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Thermomechanical Buckling of Temperature-dependent FGM Beams

Abstract
Buckling of beams made of functionally graded materials (FGM) under thermomechanical loading is analyzed herein. Properties of the constituents are considered to be functions of temperature and thickness coordinate. The derivation of the equations is based on the Timoshenko beam theory, where the effect of shear is included. It is assumed that the mechanical and thermal nonhomogeneous properties of beam vary smoothly by distribution of the power law index across the thickness of the beam. The equilibrium and stability equations for an FGM beam are derived and the existence of bifurcation buckling is examined. The beam is assumed under three types of thermal loadings; namely, the uniform temperature rise, heat conduction across the thickness, and linear distribution across the thickness. Various types of boundary conditions are assumed for the beam with combination of roller, clamped, and simply-supported edges. In each case of boundary conditions and loading, closed form solutions for the critical buckling temperature of the beam is presented. The results are compared with the isotropic homogeneous beams, that are reported in the literature, by reducing the results of the functionally graded beam to the isotropic homogeneous beam.

Keywords
Buckling, Timoshenko beam theory, Functionally graded material, Temperature Dependency

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1 INTRODUCTION

Functionally graded materials, as a branch of new materials, have attracted increasing attention in recent years. A survey in the literature reveals the existence of wealth investigations on analysis of functionally graded material beams. Among them, Kang and Lee [1] presented explicit expressions for deflection and rotation of an FGM cantilever beam subjected to an end moment. Considering the large deflection of the beam, they reported that an FGM beam can bear larger applied load than a homogeneous beam. Free vibration analysis of simply-supported functionally graded material beams is reported by Aydogdu and Taskin [2]. They used both exponential and power law form of material properties distribution to derive the governing equations. Their study includes four types of displacement fields namely, the classical beam theory, the first order theory, and the parabolic and exponential shear deformation beam theories. They concluded that, in comparison with the classical beam theory, other three types of displacement fields accurately predict the natural frequencies. Nirvana et al. [3] obtained analytical expressions for thermo-elastic analysis of three layered beams, when the middle layer is made of FGMs. A unified method to study the dynamic and static analysis of FGM Timoshenko beams is reported by Li [4]. He derived a fourth order differential equation and linked the other parameters of the beam to the solution of fourth order differential equation. His study includes the simply supported and cantilever beams. The static and free vibration analysis of layered FGM beams based on a third order shear deformation beam theory is developed by Kapuria et al. [5]. A two nodes finite element method is adopted to solve the coupled ordinary differential equations.

The mechanical and thermal buckling of beams, as a major solid structural component, have been the topic of many researches for a long period of time. Development of the new materials, such as the functionally graded materials, have necessitated more research in this area. Huang and Li [6] obtained an exact solution for mechanical buckling of FGM columns subjected to uniform axial loading based on various beam theories. Zhao et al. [7] studied the post-buckling of simply supported rod made of functionally graded materials under uniform thermal loading and nonlinear temperature distribution across the beam thickness using the numerical shooting method. They found that, under the same temperature condition, the deformation of immovably simply supported FGM rod is smaller than those of the two homogenous material rods. Also, end constrained force of FGM rod is smaller than the corresponding values of the two homogenous material rods with the small deformation. Accordingly, the stability of FGM rod is higher than those of the two homogenous material rods when there is a temperature difference. Li et al. [8] presented the post-buckling behavior of fixed-fixed FGM beams based on the Timoshenko beam theory under nonlinear temperature loading. They found the effect of shear on the critical buckling temperature of beams and used the shooting method to predict the post-buckling behavior of beams. It was found that the non-dimensional thermal axial force increases along with increase of the power law index, as the increment of metal constituent can produce more thermal expansion of beam under the same value of thermal load. Kiani and Eslami [9] discussed the buckling of functionally graded material beams under three types of thermal loading through the thickness. They examined the existence of bifurcation type buckling for various edge supports and presented their results in closed-form expressions. A semi inverse method to study the instability and vibration of axially FGM beams is carried out by Aydogdu [10]. Ke et al. [11] presented the post-buckling of a cracked beam for hinged-hinged and clamped-hinged edge conditions based on the Timoshenko beam theory. Also, Ke et al. [12] presented the free vibration and mechanical buckling of cracked beams using the first order shear deformation beam theory for three types of boundary conditions. They found that FGM beams with a smaller slenderness ratio and a lower Young's modulus ratio are much more sensitive to the edge crack. Ma and Lee [13] discussed the nonlinear behavior of FGM beams under in-plane thermal loading by means of first order shear deformation theory of beams. The derivation of the equations is based on the concept of neutral surface, where the numerical shooting method is used to solve the coupled nonlinear equations. Their study concluded that when a clamped-clamped FGM beam is subjected to uniform thermal loading, it follows the bifurcation-type buckling while the simply-supported beams do not. This feature of FGM beams, however, is ignored in some of the published works through the literature [9, 14, 15].

