\cos\omega\,t - \omega_0^2\varepsilon A^3\cos^3\omega\,t\\
&= \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right]\\
&= b_1\cos(\omega\,t) + b_3\cos(3\,\omega\,t) + \ldots\\
&\approx \tfrac{3A^3\varepsilon}{4}\left(\eta\,\omega^2 - \omega_0^2\right)\cos(\omega\,t) + \ldots .
\end{aligned} \tag{111}$$

Substituting Eq. (111) into Eq. (110) yields:

$$p^1:\quad \ddot{u}_1 + \omega^2 u_1 = \left[\frac{3A^2\varepsilon}{4}\left(\eta\omega^2 - \omega_0^2\right) + \left(1 + \gamma_1 - \omega_0^2\right)\right] \times A\cos(\omega\,t) \tag{112}$$

Avoiding secular term, gives:

$$\gamma_1 = \frac{3A^2\varepsilon}{4}\left(\omega_0^2 - \eta\omega^2\right) + \left(\omega_0^2 - 1\right) \tag{113}$$

From Eq. (75) and setting$p = 1$, we have:

$$\gamma_1 = \omega - 1 \tag{114}$$

Comparing Eqs. (113) and (114), we can obtain:

$$\omega_{HPM} = \frac{3A^2\varepsilon}{4}\left(\omega_0^2 - \eta\omega^2\right) + \omega_0^2 \tag{115}$$

Solving Eq. (115), gives:

$$\omega_{HPM} = \frac{\omega_0\sqrt{(4+3A^2\varepsilon\eta)\,(4+3A^2\varepsilon)}}{4+3A^2\varepsilon\eta}, \tag{116}$$

Table 5 Comparison of error percentages corresponding to various parameters of system

Constant parameters							Relative error %
m	A	ε	k_1	k_2	ω_{HPM}	numerical	$\frac{\omega_{HPM}-\omega_{num}}{\omega_{num}}$
1	0.5	0.5	50	5	2.220265	2.220231	0.00153
1	0.5	0.5	50	5	3.162277	3.175501	0.41644
1	2	0.5	5	5	1.889822	1.903569	0.72170
1	2	0.5	5	50	2.192645	2.195284	0.12021
3	5	1	8	16	1.612706	1.615107	0.14866
3	5	1	10	5	1.739775	1.749115	0.53398
5	10	2	12	16	1.545360	1.545853	0.03189
2	2	-0.1	10	10	1.434860	1.446389	0.00520
3	4	-0.02	30	10	1.313064	1.318370	0.40247
4	10	-0.008	6	3	0.703731	0.705412	0.23830

Table 5 represents the comparisons of angular frequencies for different parameters via numerical is presented in Table 1. The maximum relative error between the HPM results and numerical results is 0.72170 %.

5 ITERATION PERTURBATION METHOD (IPM)

The study of nonlinear oscillators is of interest to many researchers and various methods of solution have been proposed. The iteration perturbation method (IPM) is considered to be one of the powerful methods which is capable for nonlinear problems, it can converge to an accurate solution for smooth nonlinear systems. The iteration perturbation method was first proposed by He [87] in 2001 and used to give approximate solutions of the problems of nonlinear oscillators. The application of this method is used in [26, 61, 149].

5.1 Basic idea of Iteration Perturbation Method

Many researchers have devoted their attention to obtaining approximate solution of nonlinear equations in the form:

$$\ddot{u} + u + \varepsilon f(u, \dot{u}) = 0, \tag{117}$$

Subject to the following initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{118}$$

We rewrite Eq. (117) in the following form:

$$\ddot{u} + u + \varepsilon u.g(u, \dot{u}) = 0, \tag{119}$$

Where $g(u, \dot{u}) = f/u$.

We construct an iteration formula for the above equation:

$$\ddot{u}_{n+1} + u_{n+1} + \varepsilon u_{n+1}.g(u_n, \dot{u}_n) = 0, \tag{120}$$

Where we denote by u_n the n th approximate solution. For nonlinear oscillation,Eq. (120) is of Mathieu type. We will use the perturbation method to find approximately $u_{n+1}$the technique is called iteration perturbation method.

In order to assess the advantages and the accuracy of the iteration perturbation method we will consider the following examples.

Here, we will introduce a nonlinear oscillator with discontinuity in several different forms:

$$\frac{d^2u}{dt^2} + h(u) + \beta sgn(u)u = 0, \tag{121}$$

Or

$$\frac{d^2u}{dt^2} + h(u) + \beta u\,|u| = 0, \tag{122}$$

With initial conditions

$$u(0) = A, \frac{du(0)}{dt} = 0 \tag{123}$$

In this work, we assume that $h(u)$is in a polynomial form. The reason for this assumption is that the discontinuity equations found in the literature belong to this family. Since there are no small parameters in Eq. (122) the traditional perturbation methods cannot be applied directly. In the following example, we assume a linear form $h(u)$.

5.2 Application of Iteration Perturbation Method

Example 1

We let $h(u) = \alpha\, u$, in Eq. (122). We can rewrite Eq. (122) in the following form;

$$u'' + \alpha.u + \beta u\,|\,u| = 0 \tag{124}$$

To apply the Iteration Perturbation Method, the solution is expanded and the series of ε is introduced as follows:

$$u = u_0 + \sum_{i=0}^{n} \varepsilon^i u_i \tag{125}$$

$$\alpha = \omega^2 + \sum_{i=0}^{n} \varepsilon^i a_i \tag{126}$$

$$\beta = \sum_{i=0}^{n} \varepsilon^i d_i \tag{127}$$

Substituting Eqs. (125), (126) and (127) into Eq. (124) and equating the terms with the identical powers of ε, a series of linear equations are obtained. Expanding the first two linear terms becomes as follows;

$$\varepsilon^0 : \quad \ddot{u}_0 + \omega^2 u_0 = 0 \ , \quad u_0(0) = A \ , \ \dot{u}_0(0) = 0 \tag{128}$$

$$\varepsilon^1 : \quad u_1'' + \omega^2 u_1 + a_1 u_0 + d_1 u_0 \ |u_0| = 0 \ \ , \quad u_1(0) = 0 \ , \ \dot{u}_1(0) = 0 \tag{129}$$

Substituting the solution into Eq. (128), e.g.$u_0 = A\cos(\omega\, t)$, the deferential equation for u_1 becomes;

$$\begin{aligned} &u_1'' + \omega^2 u_1 + a_1 A\cos(\omega\, t) + d_1 A\cos(\omega\, t)\,|A\cos(\omega\, t)| = 0\,, \\ &u_1(0) = 0, u_1'(0) = 0 \end{aligned} \tag{130}$$

Note that the following Fourier series expansion is valid.

$$\begin{aligned} &|A\cos(\omega t)|\cos(\omega t)^{2n-1} = \sum_{k=0}^{\infty} c_{2k+1}\cos((2k+1)\omega t) \\ &= c_1\cos(\omega t) + c_3\cos(3\omega t) + ... \end{aligned} \tag{131}$$

Where c_i can be determined by Fourier series, for example,

$$
\begin{aligned}
c_1 &= \tfrac{2}{\pi}\int_0^{\pi}|\cos(\omega t)|^{2n}\cos(\omega t)d(\omega t) \\
&= \tfrac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos(\omega t)^{2n+1}d(\omega t)-\int_0^{\pi}\cos(\omega t)^{2n+1}d(\omega t)\right) \\
&= \tfrac{2}{\sqrt{\pi}}\tfrac{\Gamma(n+1)}{\Gamma(n+3/2)}
\end{aligned} \tag{132}
$$

Eq. (132) in Eq. (130) gives

$$u_1'' + \omega^2 u_1 + a_1 A\cos(\omega t) + d_1 A\sum_{k=0}^{\infty} c_{2k+1}\cos((2k+1)\omega t) = 0 \tag{133}$$

Avoiding the presence of a secular term requires that

$$a_1 + d_1 c_1 A^2 = 0 \tag{134}$$

Also, substituting $\varepsilon = 1$,into Eqs. (125) and (126) gives:

$$\alpha = \omega^2 + a_1 \tag{135}$$

$$\beta = d_1 \tag{136}$$

From Eqs. (134) ,(135)and (136), the first-order approximation to the angular frequency is:

$$\omega = \sqrt{\alpha + \frac{8\varepsilon A}{3\pi}} \tag{137}$$

Case 1:

If $\alpha = 1$,we have

$$\omega_{IPM} = \sqrt{1 + \frac{8\varepsilon A}{3\pi}} \tag{138}$$

It is the same as that obtained by the Homotopy perturbation method and the Variational method [95, 182].

Case 2:

If $\alpha = 0$, we have

$$\omega_{IPM} = \sqrt{\frac{8\varepsilon A}{3\pi}} \tag{139}$$

The obtained frequency in Eq. (139) is valid for the whole solution domain $0 < A < \infty$.

Example 2

If $h(u) = \alpha u^3$, in Eq. (122). Then we have

$$\frac{d^2u}{dt^2} + \alpha.u^3 + \beta u\,|u| = 0 \tag{140}$$

To apply the Iteration Perturbation Method, the solution is expanded and the series of ε is introduced as follows:

$$u = u_0 + \sum i = 0^n \varepsilon^i u_i \tag{141}$$

$$0 = \omega^2 + \sum_{i=0}^{n} \varepsilon^i a_i \tag{142}$$

$$1 = \sum_{i=0}^{n} \varepsilon^i d_i \tag{143}$$

Substituting Eqs. (141),(142) and (143) into Eq. (140) and equating the terms with the identical powers of ε, a series of linear equations are obtained. Expanding the first two linear terms becomes as follows;

$$\varepsilon^0: \quad \ddot{u}_0 + \omega^2 u_0 = 0\ , \quad u_0(0) = A\ , \ \dot{u}_0(0) = 0 \tag{144}$$

$$\varepsilon^1: \quad u_1'' + \omega^2 u_1 + a_1 u_0 + d_1 \alpha u_0^3 + \beta u_0\,|A\cos(\omega t)| = 0, \quad u_1(0) = 0\ , \ \dot{u}_1(0) = 0 \tag{145}$$

Substituting the solution into Eq. (144), e.g.$u_0 = A\cos(\omega t)$, the deferential equation for u_1 becomes;

$$\begin{aligned} &u_1'' + \omega^2 u_1 + a_1 A\cos(\omega t) + d_1 \alpha A^3 \cos^3(\omega t) \\ &+\beta A\cos(\omega t)\,|A\cos(\omega t)| = 0 \end{aligned} \tag{146}$$

We have the following identity;

$$\cos^3(\omega\, t) = \frac{3}{4}\cos(\omega\, t) + \frac{1}{4}\cos(3\omega\, t) \tag{147}$$

Note that the following Fourier series expansion is valid.

$$\begin{aligned} &|A\cos(\omega t)|^{2n-1}\cos(\omega t) = \sum_{k=0}^{\infty} c_{2k+1}\cos((2k+1)\omega t) \\ &= c_1\cos(\omega t) + c_3\cos(3\omega t) + ... \end{aligned} \tag{148}$$

c_i can be determined by Fourier series, for example :

$$c_1 = \frac{2}{\pi}\int_0^{\pi} |\cos(\omega t)|^{2n} \cos(\omega t) d(\omega t)$$
$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \cos(\omega t)^{2n+1} d(\omega t) - \int_0^{\pi} \cos(\omega t)^{2n+1} d(\omega t)\right) \tag{149}$$
$$= \frac{2}{\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)}$$

By means of Eqs. (147),(148) and (149) we find that

$$\begin{aligned} &u_1'' + \omega^2 u_1 + (a_1 + d_1 A^2 \tfrac{3}{4})A\cos(\omega t) + d_1 A^3 \tfrac{1}{4}\cos(3\omega t)) \\ &+A^2 \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) = 0 \end{aligned} \tag{150}$$

No secular term in $u_1$requires that

$$a_1 + d_1 \alpha A^2 \frac{3}{4} + \beta A \frac{8}{3\pi} = 0 \tag{151}$$

Also, substituting $\varepsilon = 1$,into Eqs. (142) and (143) gives:

$$0 = \omega^2 + a_1 + ... \tag{152}$$

$$1 = d_1 \tag{153}$$

From Eqs. (151) ,(152)and (153), the first-order approximation to the angular frequency is:

$$\omega_{IPM} = \sqrt{\frac{3\alpha A^2}{4} + \frac{8\beta A}{3\pi}} \tag{154}$$

Case 1:
If $\alpha = \beta, \beta = \varepsilon$ we have

$$\omega_{IPM} = \sqrt{\frac{3\beta A^2}{4} + \frac{8\varepsilon A}{3\pi}} \tag{155}$$

This agrees well with that obtained by the Homotopy perturbation method and the Variational method [95, 182].

And its period is given by

$$T_{IPM} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{3\beta A^2}{4} + \frac{8\varepsilon A}{3\pi}}} \tag{156}$$

Case 2:
If $\varepsilon = 0$, its period can be written as;

$$T_{IPM} = \frac{4\pi}{\sqrt{3}} \beta^{-\frac{1}{2}} A^{-1} \tag{157}$$

The exact period was obtained by Acton and Squire in 1985 [2].

$$T_{ex} = 7.4164\beta^{-\frac{1}{2}}A^{-1} \tag{158}$$

The maximal relative error is less than 2.2% for all$\beta > 0$!

6 ENERGY BALANCE METHOD (EBM)

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. This method was proposed by He [90] in 2002. This method can be seen as a Ritz method and leads to a very rapid convergence of the solution, and can be easily extended to other nonlinear oscillations. In short, this method yields extended scope of applicability, simplicity, flexibility in application, and avoidance of complicated numerical and analytical integration as compared to others among the previous approaches, such as, the perturbation methods, and so could widely applicable in engineering and science.Energy balance method used heavily in the literature in [17, 19, 22, 25, 55, 56, 58, 60, 65–67, 116, 120, 141, 150, 178, 209]and the references therein.

6.1 Basic idea of Energy Balance Method

In the present paper, we consider a general nonlinear oscillator in the form [90]:

$$\ddot{u} + f(u(t)) = 0 \tag{159}$$

In which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained:

$$J(u) = \int_0^t (-\frac{1}{2}\dot{u}^2 + F(u))\, dt \tag{160}$$

Where $T = \frac{2\pi}{\omega}$ is period of the nonlinear oscillator, $F(u) = \int f(u)\, du$.
Its Hamiltonian, therefore, can be written in the form;

$$H = \frac{1}{2}\dot{u}^2 + F(u) + F(A) \tag{161}$$

Or

$$R(t) = -\frac{1}{2}\dot{u}^2 + F(u) - F(A) = 0 \tag{162}$$

Oscillatory systems contain two important physical parameters, i.e. ω is the frequency and A is the amplitude of the oscillation. So let us consider such initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{163}$$

We use the following trial function to determine the angular frequency ω

$$u(t) = A\cos\omega\, t \tag{164}$$

Substituting (164) into u term of (162), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2\omega\, t + F(A\cos\omega\, t) - F(A) = 0 \tag{165}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (164) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega\, t = \frac{\pi}{4}$gives:

$$\omega = \sqrt{\frac{2(F(A)) - F(A\cos\omega\, t)}{A^2\sin^2\omega\, t}} \tag{166}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A))-F(A\cos\omega\, t)}{A^2\sin^2\omega\, t}}} \tag{167}$$

6.2 Application of Energy Balance Method

Example 1

In this section, we will consider the system with linear and nonlinear springs in series.

In Eq. (103), Its Variational principle can be easily obtained:

$$J(u) = \int_0^t \left(-\frac{1}{2}\dot{u}^2\left(1 + \frac{3}{2}\varepsilon\,\eta\, u^2\right) + \omega_0^2(\frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4)\right)dt \tag{168}$$

Its Hamiltonian, therefore, can be written in the form:

$$\begin{aligned} H &= \tfrac{1}{2}\dot{u}^2\left(1 + \tfrac{3}{2}\varepsilon\,\eta\, u^2\right) + \omega_0^2(\tfrac{1}{2}\, u^2 + \tfrac{1}{4}\,\varepsilon\, u^4) \\ &= \tfrac{1}{2}\omega_0^2 A^2 + \tfrac{1}{4}\omega_0^2\,\varepsilon\, A^4 \end{aligned} \tag{169}$$

or

$$\begin{aligned} R(t) &= \tfrac{1}{2}\dot{u}^2\left(1 + \tfrac{3}{2}\varepsilon\,\eta\, u^2\right) + \omega_0^2(\tfrac{1}{2}\, u^2 + \tfrac{1}{4}\,\varepsilon\, u^4) \\ &\quad -\tfrac{1}{2}\omega_0^2 A^2 - \tfrac{1}{4}\omega_0^2\,\varepsilon\, A^4 = 0 \end{aligned} \tag{170}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation, A. So let us consider such initial conditions:

$$u(0) = A, \qquad \dot{u}(0) = 0 \tag{171}$$

Assume that its initial approximate guess can be expressed as:

$$u(t) = A\cos\omega t \tag{172}$$

Substituting Eq. (172) into Eq. (170), yields:

$$R(t) = \frac{1}{2}(-A\omega\sin\omega t)^2(1+\frac{3}{2}\varepsilon\,\eta\,(A\cos\omega t)^2) + \omega_0^2(\frac{1}{2}\,(A\cos\omega t)^2 + \frac{1}{4}\,\varepsilon\,(A\cos\omega t)^4) - \frac{1}{2}\omega_0^2 A^2 - \frac{1}{4}\omega_0^2\,\varepsilon\,A^4 = 0 \tag{173}$$

Which trigger the following result:

$$\omega = \frac{\omega_0\sqrt{2}}{A\sin\omega t}\sqrt{\frac{-(\frac{1}{2}\,(A\cos\omega t)^2 + \frac{1}{4}\,\varepsilon\,(A\cos\omega t)^4) + \frac{1}{2}A^2 + \frac{1}{4}\varepsilon\,A^4}{\left(1+\frac{3}{2}\varepsilon\,\eta\,(A\cos\omega t)^2\right)}} \tag{174}$$

If we collocate at $\omega t = \frac{\pi}{4}$, we obtain:

$$\omega_{EBM} = \frac{\omega_0\sqrt{(4+3A^2\varepsilon\eta)\,(4+3A^2\varepsilon)}}{4+3A^2\varepsilon\eta}, \tag{175}$$

Its period can be written in the form:

$$T_{EBM} = \frac{2\pi\left(4+3A^2\varepsilon\eta\right)}{\omega_0\sqrt{(4+3A^2\varepsilon\eta)\,(4+3A^2\varepsilon)}} \tag{176}$$

To further illustrate and verify the accuracy of this approximate analytical approach, comparison of the time history oscillatory displacement responses for the system with linear and nonlinear springs in series with numerical solutions are depicted in Figures 6 and 7. Figures 6 and 7 represent the displacements of $u(t)$ for a mass with different initial conditions and spring stiffnesses.