Kiani et al. [16,17] studied the effect of applied actuator voltage on the critical buckling temperature difference of FGM beams. It is reported that the effect of applied actuator smart layers is somehow negligible on thermal buckling control. In an analytical study, Ma and Wang [18] analyzed the nonlinear response of FGM beams with shear deformation effects and obtained the exact closed-form solutions for equilibrium path of the beam. This analytical study also proves the importance of the boundary conditions on the beam equilibrium path. Fu and his co-authors [19] obtained closed-form solutions for the free vibration of thermally loaded and thermal equilibrium path of a thin FGM beam with both edges clamped. In this work the temperature dependency of the constituents is also taken into account. A single term Ritz solution along with a finite element formulation is developed by Anadrao et al. [20]. In this work the cases of a beam with both edges clamped and both edges simply-supported are analyzed.

The present work deals with the buckling analysis of FGM beams subjected to thermal or mechanical loadings. Various types of boundary conditions are assumed and the existence of bifurcation type buckling in each case is examined. Based on the concept of virtual displacements principle, three coupled differential equations are obtained as the equilibrium equations. In thermal buckling analysis, the beam is under three types of thermal loading distinctly, and closed-form solutions are obtained to evaluate the critical buckling temperatures/loads .

2 FUNCTIONALLY GRADED TIMOSHENKO BEAMS

Consider a beam of functionally graded material, where the graded properties are assumed to be through the thickness direction. The volume fractions of the constituent materials, which are assumed to ceramic of volume V_c and metal of volume V_m, may be expressed using the power law distribution as [21]

$$V_c + V_m = 1, \quad V_c = \left(\frac{1}{2} + \frac{z}{h} \right)^k \tag{1}$$

where h is the thickness of the beam and z is the thickness coordinate measured from the middle surface of the beam $-h/2 \le z \le h/2$, k is the power law index which has the value equal or greater than zero. Variation of V_c with k and z/h is shown in Figure 1.

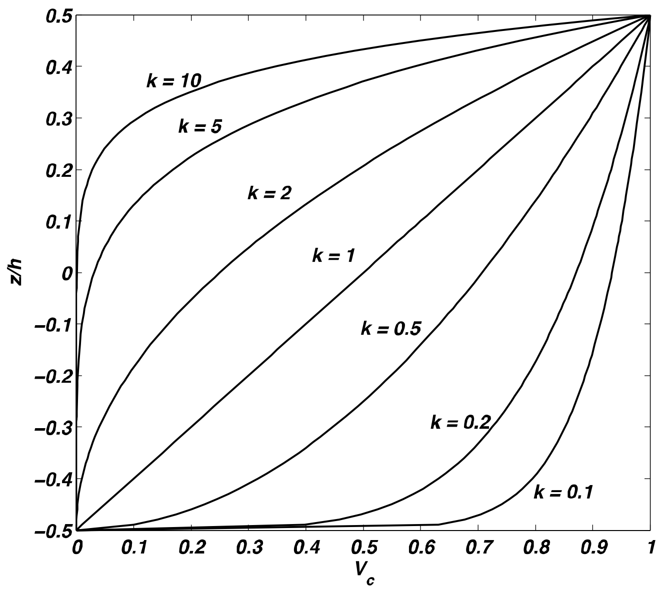

Figure 1 Distribution of ceramic volume fraction through the thickness for various power law indices

The value of k equal to zero represents a fully ceramic beam $\left(V_c = 1\right)$ and k equal to infinity represents a fully metallic beam $\left(V_c = 0\right)$. We assume that the mechanical and thermal properties of the FGM beam are distributed based on Voigt's rule [22]. Thus, the property variation of a functionally graded material using Eq. (1) is given by

$$P(z) = P_m + P_{cm}\left(\frac{1}{2} + \frac{z}{h}\right)^k \tag{2}$$

where $P_{cm} = P_c - P_m$, P_c and P_m are the corresponding properties of the metal and ceramic, respectively. In this analysis the material properties, such as Young's modulus $E(z)$, coefficient of thermal expansion $\alpha(z)$ and thermal conductivity $K(z)$ may be expressed by Eq. (2), whereas Poisson's ratio ν is considered to be constant across the thickness [21].