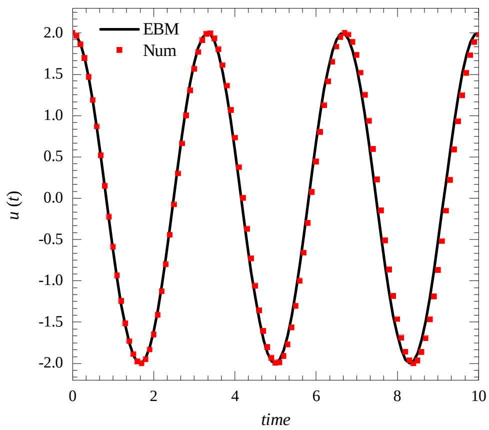

Figure 6 Comparison between approximate solutions and numerical solutions for $m = 1$, $A = 2$, $\varepsilon = 0.5$, $k_1 = 5$, $k_2 = 5$

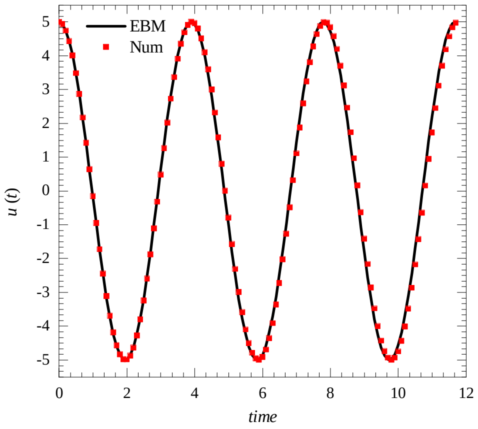

Figure 7 Comparison between approximate solutions and numerical solutions for $m = 3,\ \ A = 5,\ \ \varepsilon = 1,\ \ k_1 = 8,\ \ k_2 = 16$

Example 2

From Hamden [79], it is known that the free vibrations of an autonomous conservative oscillator with inertia and static type fifth-order non-linearties is expressed by

$$\ddot{x} + \lambda x + \varepsilon_1 x^2 \ddot{x} + \varepsilon_1 x \dot{x}^2 + \varepsilon_2 x^4 \ddot{x} + 2\varepsilon_2 x^3 \dot{x}^2 + \varepsilon_3 x^3 + \varepsilon_4 x^5 = 0, \tag{177}$$

With the initial conditions:

$$x(0) = A \quad \dot{x}(0) = 0 \tag{178}$$

Motion is assumed to start from the position of maximum displacement with zero initial velocity. λ is an integer which may take values of $\lambda = 1, 0$ or -1, and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 are positive parameters.

The solution of nonlinear equation with the Energy Balance method is:

$$\ddot{x} + \lambda x + \varepsilon_1 x^2 \ddot{x} + \varepsilon_1 x\, \dot{x}^2 + \varepsilon_2 x^4 \ddot{x} + 2\varepsilon_2 x^3 \dot{x}^2 + \varepsilon_3 x^3 + \varepsilon_4 x^5 = 0, \tag{179}$$

in which x and t are generalized dimensionless displacement and time variables, respectively. Its Variational principle can be easily obtained:

$$x(0) = A, \qquad \dot{x}(0) = 0 \tag{180}$$

in which $x$and $t$are generalized dimensionless displacement and time variables, respectively. Its Variational principle can be easily obtained:

$$J(x) = \int_0^t \left(-\frac{1}{2}\dot{x}^2 \left(1 + \varepsilon_1 x^2 + \varepsilon_2 x^4\right) + \frac{\lambda}{2}x^2 + \frac{\varepsilon_3}{4}x^4 + \frac{\varepsilon_4}{6}x^6 \right) dt \tag{181}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}\dot{x}^2\left(1 + \varepsilon_1 x^2 + \varepsilon_2 x^4\right) + \frac{\lambda}{2}x^2 + \frac{\varepsilon_3}{4}x^4 + \frac{\varepsilon_4}{6}x^6 = \frac{\lambda}{2}A^2 + \frac{\varepsilon_3}{4}A^4 + \frac{\varepsilon_4}{6}A^6 \tag{182}$$

Or

$$R(t) = \frac{1}{2}\dot{x}^2\left(1 + \varepsilon_1 x^2 + \varepsilon_2 x^4\right) + \frac{\lambda}{2}x^2 + \frac{\varepsilon_3}{4}x^4 + \frac{\varepsilon_4}{6}x^6 - \frac{\lambda}{2}A^2 - \frac{\varepsilon_3}{4}A^4 - \frac{\varepsilon_4}{6}A^6 = 0 \tag{183}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency $\omega$and the amplitude $x(t) = A\cos\omega\ t$ of oscillation, A. So let us consider such initial conditions:

$$x(0) = A, \qquad \dot{x}(0) = 0 \tag{184}$$

Assume that its initial approximate guess can be expressed as:

$$x(t) = A\cos\omega t \tag{185}$$

Substituting Eq. (185) into Eq. (183) yields:

$$\begin{aligned} R(t) = \frac{1}{2}(-A\sin\omega t)^2(1 + \varepsilon_1(A\cos\omega t)^2 + \varepsilon_2(A\cos\omega t)^4) + \frac{\lambda}{2}(A\cos\omega t)^2 + \frac{\varepsilon_3}{4}(A\cos\omega t)^4 \\ + \frac{\varepsilon_4}{6}(A\cos\omega t)^6 - \frac{\lambda}{2}A^2 - \frac{\varepsilon_3}{4}A^4 - \frac{\varepsilon_4}{6}A^6 = 0 \end{aligned} \tag{186}$$

Which trigger the following results

$$\omega = \frac{\sqrt{2}}{A\sin\omega t}\sqrt{\frac{\frac{\lambda}{2}\left(A^2 - (A\cos\omega t)^2\right) + \frac{\varepsilon_3}{4}\left(A^4 - (A\cos\omega t)^4\right) + \frac{\varepsilon_4}{6}\left(A^6 - (A\cos\omega t)^6\right)}{1 + \varepsilon_1 (A\cos\omega t)^2 + \varepsilon_2 (A\cos\omega t)^4}} \tag{187}$$

If we collocate at $\omega t = \frac{\pi}{4}$, we obtain:

$$\omega_{EBM} = \frac{\sqrt{3}}{3}\sqrt{\frac{12\lambda + 9\varepsilon_3 A^2 + 7\varepsilon_4 A^4}{4 + 2\varepsilon_1 A^2 + \varepsilon_2 A^4}} \tag{188}$$

Substituting Eq. (188) into Eq. (185) yields:

$$x(t) = A\cos\left(\frac{\sqrt{3}}{3}\sqrt{\frac{12\lambda + 9\varepsilon_3 A^2 + 7\varepsilon_4 A^4}{4 + 2\varepsilon_1 A^2 + \varepsilon_2 A^4}}\quad t\right) \tag{189}$$

The numerical solution with Runge-Kutta method for nonlinear equation is:

$$\dot{x}_1 = x_2 \quad x_1(0) = A \tag{190}$$

And

$$\dot{x}_2 = -\frac{1}{1+\varepsilon_1 x_1^2 + \varepsilon_2 x_1^4}\left(\lambda x_1 + \varepsilon_1 x_1 x_2^2 + 2\varepsilon_2 x_1^3 x_2^2 + \varepsilon_3 x_1^3 + \varepsilon_4 x_1^5\right), \quad x_2(0) = 0 \tag{191}$$

Motion is assumed to start from the position of maximum displacement with zero initial velocity. λ Is an integer which may take values of $\lambda = 1, 0$or-1, and $\varepsilon_1, \varepsilon_2, \varepsilon_3$and ε_4 are positive parameters .The values of parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4$associated for a mode is shown in Table 6.

Table 6 Values of dimensionless parameters ε_i in Eq. (189) for a mode

Mode	ε_1	ε_2	ε_3	ε_4
1	0.326845	0. 129579	0. 232598	0. 087584
2	1.642033	0.913055	0.313561	0.204297
3	4.051486	1.665232	0.281418	0.149677

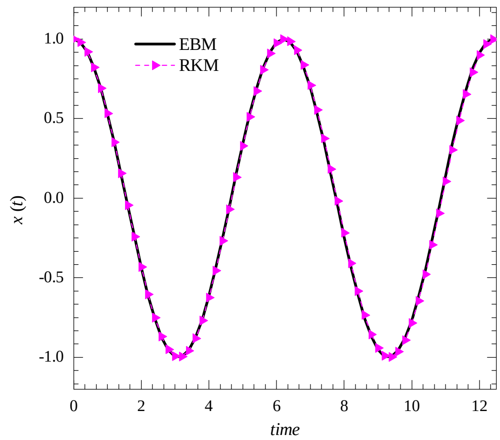

Figure 8 The Comparison between energy balance method solution and the numerical solution (Runge-Kutta method), with λ=1, A=1 for mode-1.

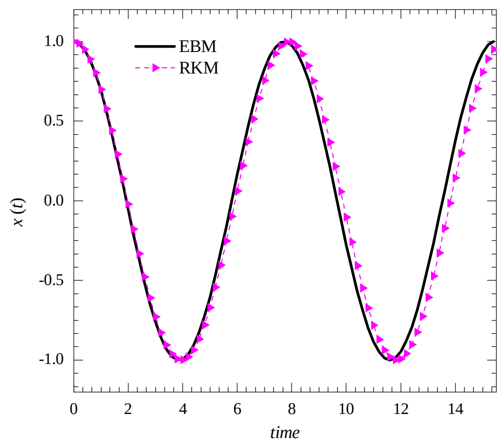

Figure 9 The comparison between energy balance method solution and the numerical solution (Runge-Kutta method), with λ=1, A=1 for mode-2.

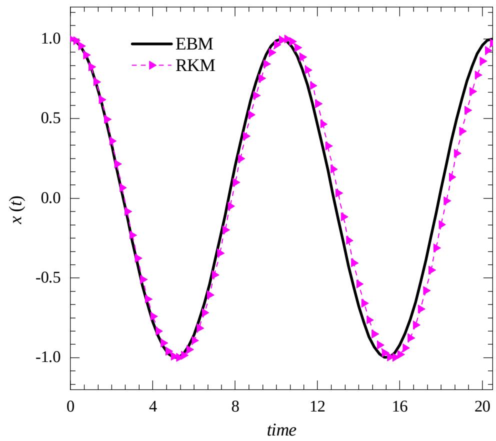

Figure 10 The comparison between energy balance method solution and numerical solution (Runge-Kutta method), with λ=1, A=1 for mode-3.

It can be seen from Figures 8-9 EBM results have a good agreement with the numerical solution for 3 modes. Figures show the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions.

Example 3

Consider a straight Euler-Bernoulli beam of length$L$, a cross-sectional area A, the mass per unit length of the beam m, a moment of inertia I, and a modulus of elasticity E that is subjected to an axial force of magnitude P as shown in Fig. 11. The equation of motion including the effects of mid-plane stretching is given by:

$$m\,\frac{\partial^2 w'}{\partial t'^2} + EI\frac{\partial^4 w'}{\partial x'^2} + \bar{P}\frac{\partial^2 w'}{\partial x'^2} - \frac{EA}{2L}\frac{\partial^2 w'}{\partial x'^2}\int_0^L \left(\frac{\partial^2 w'}{\partial x'^2}\right)^2 dx' = 0 \tag{192}$$

For convenience, the following non-dimensional variables are used:

$$x = x'/L, w = w'/\rho\,, t = t'(EI\big/ml^4)^{1/2}, P = \bar{P}L^2\big/EI \tag{193}$$

Where $\rho = (I/A)^{1/2}$ is the radius of gyration of the cross-section. As a result Eq. (192) can be written as follows:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = 0 \tag{194}$$

Assuming $w(x,t) = V(t)\,\phi\,(x)$where$\,\phi\,(x)$is the first eigenmode of the beam [189] and applying the Galerkin method, the equation of motion is obtained as follows:

$$\frac{d^2V(t)}{dt^2} + (\alpha_1 + P\alpha_2)V(t) + \alpha_3\,V^3(t) = 0 \tag{195}$$

The Eq. (195) is the differential equation of motion governing the non-linear vibration of Euler-Bernoulli beams. The center of the beam is subjected to the following initial conditions:

$$V(0) = \Delta\,, \;\; \frac{dV(0)}{dt} = 0 \tag{196}$$

Where Δ denotes the non-dimensional maximum amplitude of oscillation and α_1, α_2 and α_3 are as follows:

$$\alpha_1 = \left(\int_0^1 \left(\frac{\partial^4 \phi(x)}{\partial x^4} \right) \phi(x)\, dx \right) \Big/ \int_0^1 \phi^2(x)\, dx \tag{197a}$$

$$\alpha_2 = \left(\int_0^1 \left(\frac{\partial^2 \phi(x)}{\partial x^2} \right) \phi(x)\, dx \right) \Big/ \int_0^1 \phi^2(x)\, dx \tag{197b}$$

$$\alpha_3 = \left(\left(-\frac{1}{2} \right) \int_0^1 \left(\frac{\partial^2 \phi(x)}{\partial x^2} \int_0^1 \left(\frac{\partial^2 \phi(x)}{\partial x^2} \right)^2 dx \right) \phi(x)\, dx \right) \Big/ \int_0^1 \phi^2(x)\, dx \tag{197c}$$

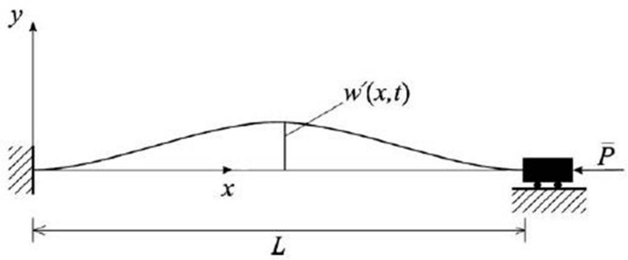

Figure 11 A schematic of an Euler-Bernoulli beam subjected to an axial load.

Variational formulation of Eq. (195) can be readily obtained as follows:

$$J(V) = \int_0^t \left(-\frac{1}{2} \frac{dV(t)}{dt} + \frac{1}{2} (\alpha_1 + P\alpha_2)\, V^2(t) + \alpha_3\, V^4(t) \right) dt\,. \tag{198}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = -\frac{1}{2} \frac{dV(t)}{dt} + \frac{1}{2} (\alpha_1 + P\alpha_2)\, V^2(t) + \alpha_3\, V^4(t) \tag{199}$$

And

$$H_{t=0} = \frac{1}{2} \Delta^2 (\alpha_1 + P\alpha_2) + \frac{1}{4} \alpha_4 \Delta^4 \tag{200}$$

$$H_t - H_{t=0} = \frac{1}{2} \frac{dV(t)}{dt} + \frac{1}{2} (\alpha_1 + P\alpha_2) V^2(t) + \alpha_3\, V^4(t) - \frac{1}{2} \Delta^2 (\alpha_1 + P\alpha_2) - \frac{1}{4} \alpha_4 \Delta^4 \tag{201}$$

We will use the trial function to determine the angular frequency ω, i.e.

$$V(t) = A\cos\omega\, t \tag{202}$$

If we substitute Eq. (204) into Eq. (203), it results the following residual equation

$$\begin{aligned}&\tfrac{1}{2}\left(-\Delta\omega\,\sin(\omega t)\right)^2+\tfrac{1}{2}\left(\alpha_1+P\alpha_2\right)\left(\Delta\,\cos(\omega\,t)\right)^2+\tfrac{1}{2}\alpha_3\left(\Delta\,\cos(\omega t)\right)^4\\&-\tfrac{1}{2}\Delta^2\left(\alpha_1+P\alpha_2\right)-\tfrac{1}{4}\alpha_3\,\Delta^4=0\end{aligned} \tag{203}$$

If we collocate at$\omega\,t=\frac{\pi}{4}$we obtain:

$$\frac{1}{4}\Delta^2\omega^2-\frac{1}{4}\Delta^2\left(\alpha_1+P\alpha_2\right)-\frac{3}{16}\alpha_3\Delta^4=0 \tag{204}$$

The non-linear natural frequency and the deflection of the beam center become as follows:

$$\omega_{NL}=\frac{\sqrt{4\left(\alpha_1+P\alpha_2\right)+3\alpha_3\Delta^2}}{2} \tag{205}$$

According to Eq. (207) and Eq. (204), we can obtain the following approximate solution:

$$V(t)=\Delta\cos\left(\frac{\sqrt{4\left(\alpha_1+P\alpha_2\right)+3\alpha_3\Delta^2}}{2}\,t\right) \tag{206}$$

Non-linear to linear frequency ratio is:

$$\frac{\omega_{NL}}{\omega_L}=\frac{1}{2}\frac{\sqrt{4\left(\alpha_1+p\alpha_2\right)+3\alpha_3\Delta^2}}{\sqrt{\alpha_1+p\alpha_2}} \tag{207}$$

Table 7 shows the comparison of non-linear to linear frequency ratio (ω_{NL}/ω_L).

Table 7 Comparison of non-linear to linear frequency ratio (ω_{NL}/ω_L).

Δ	Present Study (EBM)	Exact solution	Pade approximate P{4,2}[12]	Pade approximate P{6,4}[12]	Error % $(\omega_{EBM}-\omega_{ex})/\omega_{ex}$
0.2	1.044031	1.0438823	1.0438824	1.0438823	0.014211
0.4	1.16619	1.1644832	1.1644868	1.1644832	0.146604
0.6	1.345362	1.3397037	1.3397374	1.3397039	0.422385
0.8	1.56205	1.5505542	1.5506741	1.5505555	0.741395
1	1.802776	1.7844191	1.7846838	1.7844228	1.028712
1.5	2.462214	2.4254023	2.4261814	2.4254185	1.517775
2	3.162278	3.1070933	3.1084562	3.1071263	1.776077

To show the accuracy of Energy Balance Method (EBM) , comparisons of the time history oscillatory displacement response for Euler-Bernoulli beams with exact solutions are presented in Figs. 12 and 13.

It can be observed that the results of EBM require smaller computational effort and only a first-order approximation leads to accurate solutions. The Influence of α_3 on nonlinear to linear frequency and $\alpha_1$are presented in figures 14 and 15. It has illustrated that Energy Balance

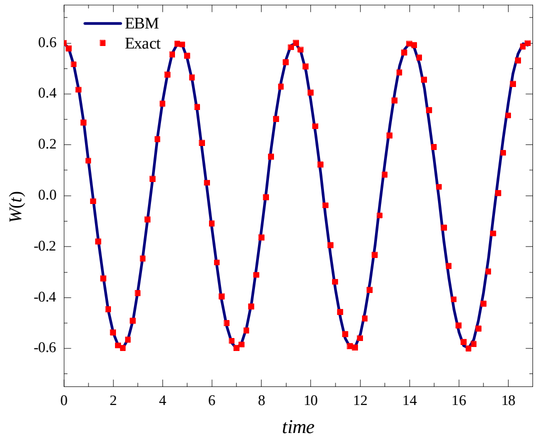

Figure 12 Comparison of analytical solution of $W(t)$based on timewith the exact solution for simply supported beam, $\Delta = 0.6$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 3$

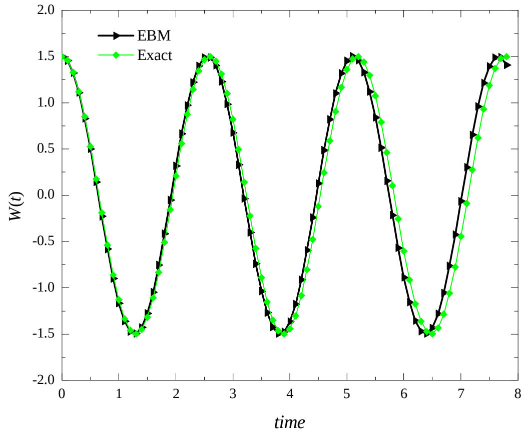

Figure 13 Comparison of analytical solution of $W(t)$based on timewith the exact solution for simply supported beam, $\Delta = 1.5$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 3$

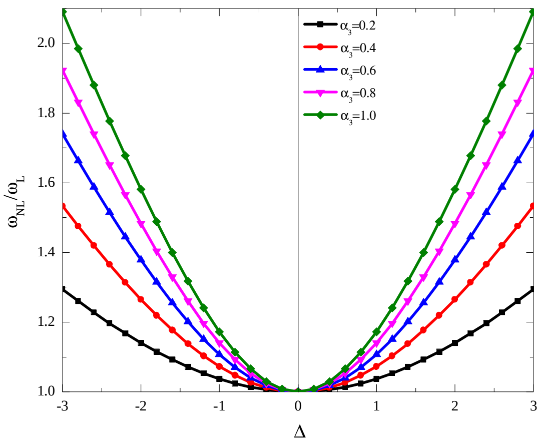

Figure 14 Influence ofα_3 on nonlinear to linear frequency base on Δ for$\alpha_1 = 1$, $\alpha_2 = 0.5$, $p = 2$

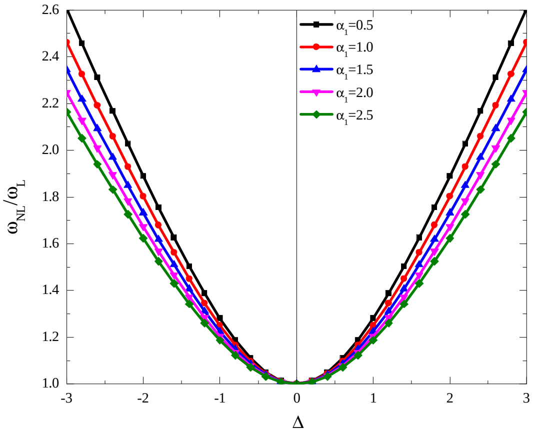

Figure 15 Influence ofα_3 on nonlinear to linear frequency base on Δ for$\alpha_2 = 1, \quad \alpha_3 = 3, \quad p = 3$

Method is a very simple method and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the Energy Balance Method can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

7 PARAMETER–EXPANSION METHOD (PEM)

Various perturbation methods have been applied frequently to analyze nonlinear vibration equations. These methods are characterized by expansions of the dependent variables in power series in a small parameter, resulting in a collection of linear deferential equations which can be solved successively. He proposed the parameter expanding method for the first time in his review article [100].The main property of the method is to use parameter-expansion technique to eliminate the secular terms and to achieve the frequency. PEM was successfully applied to various engineering problems [8, 42, 69, 97, 103, 118, 134, 147, 161, 190, 195–197, 201].

7.1 Basic idea of Parameter–Expansion Method

In order to use the PEM, we rewrite the general form of Duffing equation in the following form[100]:

$$\ddot{u} + \varepsilon u + 1 \,.\, N(u,t) = 0. \tag{208}$$

Where $N(u,t)$ includes the nonlinear term. Expanding the solution u,ε as a coefficient of u,and 1 as a coefficient of $N(u,t)$, the series of p can be introduced as follows:

$$u = u_0 + p\,u_1 + p^1\,u_2 + p^2\,u_3 + \ldots \tag{209}$$

$$\varepsilon = \omega^2 + p\,d_1 + p^1\,d_2 + p^2\,d_3 + \ldots \tag{210}$$

$$1 = p\,a_1 + p^1\,a_2 + p^2\,a_3 + ... \tag{211}$$

Substituting (209)-(211) into (208)and equating the terms with the identical powers of p, we have

$$p^0 : \ddot{u}_0 + \omega^2\,u_0 = 0\ , \tag{212}$$

$$\begin{aligned} &p^1 : \ddot{u}_1 + \omega^2\,u_1 + d_1 u_0 + a_1 N(u_0, t) = 0\ , \\ &\ \vdots \end{aligned} \tag{213}$$

Considering the initial conditions $u_0(0) = A$ and $\dot{u}_0(0) = 0$, the solution of (212) is $u_0 = A\cos(\omega\,t)$. Substituting u_0 into (213), we obtain

$$p^1 : \ddot{u}_1 + \omega^2\,u_1 + d_1\,A\cos(\omega\,t) + a_1 N(A\cos(\omega\,t), t) = 0\quad . \tag{214}$$

For achieving the secular term, we use Fourier expansion series as follows:

$$N(A\cos(\omega\,t), t) = \sum_{n=0}^{\infty} b_{2k+1}\cos((2k+1)\omega t)\,. \tag{215}$$

Substituting (215) into(214) yields;

$$p^1 : \ddot{u}_1 + \omega^2\,u_1 + (d_1\,A + \,a_1\,b_1\,)\cos(\omega\,t) = 0\quad . \tag{216}$$

For avoiding secular term, we have

$$(d_1\,A + \,a_1\,b_1\,) = 0\quad . \tag{217}$$

Setting $p = 1$ in (210)and (211) ,we have:

$$d_1\ = \varepsilon - \omega^2 A = 0\ , \tag{218}$$

$$a_1\ = 1. \tag{219}$$

Substituting (218) and (219) into (217), we will achieve the first-order approximation frequency (208) .Note that, from (211) and (219) , we can find that $a_i = 0$ for all$i = 1, 2, 3, 4, ...$

7.2 Application of Parameter–Expansion Method

Example 1

To illustrate the basic solution procedure, we consider the following nonlinear oscillator:

$$\ddot{u} + \alpha\,u + \beta\,u^3 = F_0\cos\omega\ t \quad , u(0) = A \quad , \dot{u}(0) = 0. \tag{220}$$

We rewrite it in this form

$$\ddot{u} + \alpha \,.\, u + 1.(\beta\, u^3 - F_0 \cos \omega\ t) = 0. \tag{221}$$

Assume that the solution can be expressed as a power series in an artificial Parameter to p

$$u = u_0 + pu_1 + p^2 u_2 + \ldots, \tag{222}$$

Where p is a bookkeeping parameter.

We assume that the coefficients α and 1 on the left side of Eq.(221) can be respectively expanded into a series in p:

$$\alpha = \omega^2 + p\omega_1 + p^2\omega_2 + \ldots, \tag{223}$$

$$1 = a_1 p + a_2 p^2 + \ldots\ . \tag{224}$$

Substituting Eqs.(223) and (224) into Eq. (221) and equating the terms with the identical powers p, we have:

$$p^0 : \ddot{u}_0 + \omega^2 u_0 = 0\,, \quad u_0(0) = A, \dot{u}_0(0) = 0, \tag{225}$$

$$p^1 : \ddot{u}_1 + \omega^2 u_1 + \omega_1 u_0 + a_1 \beta\, u_0^3 - a_1 F_0 \cos \omega\, t = 0 \tag{226}$$

Solving Eq.(225) , we have:

$$u_0 = A \cos \omega\ t \tag{227}$$

Substituting the result into Eq. (226),we have:

$$\ddot{u}_1 + \omega^2 u_1 + \omega_1 A \cos \omega\, t + a_1 \beta\, A^3 \cos^3 \omega\ t - a_1.F_0 \cos \omega\, t = 0 \tag{228}$$

We have the following identity

$$\cos^3(\omega\ t) = \frac{3}{4}\cos(\omega\ t) + \frac{1}{4}\cos(3\omega\ t) \tag{229}$$

And

$$\ddot{u}_1 + \omega^2 u_1 + (\omega_1 A + \frac{3}{4} a_1 \beta A^3 - a_1 F_0) \cos \omega\, t + \frac{A^3}{4}\cos 3\omega\ t = 0 \tag{230}$$

No secular terms in u_1 requires

$$\omega_1\, A + \frac{3}{4} a_1 \beta A^3 - a_1 F_0 = 0\,. \tag{231}$$

If the first-order approximation is sufficient, then we set $p = 1$and from (223)and (224) we have:

$$\alpha = \omega^2 + \omega_1, \tag{232}$$

$$1 = a_1 \,. \tag{233}$$

From Eqs. (231) , (232), (233) we obtain;

$$\omega^2 = \sqrt{\alpha + \frac{3}{4}\beta A^2 - \frac{F_0}{A}} \tag{234}$$

If we assume $\alpha = \omega_n^2, \beta = \mu$, we have:

$$\omega_{PEM} = \sqrt{\omega_n^2 + \frac{3}{4}\mu A^2 - \frac{F_0}{A}} \tag{235}$$

The same result was obtained in [162].

Example 2

Consider the following nonlinear oscillator[46, 132]:

$$\ddot{u} + \frac{u^3}{1+u^2} = 0\,, \quad u(0) = A\,, \;\; \dot{u}(0) = 0 \tag{236}$$

We rewrite it in the form

$$\ddot{u} + 0\,.\,u + 1.\ddot{u}u^2 + 1.u^3 = 0\,. \tag{237}$$

Assume that the solution can be expressed as a power series in an artificial parameter p:

$$u{=}u_0{+}p\,u_1 + p^2 u_2 + ... \tag{238}$$

Where p is a bookkeeping parameter. We assume that the coefficients 0 and 1 on the left side of Eq. (238) can be respectively expanded into a series in p

$$0{=}\omega^2{+}p\,\omega_1 + p^2\omega_2 + ... \tag{239}$$

$$1 = a_1 p + a_2 p^2 + ... \tag{240}$$

$$1 = b_1 p + b_2 p^2 + ... \tag{241}$$

Substituting Eqs. (239), (240)and (241) into Eq. (237) and equating the terms with the identical powers of p, we have

$$p^0 : \ddot{u}_0 + \omega^2 u_0 = 0, \quad u_0(0) = A \;\;, \; \dot{u}_0(0) = 0, \tag{242}$$

$$p^1 : \ddot{u}_1 + \omega^2 u_1 + \omega_1 u_0 + a_1 u_0 + a_1 \ddot{u}_0 u_0^2 + b_1 u_0^3 = 0\,, \quad u_1(0) = 0 \ \ , \ \dot{u}_1(0) = 0, \tag{243}$$

The solution of Eq. (242) can be easily obtained

$$u_0 = A\cos\omega t \tag{244}$$

Substituting the result into Eq. (243), we have:

$$p^1 : \ddot{u}_1 + \omega^2 u_1 + \left(\omega_1 A + \frac{3}{4}b_1 A^3 - \frac{3}{4}a_1\omega^2 A^3\right)\cos\,(\omega t) + \frac{1}{4}A^3\left(b_1 - a_1\omega^2\right)\cos(3\omega t) = 0. \tag{245}$$

Using Fourier series expansion, we have
No secular terms in $u_1$requires

$$\omega_1 A + \frac{3}{4}b_1 A^3 - \frac{3}{4}a_1\omega^2 A^3 = 0. \tag{246}$$

If the first-order approximation is sufficient, then we set $p = 1$ and from (239)and (240) we have

$$0{=}\omega^2{+}\,\omega_1 \tag{247}$$

$$1 = a_1. \tag{248}$$

$$1 = b_1. \tag{249}$$

From Eqs. (246),(247),(248) and (249), we have:

$$\omega_{PEM} = \sqrt{\frac{3A^2}{4+3A^2}} \tag{250}$$

Which agrees well with the exact solution The obtained frequency is valid for all $0 < A < \infty$**.**

Table 8 Comparison of approximate and exact frequencies[73].

A	ω_{PEM}	ω_{Exact}	$\frac{\omega_{PEM}-\omega_{Exact}}{\omega_{Exact}} \times 100$
0.05	0.04232	0.04326	2.172908
0.1	0.08439	0.08628	2.190542
0.5	0.38737	0.39736	2.514093
1	0.63678	0.65465	2.729703
5	0.96698	0.97435	0.756402
10	0.99092	0.9934	0.249648

Which has an excellent agreement with the exact one for all $0 < A < \infty$[132].

8 VARIATIONAL APPROACH (VA)

The study of nonlinear oscillators is an interest for many researchers, because there are many practical engineering components consisting of vibrating systems that can be modeled using oscillatory systems. Nonlinear analytical techniques for solving nonlinear problems have been dominated by different methods of investigation of these problems which appeared in numerous domains of physics and engineering. Overview of the literary texts with multiple mentions has been given by many wordsmiths utilizing miscellaneous analytical methods for solving nonlinear oscillation systems. Various variational methods have made, and will continue to make, an impact in key areas for science and technology development. The method was proposed by He in 2007[107]. He suggested a new variational method which is very effective for nonlinear oscillators. The application of this method widely used in many scientific papers [7, 16, 18, 21, 71, 119, 121, 135, 143, 151, 152, 154, 169, 175, 212].

8.1 Basic idea of Variational Approach

He suggested a variational approach which is different from the known variational methods in open literature [107]. Hereby we give a brief introduction of the method:

$$u'' + f(u) = 0 \tag{251}$$

Its variational principle can be easily established utilizing the semi-inverse method:

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}u'^2 + F(u)\right) dt \tag{252}$$

Where T is period of the nonlinear oscillator, $\partial F/\partial u = f$.Assume that its solution can be expressed as

$$u(t) = A\cos(\omega t) \tag{253}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq.(253) into Eq.(252) results in:

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left(-\tfrac{1}{2}A^2\omega^2 \sin^2 \omega t + F(A\cos\omega t)\right) dt \\ &= \tfrac{1}{\omega}\int_0^{\pi/2} \left(-\tfrac{1}{2}A^2\omega^2 \sin^2 t + F(A\cos t)\right) dt \\ &= -\tfrac{1}{2}A^2\omega \int_0^{\pi/2} \sin^2 t dt + \tfrac{1}{\omega}\int_0^{\pi/2} F(A\cos t)\, dt \end{aligned} \tag{254}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A} = 0 \tag{255}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{256}$$

But with a careful inspection, for most cases we find that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}A^2 \int_0^{\pi/2} \sin^2 t dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(A\cos t)dt < 0 \tag{257}$$

Thus, we modify conditions Eq. (255) and Eq. (256) into a simpler form:

$$\frac{\partial J}{\partial \omega} = 0 \tag{258}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

8.2 Application of Variational Approach

Example 1

We consider the physical model of nonlinear equation in the following figure with $F(t) = F_0 \sin \omega_0 t$, indicated in Fig. 16.

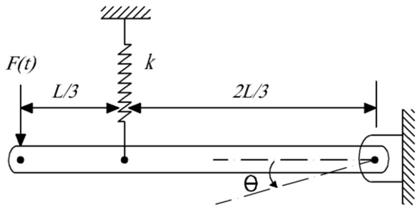

Figure 16 The physical model of nonlinear equation.

The motion equation is:

$$\ddot{\theta} + \frac{4k}{3m}\sin\theta - \frac{3F_0}{ml}\sin\omega_0 t = 0, \qquad \theta(0) = A \quad , \quad \dot{\theta}(0) = 0 \tag{259}$$

This equation is as known as Mathieu equation or the system with dependent coefficients to time. In which θ and t are generalized dimensionless displacements and time variables, respectively. And consider $F = \frac{4}{3}\frac{k}{m}$ as constant.

The approximation $sin(\theta) = \theta - (1/6)\theta^3 + (1/120)\theta^5$is used.

Its variational formulation can be readily obtained Eq. (259) as follows:

$$J(\theta) = \int_0^t \left(\frac{1}{2}\ddot{\theta}^2 + \frac{2}{3}\frac{k}{m}\theta^2 - \frac{1}{18}\frac{k}{m}\theta^4 + \frac{1}{540}\frac{k}{m}\theta^6 - \frac{3F_0 sin(\omega_0 t)}{ml}\theta \right) dt \tag{260}$$

Choosing the trial function $\theta(t) = A\cos(\omega t)$ into Eq.(260) we obtain**:**

$$J(A) = \int_0^{T/4} \left(\begin{array}{l} \frac{1}{2}A^2\omega^2 sin^2(\omega t) + \frac{2}{3}\frac{k}{m}A^2\cos^2(\omega t) - \frac{1}{18}\frac{k}{m}A^4\cos^4(\omega t) \\ +\frac{1}{540}\frac{k}{m}A^6\cos^6(\omega t) - \frac{3F_0 sin(\omega_0 t)}{ml}A\cos(\omega t) \end{array} \right) dt \tag{261}$$

The stationary condition with respect to A leads to:

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(\begin{array}{l} A\omega^2 sin^2(\omega t) + \frac{4}{3}\frac{k}{m}A\cos^2(\omega t) - \frac{2}{9}\frac{k}{m}A^3\cos^4(\omega t) \\ +\frac{1}{90}\frac{k}{m}A^5\cos^6(\omega t) - \frac{3F_0 sin(\omega_0 t)}{ml}\cos(\omega t) \end{array} \right) dt = 0 \tag{262}$$

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left(\begin{array}{l} A\omega^2 sin^2 t + \frac{4}{3}\frac{k}{m}A\cos^2 t - \frac{2}{9}\frac{k}{m}A^3\cos^4 t \\ +\frac{1}{90}\frac{k}{m}A^5\cos^6 t - \frac{3F_0 sin(\omega_0 t)}{ml}\cos t \end{array} \right) dt = 0 \tag{263}$$

Solving Eq.(263), according to ω, we have:

$$\omega^2 = \frac{\int_0^{\frac{\pi}{2}} \left(\frac{4}{3}\frac{k}{m}A\cos^2 t - \frac{2}{9}\frac{k}{m}A^3\cos^4 t + \frac{1}{90}\frac{k}{m}A^5\cos^6 t - \frac{3F_0 sin(\omega_0 t)}{ml}\cos t \right) dt}{\int_0^{\frac{\pi}{2}} A\, sin^2 t\, dt} \tag{264}$$

Then we have:

$$\omega_{VAM} = \frac{1}{12}\sqrt{\frac{1728F_0 sin\left(\frac{1}{2}\pi\omega_0\right) - 1728F_0\omega_0 + kAl\pi\left(\omega_0^2 - 1\right)\left(192 + A^4 - 24A^2\right)}{(m\omega_0^2 - m)lA\pi}} \tag{265}$$

According to Eqs. (253) and (265), we can obtain the following approximate solution:

$$\theta(t) = A\cos\left(\frac{1}{12}\sqrt{\frac{1728F_0 sin\left(\frac{1}{2}\pi\omega_0\right) - 1728F_0\omega_0 + kAl\pi\left(\omega_0^2 - 1\right)\left(192 + A^4 - 24A^2\right)}{(m\omega_0^2 - m)lA\pi}}\, t\right) \tag{266}$$

We compared the numerical solution and variational approach method for different parameters:

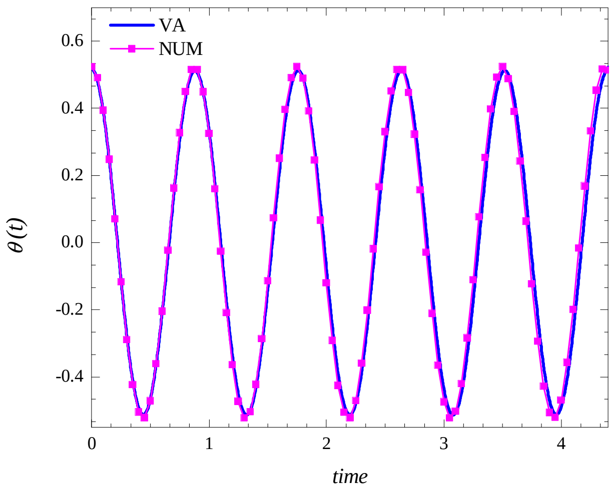

Figure 17 Comparison of analytical solution of θ based on time with the numerical solution for L=0.5 m , m=20 kg , k=800 N/m , F_0=1N , ω_0=2 rad/sec , A=π/6.