3 GOVERNING EQUATIONS

Consider a beam made of FGMs with rectangular cross section. It is assumed that the length of the beam is L, width is b, and the height is h. Rectangular Cartesian coordinates is used such that the x axis is at the left side of the beam on its middle surface and z is measured from the middle surface and is positive upward, as shown in Figure 2.

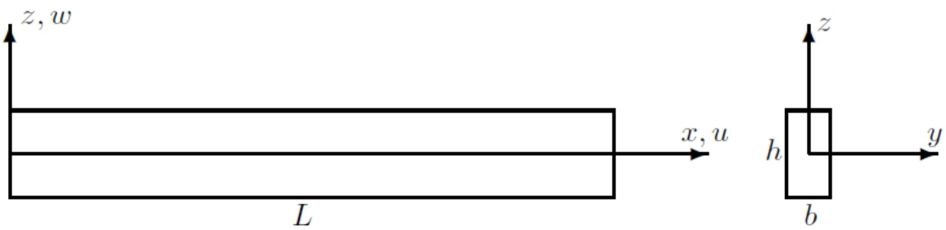

Figure 2 geometry and coordinate system of an FGM beam

The analysis of beam is based on the first order shear deformation beam theory using the Timoshenko assumptions. According to this theory, the displacement field of the beam is assumed to be [4]

$$\bar{u}(x,z) = u + z\varphi$$
$$\bar{w}(x,z) = w \tag{3}$$

where $\bar{u}(x,z)$ and $\bar{w}(x,z)$ are displacements of an arbitrary point of the beam along the x and z-directions, respectively. Here, u and w are the displacement components of middle surface and φ is the rotation of the beam cross-section, which are functions of x only. The strain-displacement relations for the beam are given in the form [11]

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial x}\right)^2$$
$$\gamma_{xz} = \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \tag{4}$$

where ε_{xx} and γ_{xz} are the axial and shear strains. Substituting Eq. (3) into Eq. (4) gives

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\varphi}{dx}$$
$$\gamma_{xz} = \varphi + \frac{dw}{dx} \tag{5}$$

The constitutive law for the material, using the linear thermo-elasticity, is given by [8]

$$\sigma_{xx} = E\left(\varepsilon_{xx} - \alpha\left(T - T_0\right)\right)$$
$$\sigma_{xz} = \frac{E}{2\left(1+\nu\right)}\gamma_{xz} \tag{6}$$

In Eqs. (6), σ_{xx} and σ_{xz} are the axial and shear stresses, T_0 is the reference temperature, and T is the temperature distribution through the beam. Eqs. (5) and (6) are combined to give the axial and shear stresses in the beam in terms of the middle surface displacements as

$$\sigma_{xx} = E\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\varphi}{dx} - \alpha\left(T - T_0\right)\right)$$
$$\sigma_{xz} = \frac{E}{2\left(1+\nu\right)}\left(\varphi + \frac{dw}{dx}\right) \tag{7}$$

The stress resultants of the beam expressed in terms of the stresses through the thickness, according to the Timoshenko beam theory, are [8]

$$N_x = \int_{-h/2}^{h/2}\sigma_{xx}dz$$
$$M_x = \int_{-h/2}^{h/2}z\sigma_{xx}dz \tag{8}$$
$$Q_{xz} = K_s\int_{-h/2}^{h/2}\sigma_{xz}dz$$

where K_s is the shear correction factor. The values of 5/6 or $\pi^2 / 12$ are used as its approximate value for the composite and FGM beams with rectangular cross section [12]. The shear correction factor is taken as $K_s = \pi^2 / 12$ for the FGM beam in this study.