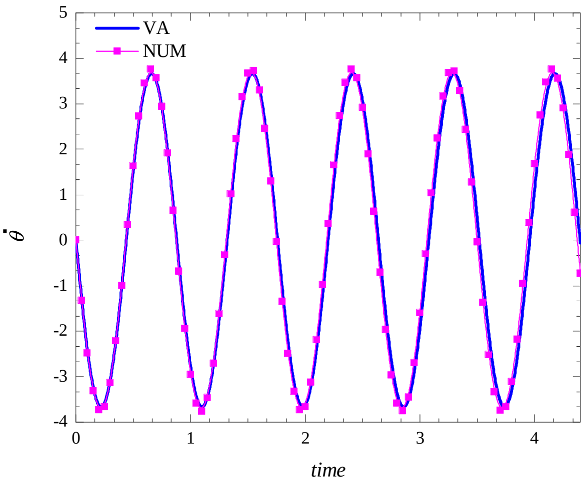

Figure 18 Comparison of analytical solution of θ based on time with the numerical solution for L=0.5 m , m=20 kg , k=800 N/m , F_0=1N , ω_0=2 rad/sec , A=$\pi/6$.

Figure 17 represents a comparison of analytical solution of $\theta(t)$ based on time with the numerical solution and Figure 18 shows comparison of analytical solution of $d\theta/dt$ based on time with the numerical solution.

Example 2

In this example, we consider the following nonlinear oscillator [71]:

$$\left(\frac{1}{12}l^2 + r^2\,\theta^2\right)\ddot{\theta} + r^2\theta\,\dot{\theta}^2 + r\,g\,\theta\,cos\,(\theta) = 0 \tag{267}$$

With the boundary conditions of:

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{268}$$

In order to apply the variational approach method to solve the above problem, the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ is used.

Its variational formulation is:

$$J(\theta) = \int_0^{T/4}\left(-\frac{1}{24}l^2\dot{\theta}^2 - \frac{1}{2}r^2\theta^2\,\dot{\theta}^2 + \frac{1}{2}\,r\,g\,\theta^2 - \frac{1}{8}\,r\,g\,\theta^4 + \frac{1}{144}g\,r\,\theta^6\right)dt \tag{269}$$

Choosing the trial function $\theta(t) = A\cos(\omega\,t)$ into Eq.(269) we obtain

$$J(A,\omega) = \int_0^{T/4}\left(\begin{array}{l} -\frac{1}{24}l^2\left(\mathrm{A}\,\omega\sin\left(\omega\mathrm{t}\right)\right)^2 - \frac{1}{2}r^2\left(A\cos\left(\omega\,t\right)\right)^2\left(\mathrm{A}\,\omega\sin\left(\omega\mathrm{t}\right)\right)^2 \\ +\frac{1}{2}\,r\,g\,\left(A\cos\left(\omega\,t\right)\right)^2 - \frac{1}{8}r\,g\,\left(A\cos\left(\omega\,t\right)\right)^4 + \frac{1}{144}g\,r\,\left(A\cos\left(\omega\,t\right)\right)^6 \end{array}\right)dt \tag{270}$$

The stationary condition with respect to A reads:

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(\begin{array}{c} -\frac{1}{12}\, l^2\omega^2 A\, sin^2\,(\omega\, t)\ - 2\, r^2\omega^2 A^3\, sin^2\,(\omega\, t)\ cos^2\,(\omega t) \\ +\, r\, g\, A\, cos^2\,(\omega\, t) - \frac{1}{2} r\, g\, A^3\, cos^4\,(\omega\, t) + \frac{1}{24} r\, g\, A^5\, cos^6\,(\omega\, t) \end{array} \right) dt = 0 \tag{271}$$

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left(\begin{array}{c} -\frac{1}{12} l^2 A\, sin^2\, t\, \omega^2 - 2\, r^2\, \omega^2\, A^3\, sin^2\, t\, cos^2\, t \\ +\, r\, g\, A\, cos^2\, t - \frac{1}{2} r\, g\, A^3\, cos^4\, t + \frac{1}{24} r\, g\, A^5 cos^6\, t \end{array} \right) dt = 0 \tag{272}$$

Then we have ;

$$\omega^2 = \frac{\int_0^{\pi/2} \left(r\, g\, A\, cos^2\, t - \frac{1}{2} r\, g\, A^3\, cos^4\, t + \frac{1}{24} r\, g\, A^5 cos^6\, t \right) dt}{\int_0^{\pi/2} \left(\frac{1}{12}\, l^2 A\, sin^2\, t + 2\, r^2\, A^3\, sin^2\, t\, cos^2\, t \right) dt} \tag{273}$$

Solving Eq. (273), according to ω, we have:

$$\omega = \frac{1}{4} \sqrt{\frac{r\, g\ \left(192 - 72\, A^2 + 5A^4\right)}{6A^2 r^2 + l^2}} \tag{274}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t) = A \cos \left(\frac{1}{4} \sqrt{\frac{r\, g\ \left(192 - 72\, A^2 + 5A^4\right)}{6A^2 r^2 + l^2}}\ t \right) \tag{275}$$

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the Variational Approach is:

$$\omega_{\,VA} = \frac{\sqrt{6}}{12} \sqrt{\frac{r\, g\, (288 - 108A^2 + 7A^4)}{6A^2 r^2 + l^2}} \tag{276}$$

The numerical solution (with Runge-Kutta method of order 4) for nonlinear equation is:

$$\begin{array}{ll} \dot{\theta} = \mathrm{y} & \theta(0) = \mathrm{A} \\ \dot{y} = -\dfrac{r^2 \theta u^2 + r\, g\, \theta\, cos(\theta)}{\frac{1}{12}\, l^2 + r^2\, \theta^2} & y(0) = 0 \end{array} \tag{277}$$

We compared the numerical solution with the variational approach in Figs 8.4 and 8.5. Fig. 19 shows the displacement of the system for l=2.5, r=0. 5, g=10, A=1. Fig. 20 represents the variation of frequency various parameters of amplitude (A).Comparing with the numerical results, it has been shown that the results of VA require smaller computational effort and only a first-order approximation of the VA leads to high accurate solutions.

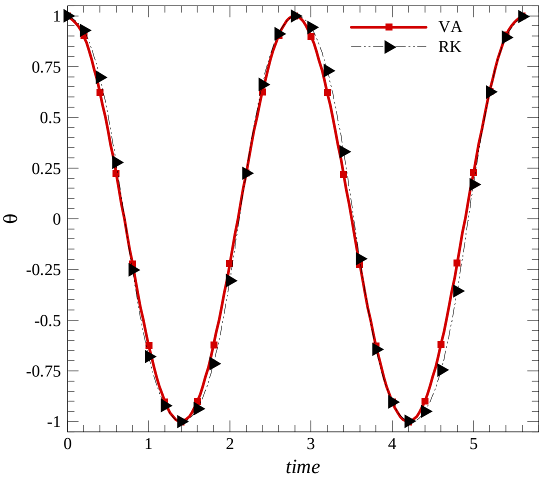

Figure 19 Comparison of (θ)of the VA solution and Runge-Kutta solution l=2.5, r=0. 5, g=10 , A=1

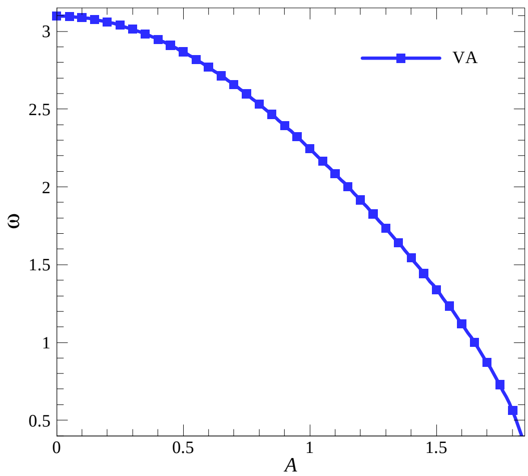

Figure 20 variation of the frequency respect to amplitude (A) for l=2.5, r=0. 5, g=10

Example 3

The mathematical pendulum is considered again as an example. The differential equation governing for the free oscillation of the mathematical pendulum is given by [138]

$$\ddot{\theta} - \Omega^2 cos\left(\theta\right) sin\left(\theta\right) + \frac{g}{r} sin\left(\theta\right) = 0 \tag{278}$$

With the boundary conditions of:

$$\theta\left(0\right) = A, \qquad \dot{\theta}\left(0\right) = 0 \tag{279}$$

In order to apply the variational approach method to solve the above problem, the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$and $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ is used.

Its variational formulation can be readily obtained as follows:

$$J(\theta) = \int_0^{T/4}\left(-\frac{1}{2}\dot{\theta}^2 - \frac{1}{2}\Omega^2\theta^2 + \frac{1}{6}\Omega^2\theta^4 - \frac{1}{48}\Omega^2\theta^6 + \frac{1}{1152}\Omega^2\theta^8 + \frac{1}{2}\frac{g}{r}\theta^2 - \frac{1}{24}\frac{g}{r}\theta^4\right)dt \tag{280}$$

Choosing the trial function $\theta(t) = A\cos(\omega t)$ into Eq.(281) we obtain

$$J(A,\omega) = \int_0^{T/4}\left(\begin{array}{l} -\frac{1}{2}\left(A\,\omega\sin(\omega t)\right)^2 - \frac{1}{2}\Omega^2\left(A\cos(\omega t)\right)^2 + \frac{1}{6}\Omega^2\left(A\cos(\omega t)\right)^4 \\ -\frac{1}{48}\Omega^2\left(A\cos(\omega t)\right)^6 + \frac{1}{1152}\Omega^2\left(A\cos(\omega t)\right)^8 + \left(\frac{1}{2}\right)\frac{g}{r}\left(A\cos(\omega t)\right)^2 \\ -\left(\frac{1}{24}\right)\frac{g}{r}\left(A\cos(\omega t)\right)^4 \end{array}\right)dt \tag{281}$$

The stationary condition with respect to A reads:

$$\frac{\partial J}{\partial A} = \int_0^{T/4}\left(\begin{array}{l} -A\,\omega^2 sin^2(\omega t) - \Omega^2 A cos^2(\omega t) + \frac{2}{3}\Omega^2 A^3 cos^4(\omega t) - \frac{1}{8}\Omega^2 A^5 cos^6(\omega t) \\ +\frac{1}{144}\Omega^2 A^7 cos^8(\omega t) + \frac{g}{r}A\,cos^2(\omega t) - \frac{1}{6}\frac{g}{r}A^3 cos^4(\omega t) \end{array}\right)dt = 0 \tag{282}$$

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2}\left(\begin{array}{l} -A\,\omega^2 sin^2 t - \Omega^2 A\,cos^2 t + \frac{2}{3}\Omega^2 A^3 cos^4 t - \frac{1}{8}\Omega^2 A^5 cos^6 t \\ +\frac{1}{144}\Omega^2 A^7 cos^8 t + \frac{g}{r}A\,cos^2 t - \frac{1}{6}\frac{g}{r}A^3 cos^4 t \end{array}\right)dt = 0 \tag{283}$$

Then we have;

$$\omega^2 = \frac{\int_0^{\pi/2}\left(\begin{array}{l} -\Omega^2 A\,cos^2 t + \frac{2}{3}\Omega^2 A^3 cos^4 t - \frac{1}{8}\Omega^2 A^5 cos^6 t + \frac{1}{144}\Omega^2 A^7 cos^8 t \\ +\frac{g}{r}A\,cos^2 t - \frac{1}{6}\frac{g}{r}A^3 cos^4 t \end{array}\right)dt}{A\int_0^{\pi/2} sin^2 t\,dt} \tag{284}$$

Solving Eq. (284), according to ω, we have:

$$\omega = \frac{1}{96}\sqrt{-9216\,\Omega^2 + 4608\,\Omega^2 A^2 - 720\,\Omega^2 A^4 + 35\,\Omega^2 A^6 + 9216\,\frac{g}{r} - 1152\,\frac{g}{r}A^2} \tag{285}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t) = A\cos\left(\frac{1}{96}\sqrt{-9216\,\Omega^2 + 4608\,\Omega^2 A^2 - 720\,\Omega^2 A^4 + 35\,\Omega^2 A^6 + 9216\,\frac{g}{r} - 1152\,\frac{g}{r}A^2}\;t\right) \tag{286}$$

To compare the results of VA, frequency obtained from VA is:

$$\omega_{VA} = \frac{\sqrt{6}}{96}\sqrt{-1536\,\Omega^2 + 768\Omega^2 A^2 - 112\,\Omega^2 A^4 + 5\,\Omega^2 A^6 + 1536\,\frac{g}{r} - 192\,\frac{g}{r}A^2} \tag{287}$$

The numerical solution (with Runge-Kutta Method of order 4) for nonlinear equation is:

$$\begin{aligned} &\dot{\theta} = \ \mathrm{y}\theta(0) = \mathrm{A} \\ &\dot{y} = \Omega^2 cos\,(\theta)\ sin\,(\theta) - \tfrac{g}{r}\,sin\,(\theta)\,y(0) = 0 \end{aligned} \tag{288}$$

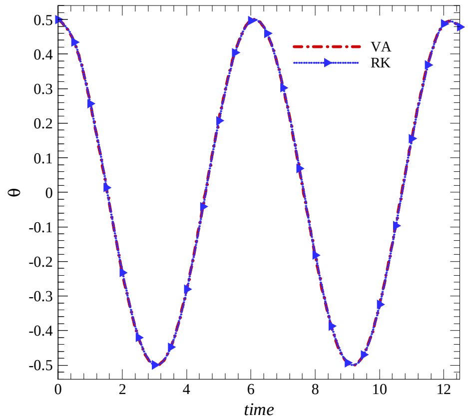

Figure 21 Comparison of displacement (θ) of the VA solution and Runge-kutta solution for Ω =1, r= 5, g=10, A=0.5

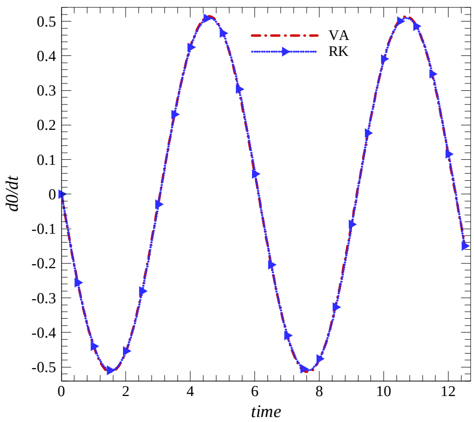

Figure 22 Comparison of velocity ($\dot{\theta}$) of the VA solution and Runge-kutta solution for Ω =1, r= 5, g=10, A=0.5

Some comparisons are presented to show the accuracy of the method. Figures 21 and 22 show comparison of analytical solution of θ and $\dot{\theta}$ based on time with the numerical solution. The variation of amplitude A on the frequency of the system is shown in figure 23. It can be approved that VA is powerful in finding analytical solutions for a wide class of nonlinear problems.

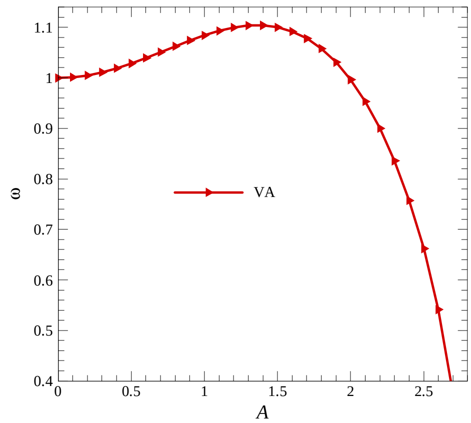

Figure 23 variation of the frequency respect to amplitude (A)for Ω =1, r= 5, g=10

9 IMPROVED AMPLITUDE-FREQUENCY FORMULATION (IAFF)

Most of engineering problems, especially some oscillation equations are nonlinear, and in most cases, it is difficult to solve such equations, especially analytically. One of the well-known methods to solve nonlinear problems is improved amplitude frequency formulation (IAFF). He in his previous review paper [100] in traduced the Ancient Chinese method including improved amplitude frequency formulation (IAFF). Geng and Cai [74]found the method to be very effective in solving strongly nonlinear oscillators. To illustrate the basic idea of the method, we consider an algebraic equation, this method applied correctly in many open literatures [1, 38, 52, 72, 106, 108, 109, 163–165, 182, 183, 200, 211, 214–216].

9.1 Basic idea of Improved Amplitude-Frequency Formulation

We consider a generalized nonlinear oscillator in the form [109]:

$$u'' + f(u) = 0, u(0) = A, u'(0) = 0, \tag{289}$$

We use two following trial functions

$$u_1(t) = A\cos(\omega_1 t), \tag{290}$$

And

$$u_2(t) = A\cos(\omega_2 t), \tag{291}$$

The residuals are

$$R_1(\omega t) = -A\omega_1^2\cos(\omega_1 t) + f\left(A\cos(\omega_1 t)\right), \tag{292}$$

And

$$R_2(\omega t) = -A\omega_2^2 \cos(\omega_2 t) + f\left(A\cos(\omega_2 t)\right), \tag{293}$$

The original Frequency-amplitude formulation reads :

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1}, \tag{294}$$

He used the following formulation [100] and Geng and Cai improved the formulation by choosing another location point [74].

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = 0) - \omega_2^2 R_1(\omega_1 t = 0)}{R_2 - R_1}, \tag{295}$$

This is the improved form by Geng and Cai.

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = \pi/3) - \omega_2^2 R_1(\omega_1 t = \pi/3)}{R_2 - R_1}, \tag{296}$$

The point is:$\cos(\omega_1 t) = \cos(\omega_2 t) = k$

Substituting the obtained ω into $u(t) = A\cos(\omega t)$, we can obtain the constant k inω^2 equation in order to have the frequency without irrelevant parameter.

To improve its accuracy, we can use the following trial function when they are required.

$$u_1(t) = \sum_{i=1}^{m} A_i \cos\ (\omega_i t), and\ \ u_2(t) = \sum_{i=1}^{m} A_i \cos(\Omega_i t) \tag{297}$$

or

$$u_1(t) = \frac{\sum_{i=1}^{m} A_i \cos(\omega_i t)}{\sum_{j=1}^{m} B_j \cos(\omega_j t)}, and\ \ u_2(t) = \frac{\sum_{i=1}^{m} A_i \cos(\Omega_i t)}{\sum_{j=1}^{m} B_j \cos(\Omega_j t)}, \tag{298}$$

But in most cases because of the sufficient accuracy, trial functions are as follow and just the first term:

$$u_1(t) = A\cos t, and\ \ u_2(t) = a\cos(\omega t) + (A - a)\cos(\omega t), \tag{299}$$

And

$$u_1(t) = A\cos t, and\ \ u_2(t) = \frac{A(1+c)\cos(\omega t)}{1 + c\cos(2\omega t)}, \tag{300}$$

Where a and c are unknown constants. In addition we can set: $\cos t = k$ in u_1, and $\cos\ (\omega t) = k$ in u_2.

9.2 Application of Improved Amplitude-Frequency Formulation

In this section, three practical examples are illustrated to show the applicability, accuracy and effectiveness of the proposed approach.

Example 1

A two-mass system connected with linear and nonlinear stiffnesses. Consider the two-mass system model as shown in Fig. 24. The equation of motion is given as [44];

$$\begin{aligned} m\ddot{x} + k_1(x-y) + k_2(x-y)^3 = 0 \\ m\ddot{y} + k_1(y-x) + k_2(y-x)^3 = 0 \end{aligned} \tag{301}$$

With initial conditions

$$\begin{aligned} x(0) = X_0, \quad \dot{x}(0) = 0, \\ y(0) = Y_0, \quad \dot{y}(0) = 0, \end{aligned} \tag{302}$$

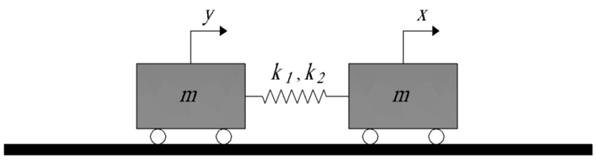

Figure 24 Two masses connected by linear and nonlinear stiffnesses.

Where double dots in Eq. (301) denote double differentiation with respect to time t, k_1 and k_2 are linear and nonlinear coefficients of the spring stiffness, respectively. Dividing Eq. (301) by mass m yields

$$\begin{aligned} \ddot{x} + \frac{k_1}{m}(x-y) + \frac{k_2}{m}(x-y)^3 = 0 \\ \ddot{y} + \frac{k_1}{m}(y-x) + \frac{k_2}{m}(y-x)^3 = 0 \end{aligned} \tag{303}$$

Introducing intermediate variables u and ν as follows [127]:

$$x := u \tag{304}$$

$$y - x := \nu \tag{305}$$

And transforming Eqs. (304) and (305) yields

$$\ddot{u} - \alpha\nu - \beta\nu^3 = 0 \tag{306}$$

$$\ddot{\nu} + \ddot{u} + \alpha\nu + \beta\nu^3 = 0 \tag{307}$$

Where $\alpha = k_1/m$ and $\alpha = k_2/m$ Eq. (306) is rearranged as follows:

$$\ddot{u} = \alpha\nu - \beta\nu^3. \tag{308}$$

Substituting Eq. (178) into Eq. (307) yields

$$\ddot{\nu} + 2\alpha\nu + 2\beta\nu^3 = 0 \tag{309}$$

With initial conditions

$$\nu(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{\nu}(0) = 0 \tag{310}$$

We use trial functions, as follows:

$$\nu_1(t) = A\cos t, \tag{311}$$

And

$$\nu_2(t) = A\cos(2t), \tag{312}$$

Respectively, the residual equations are:

$$R_1(t) = Acos(t)\left(-1 + 2\alpha + 2\beta A^2 cos^2(t)\right), \tag{313}$$

And

$$R_2(t) = 2A\,cos(2t)\left(-2 + \alpha + \beta A^2 cos^2(2t)\right), \tag{314}$$

Considering $\cos\ t_1\ = \cos\ 2t_2 = k$, we have:

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} = 2\alpha + 2\beta k^2 A^2, \tag{315}$$

We can rewrite $\nu(t) = A\cos(\omega t)$ in the form:

$$\nu(t) = A\cos\left(\sqrt{2\alpha + 2\beta k^2 A^2}\,t\right), \tag{316}$$

In view of the approximate solution, we can rewrite the main equation in the form:

$$\ddot{\nu} + (2\,\alpha + 2\,\beta\,k^2 A^2)\nu = (2\,\beta\,k^2 A^2)\nu - 2\,\beta\,\nu^3 \tag{317}$$

If by any chance $\nu(t) = A\cos\left(\sqrt{2\alpha + 2\beta k^2 A^2}\,t\right)$ is the exact solution, then the right side of Eq. (317) vanishes completely. Considering our approach which is just an approximation one, we set:

$$\int_0^{T/4} \left(2\beta k^2 A^2 \nu - 2\beta\nu^3\right)\,\cos\,\omega t\,dt = 0\,, T = 2\pi/\omega \tag{318}$$

Considering the term $\nu(t) = A\cos\left(\sqrt{2\alpha + 2\beta k^2 A^2}\, t\right)$and substituting the term to Eq. (318) and solving the integral term, we have:

$$k = \frac{1}{2}\sqrt{3}\ , \tag{319}$$

So, substituting Eq. (319) into Eq. (315), we have:

$$\omega_{IAFF} = \frac{1}{2}\sqrt{8\alpha + 6\beta A^2} \tag{320}$$

Table 9 Comparison of nonlinear frequencies in Eq. (320) with e exact solution

Constants					Results		
m	k_1	k_2	X_0	Y_0	IAFF solutionω	Exact solutionω_{Exact}	Relative error %
1	5	5	5	1	11.4018	11.1921	1.873643
1	1	1	10	-5	18.4255	18.0302	2.192433
1	10	5	20	25	14.4049	14.1514	1.791342
5	10	10	20	30	17.4356	17.0672	2.158526
10	50	-0.01	-20	40	2.1448	2.0795	3.140178

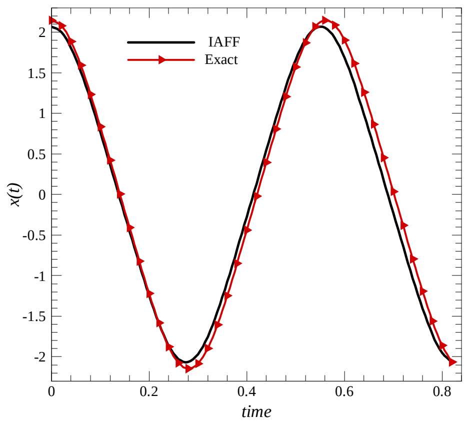

Figure 25 Comparison of the analytical approximates with the exact solution [44] for $k_1 = 5, k_2 = 5$, with $x(0) = 5$

The first-order approximate solutions is of a high accuracy and the percentage error improves significantly from lower order to higher order analytical approximations for different parameters and initial amplitudes. Hence, it is concluded that excellent agreement with the exact so. Table 9 gives the comparison of obtained results with the exact solutions for different m, k_1, k_2, and initial conditions. The maximum relative error between the IAFF results and exact results is 3.140178%.A comparison of the time history oscillatory displacementresponse for the two masses with exact solutionsare presented in Figs. 25 to 28.

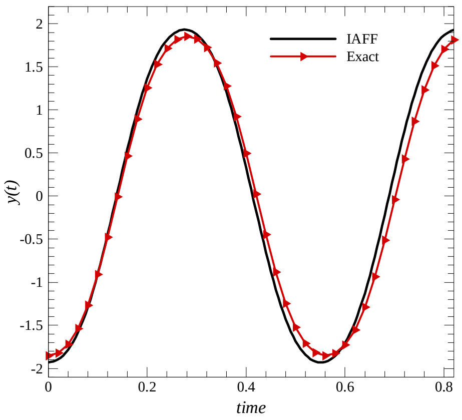

Figure 26 Comparison of the analytical approximates with the exact solution [44] for $k_1 = 5\,, k_2 = 5$, with $y(0) = 1$

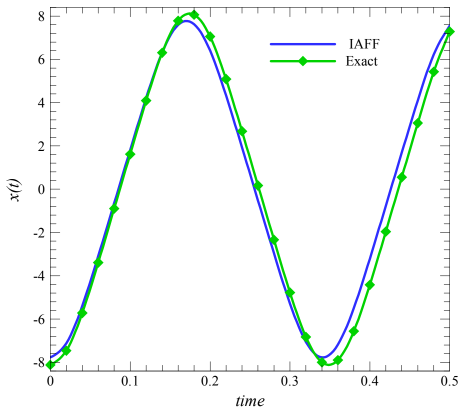

Figure 27 Comparison of the analytical approximates with the exact solution [44]for$k_1 = 5\,, k_2 = 5$, with$x(0) = 10$

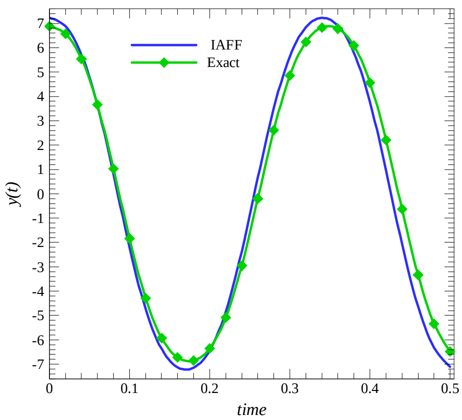

Figure 28 Comparison of the analytical approximates with the exact solution [44] for$k_1 = 5\,, k_2 = 5$, with$y(0) = -5$

Example 2

Consider a two-mass system connected with linear and nonlinear springs and fixed to a body at two ends as shown in Fig. 29 [43].

$$\begin{aligned} m\ddot{x} + k_1 x + k_2(x-y) + k_3(x-y)^3 = 0 \\ m\ddot{y} + k_1 x + k_2(y-x) + k_3(y-x)^3 = 0 \end{aligned} \tag{321}$$

With initial conditions

$$\begin{aligned} x(0) = X_0, \quad \dot{x}(0) = 0, \\ y(0) = Y_0, \quad \dot{y}(0) = 0, \end{aligned} \tag{322}$$

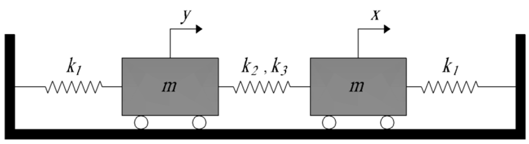

Figure 29 Two-mass system connected with the fixed bodies.

Where double dots in Eq. (321) denote double differentiation with respect to time , k_1 and k_2 are linear and nonlinear coefficients of the spring stiffness and k_3 is the nonlinear coefficient of the spring stiffness. Dividing Eq. (321) by mass m yields

$$\begin{aligned} \ddot{x} + \frac{k_1}{m}x + \frac{k_2}{m}(x-y) + \frac{k_3}{m}(x-y)^3 = 0 \\ \ddot{y} + \frac{k_1}{m}x + \frac{k_2}{m}(y-x) + \frac{k_3}{m}(y-x)^3 = 0 \end{aligned} \tag{323}$$

Like in Example 1, transforming the above equations using intermediate variables in Eqs. (304) and (305) yields;

$$\ddot{u} + \alpha u - \beta\nu - \xi\nu^3 = 0 \tag{324}$$

$$\ddot{u} + \ddot{\nu} + \alpha u - \alpha\nu + \beta\nu + \xi\nu^3 = 0 \tag{325}$$

Where $\alpha = k_1/m$,$\beta = k_2/m$ and $\xi = k_3/m$. Eq. (324) is rearranged as follows:

$$\ddot{u} = -\alpha u + \beta\nu + \xi\nu^3 \tag{326}$$

Substituting Eq. (326) into Eq. (325) yields

$$\ddot{\nu} + (\alpha + 2\beta)\nu + 2\xi\nu^3 = 0 \tag{327}$$

With initial conditions

$$\nu(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{\nu}(0) = 0 \tag{328}$$

We use trial functions, as follows:

$$\nu_1(t) = A\cos t, \tag{329}$$

And

$$\nu_2(t) = A\cos(2t), \tag{330}$$

Respectively, the residual equations are:

$$R_1(t) = Acos(t)\left(-1 + \alpha + 2\beta + 2\xi A^2 cos^2(t)\right), \tag{331}$$

And

$$R_2(t) = A\,cos(2t)\left(-4 + \alpha + 2\beta + 2\xi A^2 cos^2(2t)\right), \tag{332}$$

Considering cos t_1 = cos $2t_2 = k$, we have:

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} = \alpha + 2\beta + 2\xi k^2 A^2, \tag{333}$$

We can rewrite $\nu(t) = A\cos(\omega t)$ in the form:

$$\nu(t) = A\cos\left(\sqrt{\alpha + 2\beta + 2\xi k^2 A^2}\,t\right), \tag{334}$$

In view of the approximate solution, we can rewrite the main equation in the form:

$$\ddot{\nu} + (\alpha + 2\beta + 2\,\xi\,k^2 A^2)\,\nu = (2\,\xi\,k^2 A^2)\,\nu - 2\,\xi\,\nu^3 \tag{335}$$

If by any chance Eq. (334)is the exact solution, then the right side of Eq. (335) vanishes completely. Considering our approach which is just an approximation one, we set:

$$\int_0^{T/4} \left(2\xi k^2 A^2 \nu - 2\xi \nu^3\right)\cos\omega t\,dt = 0 \qquad\qquad T = 2\pi/\omega \tag{336}$$

Considering the term $\nu(t) = A\cos\left(\sqrt{2\alpha + 2\beta k^2 A^2}\,t\right)$and substituting the term to Eq. (336) and solving the integral term, we have:

$$k = \frac{1}{2}\sqrt{3}\ , \tag{337}$$

So, substituting Eq. (337) into Eq. (333), we have:

$$\omega_{IAFF} = \frac{1}{2}\sqrt{4\alpha + 8\beta + 6\xi A^2} \tag{338}$$

Table 10 shows an excellent agreement of the IAFF and exact solutions. From the Figs. 30 to 33, motions of the systems are periodic motions and the amplitude of vibrations is function of the initial conditions. These expressions are valid for a wide range of vibration amplitudes and initial conditions. The proposed methods are quickly convergent and can also be readily generalized to two-degree-of-freedom oscillation systems with quadratic nonlinearity by combining the transformation technique.

Table 10 Comparison of angular frequencies in Eq. (338) with exact solution.

Constants						Results		
m	k_1	k_2	k_3	X_0	Y_0	IAFF solutionω	Exact solution ω_{Exact}	Relative error
1	1	1	1	5	1	5.1961	5.1078	1.728729
1	1	1	5	5	10	13.8022	13.5121	2.146965
1	25	20	-0.05	-10	10	1.8708	1.8413	1.602129
5	10	20	30	-10	10	60.0833	58.7856	2.207513
10	50	70	90	20	-40	220.4972	215.7113	2.21866

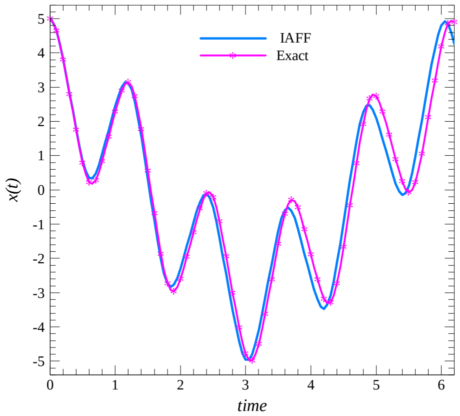

Figure 30 Comparison of the analytical approximates with the exact solution [43] for$k_1 = 5$, $k_2 = 5$, $k_3 = 1$ with $x(0) = 5$

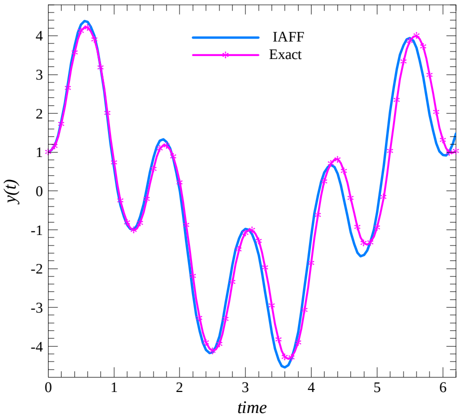

Figure 31 Comparison of the analytical approximates with the exact solution [43] for$k_1 = 5$, $k_2 = 5$, $k_3 = 1$ with $y(0) = 1$

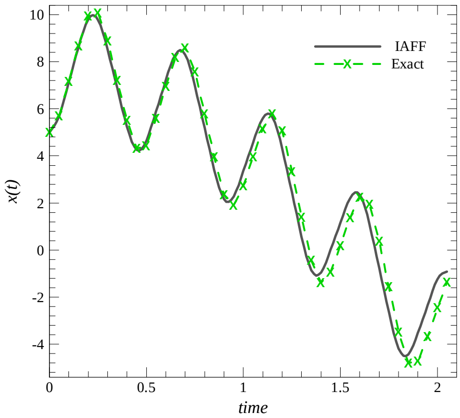

Figure 32 Comparison of the analytical approximates with the exact solution [43]for$k_1 = 5$, $k_2 = 5$, $k_3 = 5$ with $x(0) = 5$

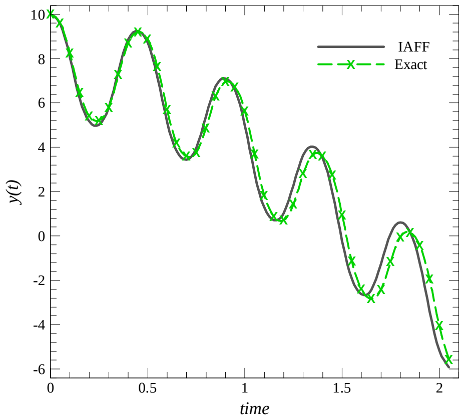

Figure 33 Comparison of the analytical approximates with the exact solution [43] for$k_1 = 1$, $k_2 = 1$, $k_3 = 5$ with $y(0) = 10$