Using Eqs. (2), (7), and (8) and noting that u, w, and φ are functions of x only, the expressions for N_x, M_x, and Q_{xz} are obtained as

$$N_x = E_1\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) + E_2\frac{d\varphi}{dx} - N_x^T$$
$$M_x = E_2\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) + E_3\frac{d\varphi}{dx} - M_x^T \tag{9}$$

$$Q_{xz} = \frac{E_1 K_s}{2(1+\nu)}\left(\varphi + \frac{dw}{dz}\right)$$

where E_1, E_2, and E_3 are stretching, coupling stretching-bending, and bending stiffnesses, respectively, and N_x^T and M_x^T are thermal force and thermal moment resultants, which are calculated using the following relations

$$E_1 = \int_{-h/2}^{h/2} E(z)dz = h\left(E_m + \frac{E_{cm}}{k+1}\right)$$

$$E_2 = \int_{-h/2}^{h/2} zE(z)dz = h^2 E_{cm}\left(\frac{1}{k+2} - \frac{1}{2k+2}\right)$$

$$E_3 = \int_{-h/2}^{h/2} z^2 E(z)dz = h^3\left(\frac{1}{12}E_m + E_{cm}\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4k+4}\right)\right) \tag{10}$$

$$N_x^T = \int_{-h/2}^{h/2} E(z)\alpha(z)\left(T - T_0\right)dz$$

$$M_x^T = \int_{-h/2}^{h/2} zE(z)\alpha(z)\left(T - T_0\right)dz$$

Note that to find the thermal force and moment resultants, the temperature distribution through the beam should be known.

The equilibrium equations of an FGM beam may be obtained through the static version of virtual displacement principle. According to this principle, since the external load is absent, an equilibrium position occurs when the first variation of strain energy function vanishes. Thus, one may write

$$\delta U = \int_0^L \int_{-b/2}^{+b/2} \int_{-h/2}^{h/2} \left(\sigma_{xx}\delta\varepsilon_{xx} + K_s \sigma_{xz}\delta\gamma_{xz}\right) dzdydx = 0 \tag{11}$$

With the aid of the stress resultant definition (9), and performing the integration by part technique to relieve the displacement gradients, the following system of equilibrium equations is obtained

$$\frac{dN_x}{dx} = 0$$
$$\frac{dM_x}{dx} - Q_{xz} = 0 \qquad (12)$$
$$\frac{dQ_{xz}}{dx} + N_x \frac{d^2w}{dx^2} = 0$$

and the boundary conditions for each side of the beam are

$$N_x = 0 \qquad or \qquad \delta u = 0$$
$$M_x = 0 \qquad or \qquad \delta \varphi = 0 \qquad (14)$$
$$Q_{xz} + N_x \frac{dw}{dx} = 0 \qquad or \qquad \delta w = 0$$

4 EXISTENCE OF BIFURCATION TYPE BUCKLING

4.1 Thermal Loading

Consider a beam made of FGMs subjected to a transversely temperature distribution. When the axial deformation is prevented in the beam, an applied thermal loading may produce an axial load. Only perfectly flat pre-buckling configurations are considered in the present work, which lead to bifurcation type buckling, otherwise beam undergoes a unique and stable equilibrium path. Now, based on Eq. (9), in the pre-buckling state, when beam is completely undeformed, and both edges are immovable, the generated pre-buckling force through the beam is equal to

$$N_{x0} = -N_x^T \qquad (14)$$

Here a subscript 0 is adopted to indicate the pre-buckling state deformation. Also, according to Eq. (9), an extra moment is produced through the beam which is equal to

$$M_{x0} = -M_x^T \qquad (15)$$

In general, this extra moment may result in deformation through the beam, except when it is vanished for some especial types of thermal loading or when boundary conditions are capable of handling the extra moments. The clamped and roller (sliding support) boundary conditions are capable of supplying the extra moments on the boundaries, while the simply-supported edge does not. Therefore, the $C-C$ and $C-R$ FGM Timoshenko beams remain un-deformed prior to buckling, while for the other types of beams with at least one simply supported edge beam commence to deflect. Also, an isotropic homogeneous beam remains un-deformed when it is subjected to uniform temperature rise, because thermal moment vanishes through the beam. Therefore, bifurcation type buckling exists for $C-C$ and $C-R$ FGM beams subjected to arbitrary transverse thermal loading. The same is true for the isotropic homogeneous beams subjected to uniform temperature rise with arbitrary case of boundary conditions.