Example 3

In order to assess the advantages and the accuracy of Improved Amplitude-frequency Formulation for solving nonlinear oscillator, we will consider the following nonlinear oscillator;

$$\ddot{u} + a\,u\dot{u}^2 + au\ddot{u} + \alpha_1 u + \alpha_2 u^3 + \alpha_3 u^5 = 0, \tag{339}$$

with the initial conditions of:

$$u(0) = A \ , \quad \dot{u}(0) = 0 \ , \tag{340}$$

We use trial functions, as follows:

$$u_1(t) = A\cos\ t, \tag{341}$$

And

$$u_2(t) = A\cos(2t), \tag{342}$$

Respectively, the residual equations are:

$$R_1(t) = A\cos(t)\left(-2\,a\,A^2\cos^2(t) + a\,A^2 - 1 + \alpha_1 + \alpha_2 A^2\cos^{\,2}(t) + \alpha_3 A^4\cos^4(t)\right), \tag{343}$$

And

$$R_2(t) = A\cos(2t)\left(-8aA^2\cos^2(2t) + 4aA^2 - 4 + \alpha_1 + \alpha_2 A^2\cos^2(2t) + \alpha_3 A^4\cos^4(2t)\right), \tag{344}$$

Considering $\cos t = \cos 2\,t = k$, we have:

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} = \frac{\alpha_1 + \alpha_2 A^2k^2 + \alpha_3 A^4k^4}{2aA^2k^2 - aA^2 + 1}, \tag{345}$$

We can rewrite $u(t) = A\cos(\omega\,t)$in the form:

$$u(t) = A\cos\left(\sqrt{\frac{\alpha_1 + \alpha_2 A^2k^2 + \alpha_3 f^4k^4}{2aA^2k^2 - aA^2 + 1}}t\right), \tag{346}$$

In view of the approximate solution, we can rewrite the main equation in the form:

$$\ddot{u} + \frac{\alpha_1 + \alpha_2 A^2k^2 + \alpha_3 f^4k^4}{2aA^2k^2 - aA^2 + 1}u = \frac{\alpha_1 + \alpha_2 A^2k^2 + \alpha_3 f^4k^4}{2aA^2k^2 - aA^2 + 1}u - au\dot{u}^2 + u\ddot{u} - \alpha_1 u - \alpha_2 u^3 - \alpha_3 u^5, \tag{347}$$

If by any chance $u(t) = A\cos\left(\sqrt{\frac{\alpha_1+\alpha_2A^2k^2+\alpha_3A^4k^4}{2aA^2k^2-aA^2+1}}t\right)$ is the exact solution, then the right side of Eq.(347) vanishes completely. Considering our approach which is just an approximation one, we set:

$$\int_0^T \left[\begin{array}{c} \frac{\alpha_1+\alpha_2A^2k^2+\alpha_3 f^4k^4}{2aA^2k^2-aA^2+1}u \\ -a\,u\,\dot{u}^2 + u\,\ddot{u} - \alpha_1 u - \alpha_2 u^3 - \alpha_3 u^5 \end{array}\right]\cos(\omega\,t)dt = 0\ , \qquad T = \frac{2\pi}{\omega}, \tag{348}$$

Considering the term $u(t) = A\cos(\omega\,t)$and substituting the term to Eq. (349) and solving the integral term, we have:

$$\begin{aligned} k^4 = {} & \tfrac{1}{16}\tfrac{1}{A^4\alpha_3^2(a\,A^2+2)^2}\,(\,5A^4\alpha_3 a + 8\alpha_1 a + 4A^2\alpha_2 a - 4\alpha_2 \\ & +(\,5\,A^8\alpha_3^2a^2 + 32\,A^4\alpha_3 a^2\alpha_1 + 16\,A^6\alpha_3 a^2\alpha_2 - 64A^4\alpha_3 a\alpha_2 + 64\alpha_1^2a^2 + 64\alpha_1 a^2A^2\alpha_2 - 64\alpha_1 a\,\alpha_2 \\ & + 16\,A^4\alpha_2^2a^2 - 32\,A^2\alpha_2^2 a + 16\,\alpha_2^2 - 20\,A^6\alpha_3^2 a + 48\,A^2\alpha_3\alpha_2 + 40\,A^4\alpha_3^2 - 96\,A^2\alpha_3\alpha_1 a\,)^{\frac{1}{2}}\,)^2\ , \end{aligned} \tag{349}$$

So, substituting Eq. (349) into Eq. (345), we have:

$$\omega = \frac{1}{2}\sqrt{\frac{5\,A^4\alpha_3 + 6\,A^2\alpha_2 + 8\,\alpha_1}{a\,A^2 + 2}}, \tag{350}$$

We can obtain the following approximate solution:

$$u(t) = A\cos\left(\frac{1}{2}\sqrt{\frac{5A^4\alpha_3 + 6A^2\alpha_2 + 8\alpha_1}{a\,A^2 + 2}}t\right), \tag{351}$$

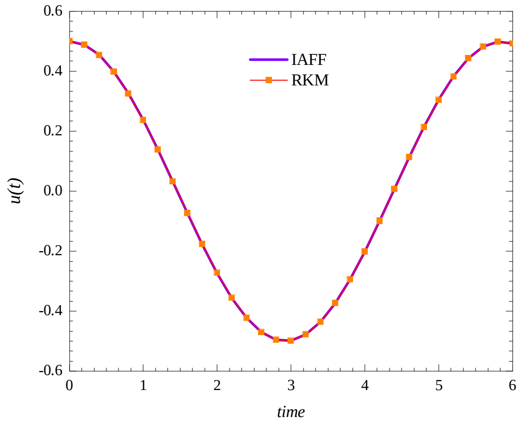

Figure 34 Comparison of displacement $u(t)$of the IAFF solution with the RKM solution$A = 0.5$, $a = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$

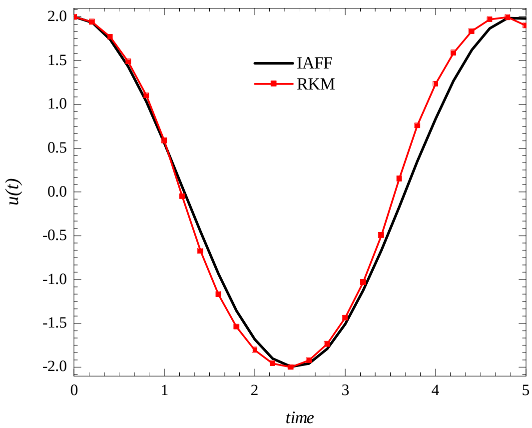

Figure 35 Comparison of $u(t)$ of the IAFF solution with the RKM solution $A = 2$, $a = 0.8$, $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.2$

Figs. 34 and 35 represent a comparison of the analytical solution of $u(t)$ based on time with the numerical solution. The time history diagram of $u(t)$ starts without an observable deviation with $A = 0.5$ and $A = 2$. The behavior of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. The best accuracy can be seen at extreme points. Although deviations of solutions are expected to increase as time progresses, the analytical solutions have adequate accuracy for the period shown

10 MAX-MIN APPROACH (MMA)

In this section, we consider a novel method called Max-Min Approach (MMA). Maximal and minimal solution thresholds of a nonlinear problem can be easily found, and an approximate solution of the nonlinear equation can be easily deduced using He Chengtian's interpolation, which has millennia history. Some examples are illustrated to show the efficiency and accuracy of the proposed method for high nonlinear vibration problems. This methodology has been utilized to achieve approximate solutions for nonlinear free vibration of conservative thick circular sector slabs. In Max-Min Approach (MMA), contrary to the conventional methods, only one iteration leads to high accuracy of solutions. Max-Min Approach (MMA) operates very well in the whole range of the parameters involved. Excellent agreement of the approximate frequencies and periodic solutions with the exact ones could be established. Some patterns are given to illustrate the effectiveness and convenience of the methodology. It has been indicated that the numerical results have same conclusion; while MMA is much easier, more convenient and more efficient than other approaches. The MMA is a novel method which alleviates drawbacks of the traditional numerical techniques. The method first was proposed by He [110].The application of this method widely used in many scientific papers [13, 20, 23, 70, 73, 171, 188, 207].

10.1 Basic idea of Max-Min Approach

We consider a generalized nonlinear oscillator in the form

$$\ddot{u} + u\,f(u) = 0, u(0) = A, \dot{u}(0) = 0, \tag{352}$$

Where $f(u)$ is a non-negative function of u. According to the idea of the max–min method, we choose a trial-function in the form

$$u(t) = A\cos(\omega\, t), \tag{353}$$

Where ω the unknown frequency to be further is determined.

Observe that the square of frequency, ω^2, is never less than that in the solution

$$u_1(t) = A\cos(\sqrt{f_{\min}}\, t), \tag{354}$$

of the following linear oscillator

$$\ddot{u} + u\,f_{\min} = 0, u(0) = A, \dot{u}(0) = 0, \tag{355}$$

Where $f_{\min}$ is the minimum value of the function$f(u)$.

In addition, ω^2 never exceeds the square of frequency of the solution

$$u_1(t) = A\cos\left(\sqrt{f_{\max}}\,t\right), \tag{356}$$

of the following oscillator

$$\ddot{u} + u\,f_{\max} = 0, u(0) = A, \dot{u}(0) = 0, \tag{357}$$

Where $f_{\max}$ is the maximum value of the function $f(u)$.

Hence, it follows that

$$\frac{f_{\min}}{1} < \omega^2 < \frac{f_{\max}}{1}. \tag{358}$$

According to He Chentian interpolation [110, 112], we obtain

$$\omega^2 = \frac{m\,f_{\min} + n\,f_{\max}}{m+n}, \tag{359}$$

Or

$$\omega^2 = \frac{f_{\min} + k\,f_{\max}}{1+k}, \tag{360}$$

Where m and n are weighting factors, $k = n/m$. So the solution of Eq. (352) can be expressed as

$$u(t) = A\,\cos\sqrt{\frac{f_{\min} + k\,f_{\max}}{1+k}}\,t, \tag{361}$$

The value of k can be approximately determined by various approximate methods [105, 110, 112]. Among others, hereby we use the residual method [110]. Substituting (361) into (352) results in the following residual:

$$R(t;k) = -\omega^2 A\,\cos(\omega\,t) + (A\,\cos(\omega\;t))\cdot f\,(A\,\cos(\omega\,t)) \tag{362}$$

Where $\omega = \sqrt{\frac{f_{\min}+k\,f_{\max}}{1+k}}$

If, by chance, Eq. (361) is the exact solution, then $R(t;k)$ is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^T R(t;k)\,\cos\sqrt{\frac{f_{\min} + k\,f_{\max}}{1+k}}\,t\,dt = 0, \tag{363}$$

where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{f_{\max} - f_{\min}}{1 - \sqrt{\frac{A}{\pi}\int_0^{\pi}\cos^2 x.f\,(A\,\cos\,x)\,dx}}. \tag{364}$$

Substituting the above equation into Eq. (361), we obtain the approximate solution of Eq. (352).

10.2 Application of Max-Min Approach

In this section, three examples are illustrated and solved to show the applicability, accuracy and effectiveness of Max-Min Approach.

Example 1

We can re-write Eq. (309) from the previous section in the following form;

$$\ddot{\nu} + (2\alpha + 2\beta\nu^2)\nu = 0. \tag{365}$$

We choose a trial-function in the form

$$\nu = A\cos(\omega t) \tag{366}$$

Where ω the frequency to be is determined the maximum and minimum values of $2\alpha+2\beta\nu^2$ will be $2\alpha + 2\beta A^2$ and 2α respectively, so we can write:

$$\frac{2\alpha}{1} < \omega^2 = 2\alpha + 2\beta\nu^2 < \frac{2\alpha + 2\beta A^2}{1} \tag{367}$$

According to He Chengtian's inequality , we have

$$\omega^2 = \frac{m.2\alpha + n.(2\alpha + 2\beta A^2)}{m+n} = 2\alpha + 2k\ \beta A^2 \tag{368}$$

Where m and $n$are weighting factors, $k = n/m + n$. Therefore the frequency can be approximated as:

$$\omega = \sqrt{2\alpha + 2k\ \beta A^2} \tag{369}$$

Its approximate solution reads

$$\nu = A\cos\sqrt{2\alpha + 2k\ \beta A^2}\, t \tag{370}$$

In view of the approximate solution, Eq.(370) we re-write Eq.(365) in the form

$$\ddot{\nu} + (2\alpha + 2k\ \beta A^2)\nu = (2\alpha + 2k\ \beta A^2)\nu - 2\beta\nu^3 \tag{371}$$

If by any chance Eq.(370) is the exact solution, then the right side of Eq.(371) vanishes completely. Considering our approach which is just an approximation one, we set:

$$\int_0^{T/4} \left(2k\ \beta A^2\nu - 2\beta\nu^3\right)\cos\omega t\, dt = 0 \tag{372}$$

Where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{3}{4} \tag{373}$$

Finally the frequency is obtained as

$$\omega = \frac{1}{2}\sqrt{8\alpha + 6\beta A^2} \tag{374}$$

According to Eqs. (366) and (374) , we can obtain the following approximate solution:

$$\nu(t) = A\cos\left(\frac{1}{2}\sqrt{8\alpha + 6\beta A^2}\,t\right) \tag{375}$$

The first-order analytical approximation for $u(t)$is

$$u(t) = \iint(\alpha\nu + \beta\nu^3)dt\,dt = -\frac{1}{9\omega^2}A\cos(\omega t)\left(9\alpha + 6\beta A^2 + A\beta\cos^2(\omega t)\right). \tag{376}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$\begin{aligned} x(t) &= u(t) \\ x(t) &= u(t) + A\cos(\omega t) \end{aligned} \tag{377}$$

Table 11 Comparison of frequency corresponding to various parameters of the system.

Constant parameters					Approximate Solution	Exact solution	Relative error %
m	k_1	k_2	X_0	Y_0	ω_{MMA}	ω_{Exact}[44]	$\frac{\omega_{MMA}-\omega_{Ex}}{\omega_{Ex}}$
1	0.5	0.5	1	5	3.605551	3.539243	1.873506
1	1	1	5	1	5.09902	5.005246	1.873506
5	2	0.5	5	10	4.421538	4.333499	2.031592
10	5	5	10	20	8.717798	8.533586	2.158667
20	40	50	20	10	19.46792	19.05429	2.17082
50	100	50	-10	20	36.79674	36.00234	2.206522

From table 11, the relative error of the MMA is 2.2065% for the first-order analytical approximations, for different values of m, k_1, k_2, X_0 and Y_0. The first-order approximate solution gives an excellent agreement with the exact one.To further illustrate and verify the accuracy of this approximate analytical approach, a comparison of the time history oscillatory displacement and velocity responses for the two masses with exact solutions is depicted in Figs. 36 and 37. Figs. 38 and 39 represent the effects of amplitude on the phase plan of the system. It is apparent that the first-order analytical approximations show excellent agreement with the exact solution using the Jacobi elliptic function.

Example 2

A two-mass system connected with linear and nonlinear stiffnesses fixed to the body was solved by IAFF is considered again in this section. We can re-write Eq. (327) in the following form;

$$\ddot{\nu} + \left((\alpha + 2\beta) + 2\xi\nu^2\right)\nu = 0 \tag{378}$$

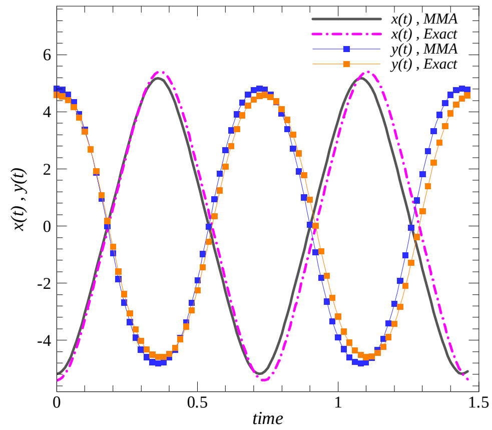

Figure 36 Comparison of analytical solution of displacement $x(t)$and $y(t)$based on time $t$with the exact solution[44] for $m = 10, \quad k_1 = 5, \quad k_2 = 5\,, \quad X_0 = 10, \quad Y_0 = 20$

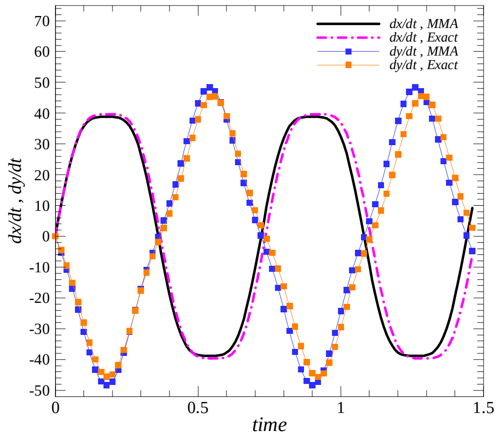

Figure 37 Comparison of analytical solution of dx/dt and dy/dt based on time t with the exact solution [44] for $m = 10, k_1 = 5, k_2 = 5\, X_0 = 10, Y_0 = 20$

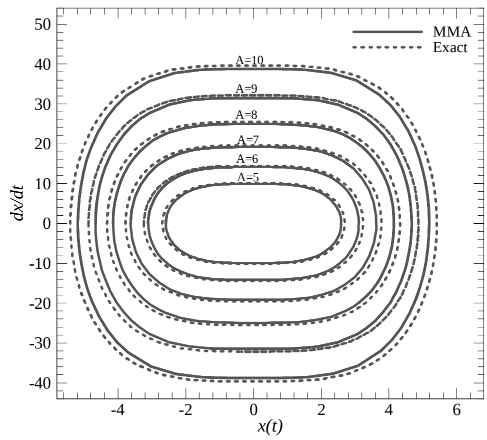

Figure 38 Comparison of analytical solution of dx/dt based on $x(t)$ with the exact solution [44] for $m = 10, \quad k_1 = 5, \quad k_2 = 5$

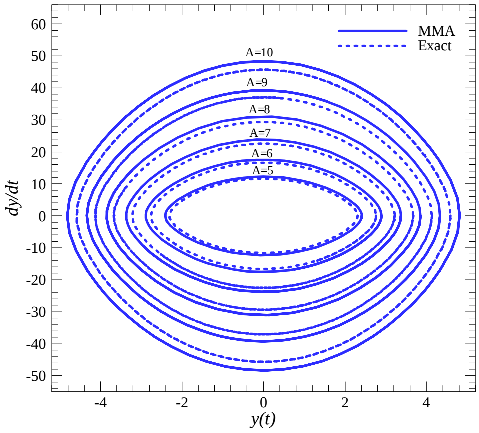

Figure 39 Comparison of analytical solution of dy/dt based on $y(t)$ with the exact solution [44] for $m = 10,\ k_1 = 5,\ k_2 = 5$

We choose a trial-function in the form

$$\nu = A\cos(\omega t) \tag{379}$$

Where ω the frequency to be is determined the maximum and minimum values of $\alpha + 2\beta + 2\xi\nu^2$ will be $\alpha + 2\beta + 2\xi A^2$ and $\alpha + 2\beta$ respectively, so we can write:

$$\frac{\alpha + 2\beta}{1} < \omega^2 = \alpha + 2\beta + 2\xi\nu^2 < \frac{\alpha + 2\beta + 2\xi A^2}{1} \tag{380}$$

According to He Chengtian's inequality , we have

$$\omega^2 = \frac{m.(\alpha + 2\beta) + n.(\alpha + 2\beta + 2\xi A^2)}{m+n} = \alpha + 2\beta + 2\xi k A^2 \tag{381}$$

Where m and n are weighting factors, $k = n/m + n$. Therefore the frequency can be approximated as:

$$\omega = \sqrt{\alpha + 2\beta + 2\xi k A^2} \tag{382}$$

Its approximate solution reads

$$\nu = A\cos\sqrt{\alpha + 2\beta + 2\xi k A^2}\,t \tag{383}$$

In view of the approximate solution, Eq. (382) we re-write Eq. (378)in the form;

$$\ddot{\nu} + \left(\alpha + 2\beta + 2\xi k A^2\right)\nu = \left(2\xi k A^2\right)\nu - 2\xi\nu^3 \tag{384}$$

If by any chance Eq. (383) is the exact solution, then the right side of Eq.(384) vanishes completely. Considering our approach which is just an approximation one, we set:

$$\int_0^{T/4} \left(2\xi k A^2 \nu - 2\xi \nu^3\right) \cos \omega t \, dt = 0 \tag{385}$$

Where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{3}{4} \tag{386}$$

Finally the frequency is obtained as

$$\omega = \frac{1}{2}\sqrt{4\alpha + 8\beta + 6\xi A^2} \tag{387}$$

According to Eqs. (387) and (379) , we can obtain the following approximate solution:

$$\nu(t) = A \cos\left(\frac{1}{2}\sqrt{4\alpha + 8\beta + 6\xi A^2}\, t\right) \tag{388}$$

The first-order analytical approximation for $u(t)$is

$$\begin{aligned} u(t) = & \frac{-cos(\sqrt{\alpha}t)\left(-X_0\,\alpha^2+10X_0\,\alpha\omega^2-9X_0\,\omega^4+\xi A^3\alpha-7\xi A^3\omega^2-9A\beta\omega^2+A\alpha\beta\right)}{\alpha^2-10\alpha\omega^2+9\omega^4} \\ & -\frac{27A\left(cos(\omega t)\left((\xi A^2+\frac{4}{3}\beta)(\omega^2-\frac{1}{9}\alpha)\right)+cos(3\omega t)\left(\frac{1}{27}\xi A^2(\omega^2-\alpha)\right)\right)}{4\alpha^2-40\alpha\omega^2+36\omega^4} \end{aligned} \tag{389}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$\begin{aligned} x(t) &= u(t) \\ x(t) &= u(t) + A\cos(\omega\, t) \end{aligned} \tag{390}$$

Table 12 Comparison of frequency corresponding to various parameters of system

Constant parameters						Approximate Solution	Exact Solution	Relative error %
m	k_1	k_2	k_3	X_0	Y_0	ω_{MMA}	$\omega_{\ Exact}$[43]	$\frac{\omega_{MMA}-\omega_{Ex}}{\omega_{Ex}}$
1	0.5	0.5	0.5	1	5	3.674235	3.611743	1.730234
1	1	1	2	5	1	7.141428	7.004694	1.952045
5	2	0.5	5	5	10	6.17252	6.042804	2.146618
10	5	5	10	10	20	12.30853	12.04665	2.173874
20	40	50	50	20	10	19.54482	19.13632	2.134672
50	100	50	100	-10	20	52.00000	50.87391	2.213492

Example 3

Table 12 gives the comparison of obtained results with exact ones are tabulated in Table 12 for different value of m, k_1, k_2, k_3 and initial conditions. Comparisons of results for different parameters via numerical and MMA are presented in Figures 40 to 43. From figures 40 and 41, it is obvious that the motion of the system is periodic. Figures 42 and 43 represent comparison

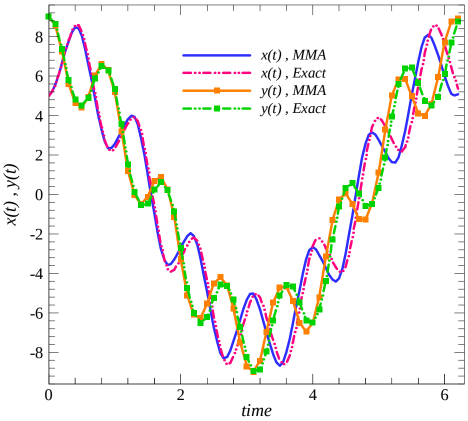

Figure 40 Comparison of analytical solution of displacement $x(t)$and $y(t)$based on time t with the exact solution [43]for $m = 1, \quad k_1 = 1, \quad k_2 = 1\,, \; k_3 = 2, \;\; X_0 = 5, \quad Y_0 = 1$

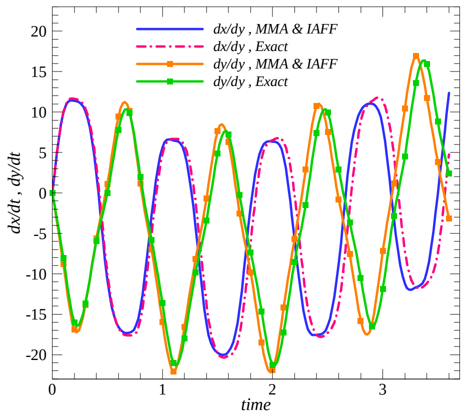

Figure 41 Comparison of analytical solution of dx/dt and dy/dt based on time t with the exact solution [43]for $m = 1, \quad k_1 = 1, \quad k_2 = 1\,, \; k_3 = 2, \;\; X_0 = 5, \quad Y_0 = 1$

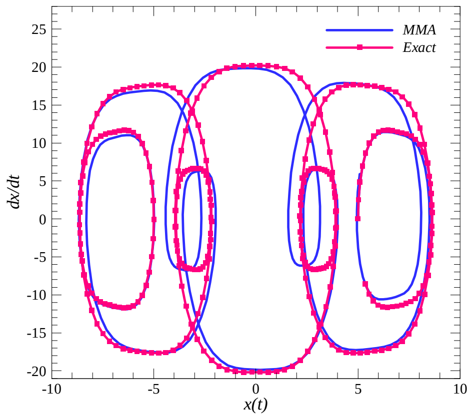

Figure 42 Comparison of analytical solution of dx/dt based on $x(t)$ with the exact solution [43]for $m = 1, \quad k_1 = 1, \quad k_2 = 1\,, \; k_3 = 2, \;\; X_0 = 5, \quad Y_0 = 1$

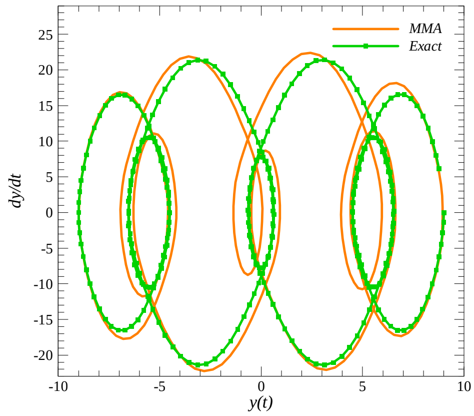

Figure 43 Comparison of analytical solution of dy/dt based on $y(t)$ with the exact solution [43]for $m=1$, $k_1=1$, $k_2=1$, $k_3=2$, $X_0=5$, $Y_0=1$

of analytical solution of dx/dt and dy/dt based on time with the numerical solution for different parameters of the system.

We consider geometrically non-linear Tapered beams.In dimensionless form, Goorman is given the governing differential equation corresponding to fundamental vibration mode of a tapered beam [78]:

$$\left(\frac{d^2u}{dt^2}\right)+\varepsilon_1\left(u^2\left(\frac{d^2u}{dt^2}\right)+u\left(\frac{du}{dt}\right)^2\right)+u+\varepsilon_2u^3=0 \tag{391}$$

Where u is displacement and ε_1 and ε_2 are arbitrary constants. Subject to the following initial conditions:

$$u(0)=A, \qquad \frac{du(0)}{dt}=0 \tag{392}$$

We can re-write Eq. (391) in the following form

$$\left(\frac{d^2u}{dt^2}\right)+\left(\frac{1+\varepsilon_1\left(\frac{du}{dt}\right)^2+\varepsilon_2u^2}{1+\varepsilon_1u^2}\right)u=0 \tag{393}$$

We choose a trial-function in the form

$$u=A\cos(\omega t) \tag{394}$$

Where ω the frequency to be is determined.

By using the trial-function, the maximum and minimum values of ω^2 will be:

$$\begin{aligned}\omega_{\min}&=\tfrac{1+\varepsilon_1A^2\omega^2}{1},\\ \omega_{\max}&=\tfrac{1+\varepsilon_2A^2}{1+\varepsilon_1A^2}.\end{aligned} \tag{395}$$

So we can write:

$$\frac{1+\varepsilon_1 A^2\omega^2}{1} < \omega^2 < \frac{1+\varepsilon_2 A^2}{1+\varepsilon_1 A^2} \tag{396}$$

According to the Chengtian's inequality , we have

$$\omega^2 = \frac{m.\left(1+\varepsilon_1 A^2\omega^2+\varepsilon_2 A^2\right)+n.\left(1+\varepsilon_1 A^2\omega^2\right)}{m+n} = 1+\varepsilon_1 A^2\omega^2 + k\ \ \varepsilon_2 A^2 \tag{397}$$

Where m and n are weighting factors, $k = n/m+n$. Therefore the frequency can be approximated as:

$$\omega = \sqrt{\frac{1+k\ \ \varepsilon_2 A^2}{1-\varepsilon_1 A^2}} \tag{398}$$

Its approximate solution reads

$$u = A\cos\sqrt{\frac{1+k\ \ \varepsilon_2 A^2}{1-\varepsilon_1 A^2}}\,t \tag{399}$$

In view of the approximate solution, Eq. (393), we re-write Eq.(393) in the form

$$\frac{d^2u}{dt^2} + \left(\frac{1+k\ \ \varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)u = \left(\frac{d^2u}{dt^2}\right) + \varepsilon_1\left(u^2\left(\frac{d^2u}{dt^2}\right) + u\left(\frac{du}{dt}\right)^2\right) + u + \varepsilon_2 u^3 + \Psi \tag{400}$$

$$\Psi = \left(\frac{1+k\ \ \varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)u - \varepsilon_1 u^2\left(\frac{d^2u}{dt^2}\right) - \varepsilon_1 u\left(\frac{du}{dt}\right)^2 - u - \varepsilon_2 u^3 \tag{401}$$

Substituting the trial function into Eq. (401), and using Fourier expansion series, it is obvious that:

$$\begin{aligned}\Psi &= \left(\tfrac{1+k\ \ \varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right)(A\cos\omega t) - \left(2\omega^2\varepsilon_1 A^2 cos^2(\omega t) - \varepsilon_1 A^2\omega^2 - 1 - \varepsilon_2 A^2 cos^2(\omega t)\right)A\,cos(\omega t)\\ &= \textstyle\sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\,\omega t\right] = b_1\cos(\omega t) + b_3\cos(3\omega t) + \ldots \approx b_1\cos(\omega t)\end{aligned} \tag{402}$$

For avoiding secular term we set $b_1 = 0$

$$\int_0^{T/4}\left(\left(\frac{1+k\ \ \varepsilon_2 A^2}{1-\varepsilon_1 A^2}\right) - \left(2\omega^2\varepsilon_1 A^2 cos^2(\omega t) - \varepsilon_1 A^2\omega^2 - 1 - \varepsilon_2 A^2 cos^2(\omega t)\right)\right)A\,cos(\omega t)\,dt = 0 \tag{403}$$

Where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = -\frac{\left(\varepsilon_1\omega^2 - \varepsilon_1^2 A^2\omega^2 + 3\varepsilon_1 - 2\varepsilon_2 + 2\varepsilon_2 A^2\varepsilon_1\right)}{3\varepsilon_2} \tag{404}$$

Substituting Eq. (404) into Eq. (398), yields

$$\omega = \frac{\sqrt{(3+\varepsilon_1 A^2)(2\varepsilon_2 A^2+3)}}{(3+\varepsilon_1 A^2)} \tag{405}$$

According to Eqs. (405) and (394), we can obtain the following approximate solution:

$$u(t) = A\cos\left(\frac{\sqrt{(3+\varepsilon_1 A^2)(2\varepsilon_2 A^2+3)}}{(3+\varepsilon_1 A^2)}\,t\right) \tag{406}$$

The exact frequency ω_e for a dynamic system governed by Eq. (391) can be derived, as shown in Eq. (407), as follows:

$$\omega_{Exact} = 2\pi \Big/ 4\sqrt{2}\,A \int_0^{\pi/2} \frac{\sqrt{1+\varepsilon_1 A^2 cos^2 t}\; sin\, t}{\sqrt{A^2\,(1-cos^2 t)\,(\varepsilon_2 A^2 cos^2 t+\varepsilon_2 A^2+2)}}\,dt \tag{407}$$

To demonstrate the accuracy of the MMA, the procedures explained in previous sections are applied to obtain natural frequency and corresponding displacement of tapered beams. A comparison of obtained results from the Max-Min Approach and the exact one is tabulated in table 13 for different parameters A,ε_1 and ε_2.

Table 13 Comparison of frequency corresponding to various parameters of system

Constant parameters			Approximate solution	Exact solution	Relative error %
A	ε_1	ε_2	ω_{MMA}	ω_{Exact}	$\frac{\omega_{MMA}-\omega_{Ex}}{\omega_{Ex}}$
2	0.1	0.5	1.43486	1.44100	0.42665
2	0.5	1	1.48323	1.44506	2.64192
2	5	10	1.8996	1.85323	2.50516
2	10	50	3.06138	3.0103	1.69512
10	0.1	0.5	2.81479	2.73523	2.90861
10	0.5	1	1.95708	1.92710	1.55604
10	5	10	1.99552	1.98950	0. 1842
10	10	50	3.15801	3.15265	0.17001

Figs. 44 and 45 represent the high accuracy of the MMA with the exact one for $\varepsilon_1 = 0.1\,\varepsilon_2 = 0.5$ and $\varepsilon_1 = 0.5\,\varepsilon_2 = 0.1$. The effect of small parameters ε_2 and ε_1 on the frequency corresponding to various parameters of amplitude (A) has been studied in Figs. 46 and 47.It is evident that MMA shows excellent agreement with the numerical solution using the exact solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions.

11 HAMILTONIAN APPROACH (HA)

Investigate of nonlinear problems which are arisen in many areas of physics and engineering, especially some oscillation equations are nonlinear, and in most cases it is difficult to solve such equations, especially analytically. Previously, He had introduced the Energy Balance method based on collocation and the Hamiltonian. This approach is very simple but strongly depends

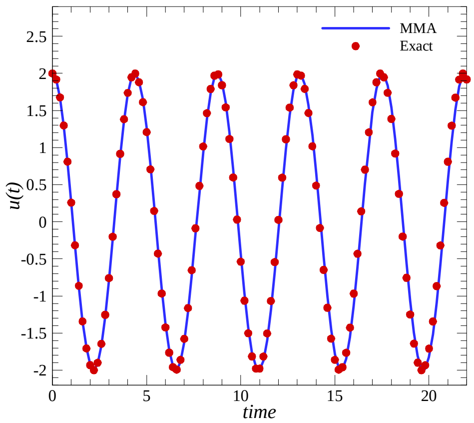

Figure 44 Comparison of analytical solutions of $u(t)$ based on t with the exact solution for $\varepsilon_1 = 0.1\,\varepsilon_2 = 0.5\,A = 2$

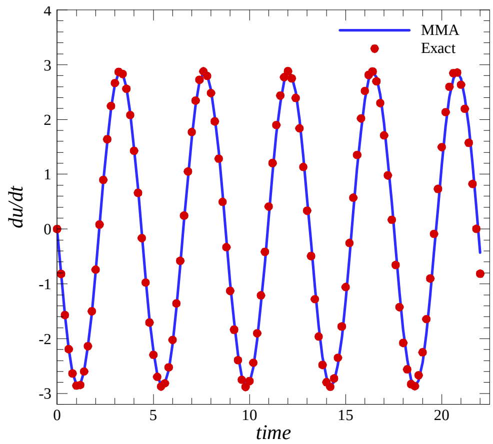

Figure 45 Comparison of analytical solutions of du/dt based on time with the exact solution for $\varepsilon_1 = 0.5\,,\quad \varepsilon_2 = 0.1\,,\,A = 2,$

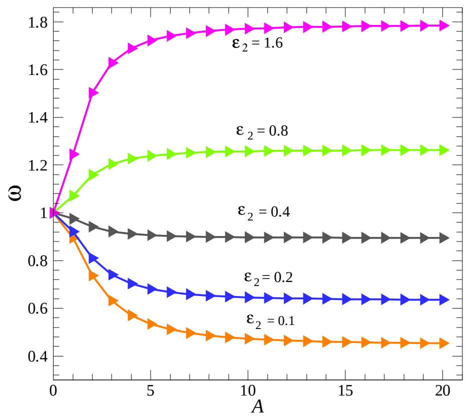

Figure 46 Comparison of frequency corresponding to various parameters of amplitude (A) and $\varepsilon_1 = 1$

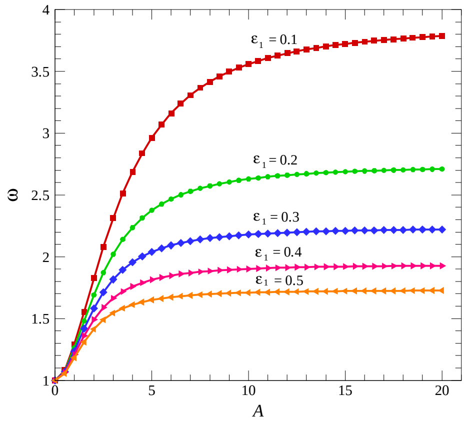

Figure 47 Comparison of frequency corresponding tovarious parameters of amplitude (A) and $\varepsilon_2 = 1$

upon the chosen location point. Recently, He [111]has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. Application of this method can be found in many literatures [124, 140, 198, 199, 203–205].

11.1 Basic idea of Hamiltonian Approach

In order to clarify this approach, consider the following general oscillator;

$$\ddot{u} + f(u, \dot{u}, \ddot{u}) = 0 \tag{408}$$

With initial conditions:

$$u(0) = A, \ , \dot{u}(0) = 0. \tag{409}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (408), which reads;

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}\dot{u}^2 + F(u) \right\} dt \tag{410}$$

Where T is period of the nonlinear oscillator, $\partial F / \partial u = f$.