4.2 Mechanical Loading in Thermal Field

Consider an FGM beam in thermal field which is subjected to an in-plane axial load P, and operates in thermal field. The left side of the beam is immovable, while the right hand side is movable and undergoes an in-plane force P

When the beam exhibits the bifurcation-type of buckling, it remains un-deformed in primary equilibrium path. Based to the first equilibrium equation, the pre-buckling force resultant is equal to

$$N_{x0} = -\frac{P}{b} \tag{16}$$

Based on the definition of force resultants, when the lateral deflection is ignored, the induced mechanical moment due to the applied in-plane force is equal to

$$M_{x0} = -\frac{P}{b}\frac{E_2}{E_1} + N_x^T \frac{E_2}{E_1} - M_x^T \tag{17}$$

The existence of bifurcation type buckling depends on the vanishing of the extra bending moment in Eq. (17). In the following general cases are studied

Case 1: For the case when an FGM beam is subjected to axial load only, $C-C$ and $C-R$ cases follow the branching type of buckling. Otherwise the induced moment in Eq. (17) results in the initial deflection.

Case 2: For the case of reduction of an FGM beam to an isotropic homogeneous one that is subjected to uniform temperature rise loading, $M_{x0}=0$ and bifurcation occurs for any arbitrary case of out-of-plane boundary conditions.

Case 3: For the case of reduction of an FGM beam to an isotropic homogeneous one that is subjected to heat conduction across the thickness, $M_{x0} = -M_x^T$ and bifurcation occurs only for the especial cases of $C-C$ and $C-R$ end supports. This is due to the ability of clamped and roller edges to supply the extra moment in pre-buckling state.

Case 4: For the case of an FGM beam that is subjected to arbitrary case of thermal loading in the presence of axial in-plane load, M_{x0} is given by Eq. (17). Generally this moment does not vanish and similar to the previous case only $C-C$ and $C-R$ end supports exhibit the bifurcation-type of buckling. In other combinations of edge supports, beam initially start to lateral deflection at the onset of thermal loading.

5 STABILITY EQUATIONS

To derive the stability equations, the adjacent-equilibrium criterion is used. Assume that the equilibrium state of a functionally graded beam in pre-buckling state is defined in terms of the displacement components u_0, w_0 and φ_0. The displacement components of a neighboring stable state differ by u_1, w_1 and φ_1 with respect to the equilibrium position. Thus, the total displacements of a neighboring state are [23]

$$\begin{aligned} u &= u_0 + u_1 \\ w &= w_0 + w_1 \\ \varphi &= \varphi_0 + \varphi_1 \end{aligned} \tag{18}$$

Similar to the displacements, the force and moment resultants of a neighboring state may be related to the state of equilibrium as

$$\begin{aligned} N_x &= N_{x0} + N_{x1} \\ M_x &= M_{x0} + M_{x1} \\ Q_{xz} &= Q_{xz0} + Q_{xz1} \end{aligned} \tag{19}$$

Here, stress resultants with subscript 1 represent the linear parts of the force and moment resultant increments corresponding to u_1, w_1 and φ_1. The stability equations may be obtained by substituting Eqs. (18) and (19) in Eq. (12). Upon substitution, the terms in the resulting equations with subscript 0 satisfy the equilibrium conditions and therefore drop out of the equations. Also, the non-linear terms with subscript 1 are ignored because they are small compared to the linear terms. The remaining terms form the stability equations as

$$E_1 \frac{d^2 u_1}{dx^2} + E_2 \frac{d^2 \varphi_1}{dx^2} = 0$$

$$\boldsymbol{E}_2\frac{\boldsymbol{d}^2\boldsymbol{u}_1}{\boldsymbol{dx}^2}+\boldsymbol{E}_3\frac{\boldsymbol{d}^2\varphi_1}{\boldsymbol{dx}^2}-\frac{\boldsymbol{E}_1\boldsymbol{K}_s}{2\left(1+\nu\right)}\left(\varphi_1+\frac{\boldsymbol{dw}_1}{\boldsymbol{dx}}\right)=0 \tag{20}$$

$$\frac{\boldsymbol{E}_1\boldsymbol{K}_s}{2\left(1+\nu\right)}\left(\frac{\boldsymbol{d}\varphi_1}{\boldsymbol{dx}}+\frac{\boldsymbol{d}^2\boldsymbol{w}_1}{\boldsymbol{dx}^2}\right)+\boldsymbol{N}_{x0}\frac{\boldsymbol{d}^2\boldsymbol{w}_1}{\boldsymbol{dx}^2}=0$$

Combining Eqs. (20) by eliminating $\boldsymbol{u}_1$ and φ_1 provides an ordinary differential equation in terms of $\boldsymbol{w}_1$ which is the stability equation of an FGM beam under transverse thermal loadings

$$\frac{\boldsymbol{d}^4\boldsymbol{w}_1}{\boldsymbol{dx}^4}+\mu^2\frac{\boldsymbol{d}^2\boldsymbol{w}_1}{\boldsymbol{dx}^2} \tag{21}$$

with

$$\mu^2=\frac{\boldsymbol{E}_1\boldsymbol{N}_{x0}}{\left(\boldsymbol{E}_1\boldsymbol{E}_3-\boldsymbol{E}_2^2\right)\left(1-2\boldsymbol{N}_{x0}\frac{1+\nu}{\boldsymbol{E}_1\boldsymbol{K}_s}\right)} \tag{22}$$

The stress resultants with subscript 1 are linear parts of resultants that correspond to the neighboring state. Using Eqs. (9) and (18) the expressions for, $\boldsymbol{N}_{x1}$, $\boldsymbol{M}_{x1}$ and $\boldsymbol{Q}_{xz1}$ become

$$\boldsymbol{N}_{x1}=\boldsymbol{E}_1\frac{\boldsymbol{du}_1}{\boldsymbol{dx}}+\boldsymbol{E}_2\frac{\boldsymbol{d}\varphi_1}{\boldsymbol{dx}}$$

$$\boldsymbol{M}_{x1}=\boldsymbol{E}_2\frac{\boldsymbol{du}_1}{\boldsymbol{dx}}+\boldsymbol{E}_3\frac{\boldsymbol{d}\varphi_1}{\boldsymbol{dx}} \tag{23}$$

$$\boldsymbol{Q}_{xz1}=\frac{\boldsymbol{E}_1\boldsymbol{K}_s}{2\left(1+\nu\right)}\left(\varphi_1+\frac{\boldsymbol{dw}_1}{\boldsymbol{dx}}\right)$$

When temperature distribution through the beam is along the thickness direction only, the parameter μ is constant. In this case the exact solution of Eq. (2) is

$$\boldsymbol{w}_1(\boldsymbol{x})=\boldsymbol{C}_1\sin\left(\mu\boldsymbol{x}\right)+\boldsymbol{C}_2\cos\left(\mu\boldsymbol{x}\right)+\boldsymbol{C}_3\boldsymbol{x}+\boldsymbol{C}_4 \tag{24}$$

Using Eqs. (20), (23), and (24), the expressions for $\boldsymbol{u}_1$, φ_1, $\boldsymbol{N}_{x1}$, $\boldsymbol{M}_{x1}$, and $\boldsymbol{Q}_{xz1}$ become

$$\varphi_1(x) = -S\left(\mu\right)\left(C_1\cos\left(\mu x\right) - C_2\sin\left(\mu x\right)\right) - C_3$$

$$u_1(x) = \frac{E_2}{E_1}S\left(\mu\right)\left(C_1\cos\left(\mu x\right) - C_2\sin\left(\mu x\right)\right) + C_5 x + C_6$$

$$M_{x1}(x) = \mu S(\mu)\frac{E_1E_3 - E_2^2}{E_1}\left(C_1\sin\left(\mu x\right) + C_2\cos\left(\mu x\right)\right) + E_2C_5 \tag{25}$$

$$Q_{xz1}(x) = \frac{E_1K_s}{2\left(1+\nu\right)}\left(\mu - S\left(\mu\right)\right)\left(C_1\cos\left(\mu x\right) - C_2\sin\left(\mu x\right)\right)$$

$$N_{x1}(x) = E_1C_5$$

With

$$S(\mu) = \frac{\mu}{1 + 2\left(1+\nu\right)\mu^2\,\dfrac{E_1E_3 - E_2^2}{E_1^2K_s}} \tag{26}$$

The constants of integration C_1 to C_6 are obtained using the boundary conditions of the beam. Also, the parameter μ must be minimized to find the minimum value of N_{x0} associated with the thermal or mechanical buckling load. Five types of boundary conditions are assumed for the FGM or homogeneous beam with combination of the roller, simply supported, and clamped edges. Boundary conditions in each case are listed in Table 1.