In the Eq (410), $\frac{1}{2}\dot{u}^2$ is kinetic energy and *F (u)* potential energy, so the Eq (410) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads ;

$$H(u) = \frac{1}{2}\dot{u}^2 + F(u) = \text{constant} \tag{411}$$

From Eq. (411), we have;

$$\frac{\partial H}{\partial A} = 0 \tag{412}$$

Introducing a new function, $\bar{H}(u)$, defined as;

$$\bar{H}(u) = \int_{0}^{T/4} \int \left\{ \frac{1}{2}\dot{u}^2 + F(u) \right\} dt = \frac{1}{4} T H \tag{413}$$

Eq. (412) is, then, equivalent to the following one;

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial T}\right) = 0 \tag{414}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)}\right) = 0 \tag{415}$$

From Eq.(415) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

11.2 Application of Hamiltonian Approach

We have considered three examples in this section to show the application of the proposed method.

Example 1

To illustrate the basic procedure of the present method, we consider an $u^{1/3}$ force nonlinear oscillator:

$$\ddot{u} + au + bu^3 + cu^{1/3} = 0, \qquad u(0) = A, \quad \dot{u}(0) = 0 \tag{416}$$

The Hamiltonian of Eq. (416) is constructed as:

$$H = \frac{1}{2}\dot{u}^2 + \frac{1}{2}au^2 + \frac{1}{4}bu^4 + \frac{3}{4}cu^{4/3} \tag{417}$$

Integrating Eq.(417) with respect to t from 0 to $T/4$, we have;

$$\bar{H} = \int_0^{T/4} \left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}au^2 + \frac{1}{4}bu^4 + \frac{3}{4}cu^{4/3}\right) dt \tag{418}$$

Assume that the solution can be expressed as:

$$u(t) = A\cos(\omega t) \tag{419}$$

Substituting Eq.(419) into Eq. (418), we obtain:

$$\begin{aligned}\bar{H} &= \int_0^{T/4}\left(\tfrac{1}{2}A^2\omega^2 sin^2\left(\omega t\right)+\tfrac{1}{2}a\,A^2cos^2\left(\omega t\right)+\tfrac{1}{4}b\,A^4cos^4\left(\omega t\right)+\tfrac{3}{4}cA^{4/3}cos^{4/3}\left(\omega t\right)\right)dt\\ &= \int_0^{\pi/2}\left(\tfrac{1}{2}A^2\omega\, sin^2 t+\tfrac{1}{2\omega}a\,A^2cos^2 t+\tfrac{1}{4\omega}bA^4cos^4 t+\tfrac{3}{4\omega}cA^{4/3}cos^{4/3}t\right)dt\\ &= \tfrac{1}{8}\omega A^2\pi+\tfrac{1}{8}a\,A^2\tfrac{\pi}{\omega}+\tfrac{3}{64}bA^4\tfrac{\pi}{\omega}+0.12267\,c\,A^{4/3}\tfrac{\pi^{3/2}}{\omega}\end{aligned} \tag{420}$$

Setting:

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial\left(1/\omega\right)}\right)=-\frac{1}{4}\omega^2 A\,\pi+\frac{1}{4}a\,A\,\pi+\frac{3}{64}bA^4\pi+0.16356\,\mathrm{c}\,A^{1/3}\pi^{3/2} \tag{421}$$

Solving the above equation, an approximate frequency as a function of amplitude equals;

$$\omega_{HA}=\sqrt{a+\frac{3}{4}A^2b+\frac{0.654236\,c\sqrt{\pi}}{A^{2/3}}} \tag{422}$$

Hence, the approximate solution can be readily obtained;

$$u(t)=A\cos\left(\sqrt{a+\frac{3}{4}A^2b+\frac{0.654236\,c\sqrt{\pi}}{A^{2/3}}}\;\;t\right) \tag{423}$$

The same result was obtained by He [107].

Example 2

Considering the governing equation of motion for the Duffing-harmonic oscillator:

$$\ddot{u}+\frac{u^3}{1+u^2}=0,\qquad u\left(0\right)=A,\;\;\dot{u}\left(0\right)=0 \tag{424}$$

The Hamiltonian of Eq. (424) is constructed as:

$$H=\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2-\frac{1}{2}\log\left(1+u^2\right) \tag{425}$$

Integrating Eq.(425) with respect to t from 0 to $T/4$, we have;

$$\bar{H}=\int_0^{T/4}\left(\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2-\frac{1}{2}\log\left(1+u^2\right)\right)dt \tag{426}$$

Assume that the solution can be expressed as:

$$u(t)=A\cos(\omega\,t) \tag{427}$$

Substituting Eq.(427) into Eq. (426), we obtain:

$$\begin{aligned}\bar{H} &= \int_0^{T/4}\left(\tfrac{1}{2}A^2\omega^2 sin^2\left(\omega t\right)+\tfrac{1}{2}\,A^2cos^2\left(\omega t\right)-\tfrac{1}{2}\log\left(1+A^2\cos^2\left(\omega t\right)\right)\right)dt\\ &= \int_0^{\pi/2}\left(\tfrac{1}{2}A^2\omega sin^2 t+\tfrac{1}{2\omega}\,A^2cos^2 t-\tfrac{1}{2\omega}\log\left(1+A^2\cos^2 t\right)\right)dt\end{aligned} \tag{428}$$

Setting:

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial\left(1/\omega\right)}\right) = 0 \tag{429}$$

Solving the above equation, an approximate frequency as a function of amplitude equals;

$$\omega_{HA} = \sqrt{\frac{\int_0^{\pi/2}\left\{\frac{\cos^2 t}{1+A^2\cos^2 t}\right\}dt}{\int_0^{\pi/2}\sin^2 t\,dt}} \tag{430}$$

The exact frequency is given by[132]:

$$\omega_{Ex} = \frac{2\pi}{4\int_0^A \frac{du}{\sqrt{[\log(A^2+1)-\log(u^2+1)]}}} \tag{431}$$

Table 14 Comparison of frequency Hamiltonian approach and exact solution

A	ω_{ex}	ω_{HA}	Relative error (%)
0.01	0.00847	0.00865	2.12515
0.1	0.08439	0.08624	2.192203
1	0.63678	0.64359	1.06944
10	0.99092	0.99095	0.00303
100	0.9999	0.9999	0.0001

From Table 14, the maximum relative error is 2.192203%.

Example 3

The Hamiltonian of Eq. (391) is constructed as;

$$H = \frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}\varepsilon_1\left(\frac{du}{dt}\right)^2 u^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon_2 u^4 \tag{432}$$

Integrating Eq. (432) with respect to t from 0 to $T/4$, we have;

$$\bar{H} = \int_0^{T/4}\left(\frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}\varepsilon_1\left(\frac{du}{dt}\right)^2 u^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon_2 u^4\right)dt \tag{433}$$

Assume that the solution can be expressed as;

$$u(t) = A\cos(\omega t) \tag{434}$$

Substituting Eq. (434) into Eq. (433), we obtain;

$$\begin{aligned}\bar{H} &= \int_0^{T/4}\left(\tfrac{1}{2}\mathrm{A}^2\omega^2\sin^2(\omega \mathrm{t}) + \tfrac{1}{2}\varepsilon_1\,\mathrm{A}^4\omega^2\sin^2(\omega \mathrm{t})\cos^2(\omega t) + \tfrac{1}{2}\mathrm{A}^2\cos^2(\omega t) + \tfrac{1}{4}\varepsilon_2\mathrm{A}^4\cos^4(\omega t)\right)dt \\ &= \int_0^{\pi/2}\left(\tfrac{1}{2}\mathrm{A}^2\omega\sin^2 t + \tfrac{1}{2}\varepsilon_1\,\mathrm{A}^4\omega\sin^2 t\,\cos^2 t + \tfrac{1}{2\omega}\mathrm{A}^2\cos^2 t + \tfrac{1}{4\omega}\varepsilon_2\mathrm{A}^4\cos^4 t\right)dt \\ &= \tfrac{1}{8}\omega A^2\pi + \tfrac{1}{32}\omega A^4\varepsilon_1\pi\tfrac{1}{8\omega}A^2\pi + \tfrac{3}{64\omega}A^4\varepsilon_2\pi\end{aligned} \tag{435}$$

Setting:

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)}\right) = -\frac{1}{4}A\,\pi\,\omega^2 - \frac{1}{8}\varepsilon_1 A^3\,\pi\,\omega^2 + \frac{1}{4}A\,\pi + \frac{3}{16}\varepsilon_2\,A^3\,\pi \tag{436}$$

Solving the above equation, an approximate frequency as a function of amplitude equals;

$$\omega_{HA} = \frac{\sqrt{2}}{2}\frac{\sqrt{(\varepsilon_1\,A^2+2)\,(3\,\varepsilon_2\,A^2+4)}}{(\varepsilon_1\,A^2+2)} \tag{437}$$

Hence, the approximate solution can be readily obtained;

$$u(t) = A\cos\left(\frac{\sqrt{2}}{2}\frac{\sqrt{(2+\varepsilon_1\,A^2)\,(4+3\,\varepsilon_2\,A^2)}}{(2+\varepsilon_1\,A^2)}\;t\right) \tag{438}$$

Table 15 Comparison of frequency corresponding to various parameters of system

Constant parameters			Approximate solution	Exact solution	Relative error %
A	ε_1	ε_2	ω_{HA}	ω_{Exact}	$\frac{\omega_{EX}-\omega_{HA}}{\omega_{Ex}}$
0.1	0.1	0.1	1.0001	1.0005	0.0374
0.1	1	0.2	0.9983	0.9983	0.0002
0.5	0.5	1	1.0572	1.0573	0.0084
0.5	1	0.5	0.9860	0.9870	0.1018
1	1	1	1.0801	1.0904	0.9382
1	0.5	0.2	0.9592	0.9623	0.3262
2	0.4	0.2	0.9428	0.9593	1.7212
2	1	0.8	1.0646	1.0917	2.4853
2	1	0.2	0.7303	0.7504	2.6846

The maximum relative error of Hamiltonian approach 2.6846 % for different values of $A, \varepsilon_1, \varepsilon_2$ in comparison with the exact one.

12 HOMOTOPY ANALYSIS METHOD (HAM)

Homotopy analysis is a general analytic method for solving the non-linear differential equations. The HAM transforms a non-linear problem into an infinite number of linear problems with embedding an auxiliary parameter (q) that typically ranges from zero to one. As q increases from 0 to 1, the solution varies from the initial guess to the exact solution. By suitable choice of the auxiliary parameter (q), we can obtain reasonable solutions for large modulus. This method is a strong and easy-to-use analytic tool for investigating nonlinear problems, which does not need small parameters. In 1992, Liao employed the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [128]. This method has been successfully applied to solve many types of nonlinear problems by others [4, 6, 40, 41, 49, 51, 53, 114, 126, 129–131, 155–159, 172, 193, 194, 213]. The basic idea of HAM is introduced and then its application in nonlinear vibration is studied.

12.1 Basic idea of Homotopy Analysis Method

To illustrate the basic ideas of the HAM, consider the following non-linear differential equation:

$$N\left[u(t)\right] = 0, \tag{439}$$

Where N is a nonlinear operator, t denotes the independent variable and $u(t)$ is an unknown variable. The homotopy function is constructed as follows:

$$\bar{H}(\phi; q, \hbar, H(t)) = (1-q)L\left[\phi(t;q) - u_0(t)\right] - q\hbar H(t)N\left[\phi(t;q)\right] \tag{440}$$

where ϕ , $\hbar$ and $H(t)$ are a function of t and q , the non-zero auxiliary parameter, is a non-zero auxiliary function, respectively. The parameter L denotes an auxiliary linear operator. As q increases from 0 to 1, the $\phi(t;q)$ varies from the initial approximation to the exact solution. In the other words, $\phi(t;0) = u_0(t)$is the solution of the $\bar{H}(\phi, q, \hbar, H(t))\big|_{q=0} = 0$ and $\phi(t;1) = u_0(t)$ is the solution of the $\bar{H}(\phi, q, \hbar, H(t))\big|_{q=1} = 0$. Enforcing $\bar{H}(\phi, q, \hbar, H(t)) = 0$, the zero-order deformation is constructed as:

$$(1-q)L\left[\phi(t,q) - u_0(t)\right] = q\hbar H(t)N\left[\phi(t,q)\right], \tag{441}$$

with the following initial conditions:

$$\phi(0;q) = a\ , \quad \frac{d\phi(0,q)}{dt} = 0\,. \tag{442}$$

The functions $\phi(t,q)$ and $\omega(q)$can be expanded as power series of q using Taylor's theorem as;

$$\phi(t,q) = \phi(t,0) + \sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\phi(t;q)}{\partial q^m}\Big|_{q=0}\ q^m = u_0(\tau) + \sum_{m=1}^{\infty}u_m(t)q^m \tag{443}$$

$$\omega(q) = \omega_0 + \sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\omega(q)}{\partial q^m}\Big|_{q=0}\ q^m = \omega_0 + \sum_{m=1}^{\infty}\omega_m q^m \tag{444}$$

Where $u_m(t)$ and ω_m are called the m-order deformation derivations.

Differentiating zero-order deformation equation with respect to q and the setting $q = 0$,yields the first order deformation equation($m = 1$)which gives the first-order approximation of the $u(t)$as follows:

$$L\left[u_1(t)\right] = \hbar H(t)N\left[u_0(t), \omega_0\right]\big|_{q=0}\ , \tag{445}$$

with the following initial conditions:

$$u_1(0) = 0\ ,\ \dot{u}_1(0) = 0 \tag{446}$$

The higher order approximations of the solution can be obtained by calculating the m-order (m>1) deformation equation. The m-order deformation equation can be calculated by differentiating Eqs. (443) and (444) m times with respect to q as follows:

$$L\left[u_m(t) - u_{m-1}\right] = \hbar H(t) R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}), \tag{447}$$

Where the $\vec{u}_{m-1}, \vec{\omega}_{m-1}$and $R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1})$are defined as follows:

$$R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N\left[\phi(t,q)\right], \omega(q)}{\partial q^{m-1}}\bigg|_{q=0}, \tag{448}$$

$$\vec{u}_{m-1} = \{\vec{u}_0, \vec{u}_1, \vec{u}_2, ..., \vec{u}_{m-1}\} \tag{449}$$

$$\vec{\omega}_{m-1} = \{\omega_0, \omega_1, \omega_2, ..., \omega_{m-1}\} \tag{450}$$

Subject to the following initial conditions:

$$u_m(0) = \dot{u}_m(0) = 0. \tag{451}$$

12.2 Application of Homotopy Analysis Method

Example 1

Consider the following Duffing equation ;

$$\ddot{u} + \alpha u + \beta u^3 = 0 \quad u(0) = A \;, \dot{u}(0) = 0 \tag{452}$$

Under the transformation $\tau = \omega t$ and $W(\tau) = u(t)$ Eq. (452) becomes as follows:

$$\omega^2 \ddot{W} + \alpha W + \beta W^3 = 0 \tag{453}$$

The zero-order deformation equation can be written as below:

$$(1-q)\, L\left[\phi(\tau; q) - W_0(\tau)\right] = q h \hbar(\tau) N[\phi(\tau; q)] \tag{454}$$

In which;

$$N[\phi(\tau;q)] = \omega^2 \frac{\partial^2 \phi(\tau;q)}{\partial \tau^2} + \alpha\phi(\tau;q) + \beta\phi(\tau;q)^3 = 0 \tag{455}$$

We chose the following auxiliary linear operator as:

$$L[\phi(\tau;q)] = \omega_0^2 \left[\frac{\partial^2 \phi(\tau;q)}{\partial \tau^2} + \phi(\tau;q)\right] \tag{456}$$

We employ Taylor expansion series for $\phi(t;q)$ and $\omega(q)$as

$$\phi(\tau;q) = \phi(\tau;0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m} |_{q=0} \ q^m = W_0(\tau) + \sum_{m=1}^{\infty} W_m(\tau) q^m \tag{457}$$

$$\omega(q) = \omega_0 + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \omega(q)}{\partial q^m} |_{q=0} \ q^m = \omega_0 + \sum_{m=1}^{\infty} \omega_m q^m \tag{458}$$

In order to satisfy the initial conditions, the initial guess of $W(\tau)$ is chosen as follows:

$$\omega_0(\tau) = W_{\max} \cos(\tau) \tag{459}$$

In our case, to obtain the first-order approximation, the function of $W_1(\tau)$ can be expressed as

$$L[W_1(t)] = h\hbar(t) N[\phi(t;q)] |_{q=0} \tag{460}$$

$$W_1(0) = 0 \ , \ \frac{dW_1(0)}{dt} = 0 \tag{461}$$

Assuming $\hbar_1 = -1$, $h(t) = 1$and after substituting Eq. (459) in Eq. (460), one would get:

$$\omega_0^2(\ddot{W}_1 + W_1) = W_{\max} \cos(\tau)(\omega_0^2 - \alpha - \frac{3}{4}\beta W_{\max}^2) - \frac{\beta W_{\max}^3}{4} \cos(3\tau) \tag{462}$$

$$W_1(0) = 0 \ , \ \dot{W}_1(0) = 0 \tag{463}$$

Eliminating the secular term, we have:

$$\omega_0 = \sqrt{\alpha + \frac{3}{4}\beta W_{\max}^2} \tag{464}$$

The same result was obtained in the first example of section 2.
Solving Eqs. (462) and (463), the $W_1(\tau)$ is obtained as follows:

$$W_1(\tau) = -\frac{1}{32\omega_0^2} \beta W_{\max}^3 (\cos(\tau) - \cos(3\tau)) \tag{465}$$

Thus the first-order approximation of the $W(\tau)$yields to:

$$W(\tau) = W_0(\tau) + W_1(\tau) \tag{466}$$

In which:

$$\tau = \omega t \ , \ \omega = \omega_0 \tag{467}$$

13 CONCLUSIONS

It has reviewed new asymptotic methodologies throughout numerous examples. The analytical solutions yield a thoughtful and insightful understanding of the effect of system parameters and initial conditions. Also, Analytical solutions give a reference frame for the verification and validation of other numerical approaches.

Variational Iteration Method (VIM),Homotopy Perturbation Method (HPM), Energy Balance Method (EBM),Parameter-Expansion Method (PEM) ,Variational Approach (VA),Improved Amplitude Frequency Formulation (IAFF),Max-Min Approach (MMA),Hamiltonian Approach (HA) and Homotopy Analysis Method (HAM) are suitable not only for weak nonlinear problems, but also for strong nonlinear problems as it is indicated in this review. The most significant feature of those methods is their excellent accuracy for the whole range of oscillation amplitude values. Also, it can be used to solve other conservative truly nonlinear oscillators with complex nonlinearities. The solutions are quickly convergent and its components can be simply calculated. Also, compared to other analytical methods, it can be observed that the results of those methods require smaller computational effort and only the one iteration leads to accurate solutions. The successful implementations of the mentioned methods for the large amplitude nonlinear oscillation problem were considered in this review. All reviewed methods can be applied to various kinds of weak and strong nonlinear problems, and the examples studied in this review can be utilized as paradigms for oscillator problems. Through nonlinear oscillators, all the reviewed methods yield high accurate approximate periods which indicated above.

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