Table 1 Boundary conditions for FGM Timoshenko beams under thermal loading. C indicates clamped, S shows simply-supported and R is used for roller edge. For mechanical buckling case, u_1 should be replaced by N_{x1}

Edge supports	B.Cs at $x = 0$	B.Cs at $x = L$
$C - C$	$u_1 = w_1 = \varphi_1 = 0$	$u_1 = w_1 = \varphi_1 = 0$
$S - S$	$u_1 = w_1 = M_{x1} = 0$	$u_1 = w_1 = M_{x1} = 0$
$C - S$	$u_1 = w_1 = \varphi_1 = 0$	$u_1 = w_1 = M_{x1} = 0$
$C - R$	$u_1 = w_1 = \varphi_1 = 0$	$u_1 = \varphi_1 = Q_{xz1} + N_{x0}\dfrac{dw_1}{dx} = 0$
$S - R$	$u_1 = w_1 = M_{x1} = 0$	$u_1 = \varphi_1 = Q_{xz1} + N_{x0}\dfrac{dw_1}{dx} = 0$

Let us consider a beam with both edges clamped under thermal loading. Using Eqs. (24) and (25), the constants $\boldsymbol{C}_1$ to $\boldsymbol{C}_6$ must satisfy the system of equations

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ \sin(\mu \boldsymbol{L}) & \cos(\mu \boldsymbol{L}) & \boldsymbol{L} & 1 & 0 & 0 \\ -\boldsymbol{S}(\mu) & 0 & -1 & 0 & 0 & 0 \\ -\boldsymbol{S}(\mu)\cos(\mu \boldsymbol{L}) & \boldsymbol{S}(\mu)\sin(\mu \boldsymbol{L}) & -1 & 0 & 0 & 0 \\ \frac{\boldsymbol{E}_2}{\boldsymbol{E}_1}\boldsymbol{S}(\mu) & 0 & 0 & 0 & 0 & 1 \\ \frac{\boldsymbol{E}_2}{\boldsymbol{E}_1}\boldsymbol{S}(\mu)\cos(\mu \boldsymbol{L}) & -\frac{\boldsymbol{E}_2}{\boldsymbol{E}_1}\boldsymbol{S}(\mu)\sin(\mu \boldsymbol{L}) & 0 & 0 & \boldsymbol{L} & 1 \end{pmatrix} \begin{Bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_3 \\ \boldsymbol{C}_4 \\ \boldsymbol{C}_5 \\ \boldsymbol{C}_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{27}$$

To have a nontrivial solution, the determinant of coefficient matrix must be set equal to zero, which yields

$$\boldsymbol{S}(\mu)\boldsymbol{L}\left(2 - 2\cos(\mu \boldsymbol{L}) - \boldsymbol{L}\boldsymbol{S}(\mu)\sin(\mu \boldsymbol{L})\right) = 0 \tag{28}$$

The smallest positive value of μ which satisfies Eq. (28) is $\mu_{\min} = \frac{6.28319}{\boldsymbol{L}}$. It can be seen easily that for the other types of boundary conditions, except $\boldsymbol{C}-\boldsymbol{S}$ case, the nontrivial solution leads to an exact parameter for μ. Using an approximate solution given in [24] for the critical axial force of $\boldsymbol{C}-\boldsymbol{S}$ beams, the critical force for an FGM Timoshenko beam with arbitrary boundary conditions can be expressed as below

$$\boldsymbol{N}_{x0,cr} = \frac{\frac{\boldsymbol{p}}{\boldsymbol{L}^2}\left(\boldsymbol{E}_3 - \frac{\boldsymbol{E}_2^2}{\boldsymbol{E}_1}\right)}{1 + \boldsymbol{q}\frac{1+\nu}{\boldsymbol{K}_s \boldsymbol{L}^2}\left(\frac{\boldsymbol{E}_3}{\boldsymbol{E}_1} - \left(\frac{\boldsymbol{E}_2}{\boldsymbol{E}_1}\right)^2\right)} \tag{29}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are constants depending upon the boundary conditions and are listed in Table 2

Table 2 Constants of formula (29) which are related to boundary conditions.

Parameter	$C-C$	$S-S$	$C-S$	$S-R$	$C-R$
p	39.47842	9.86960	20.19077	2.46740	9.86960
q	78.95684	19.73920	44.41969	4.93480	19.73920

When the critical buckling force resultant is obtained, in the case of mechanical loading, the total compressive load may be evaluated by Eq. (16). For the case of thermal buckling analysis, however, temperature profile should be known.

6 TYPES OF THERMAL LOADING

6.1 Uniform temperature rise (UTR)

Consider a beam which is at reference temperature T_0. When the axial displacement is prevented, the uniform temperature may be raised to $T_0 + \Delta T$ such that the beam buckles. Substituting $T_0 + \Delta T$ into Eq. (10) gives

$$N_x^T = h\Delta T\left(E_m\alpha_m + \frac{E_{cm}\alpha_m + E_m\alpha_{cm}}{k+1} + \frac{E_{cm}\alpha_{cm}}{2k+1}\right) \tag{30}$$

Considering Eq. (30), the critical buckling temperature difference ΔT_{cr}^{UTR} is expressed in the form

$$\Delta T_{cr}^{UTR} = \frac{\frac{p}{\alpha_m}\left(\frac{h}{L}\right)^2 F(k,\xi)}{G(k,\xi,\zeta)\left[1 + q\frac{1+\nu}{K_s}\left(\frac{h}{L}\right)^2 E(k,\xi)\right]} \tag{31}$$

where $\xi = \frac{E_{cm}}{E_m}$ and $\zeta = \frac{\alpha_{cm}}{\alpha_m}$. Also, the functions $E(k,\xi)$, $F(k,\xi)$, and $G(k,\xi,\zeta)$ are defined as

$$F(k,\xi) = \frac{1}{12} + \frac{\xi(k^2 + k + 2)}{4(k+1)(k+2)(k+3)} - \frac{\xi^2 k^2}{4(k+1)(k+2)^2(k+1+\xi)}$$

$$E(k,\xi) = \frac{k+1}{12(k+1+\xi)} + \frac{\xi(k^2 + k + 2)}{4(k+2)(k+3)(k+1+\xi)} - \frac{\xi^2 k^2}{4(k+2)^2(k+1+\xi)^2} \tag{32}$$

$$G(k,\xi,\zeta) = 1 + \frac{\xi + \zeta}{k+1} + \frac{\xi\zeta}{2k+1}$$

6.2 Linear temperature distribution (LTD)

Consider a thin FGM beam which the temperature in ceramic-rich and metal-rich surfaces are T_c and T_m, respectively. The temperature distribution for the given boundary conditions is obtained by solving the heat conduction equation along the beam thickness. If the beam thickness is thin enough, the temperature distribution is approximated linear through the thickness. So the temperature as a function of thickness coordinate z can be written in the form

$$T = T_m + (T_c - T_m)(\frac{1}{2} + \frac{z}{h}) \tag{33}$$

Substituting Eq. (33) into Eq. (10) gives the thermal force as

$$\begin{aligned} N_x^T &= h(T_m - T_0)\left(E_m\alpha_m + \frac{E_{cm}\alpha_m + E_m\alpha_{cm}}{k+1} + \frac{E_{cm}\alpha_{cm}}{2k+1} \right) \\ &+ h\Delta T\left(\frac{E_m\alpha_m}{2} + \frac{E_{cm}\alpha_m + E_m\alpha_{cm}}{k+2} + \frac{E_{cm}\alpha_{cm}}{2k+2} \right) \end{aligned} \tag{34}$$

where $\Delta T = T_c - T_m$. Combining Eqs. (29) and (34) gives the final form of the critical buckling temperature difference through the thickness as

$$\Delta T_{cr}^{LTD} = \frac{\frac{p}{\alpha_m}\left(\frac{h}{L}\right)^2 F(k,\xi)}{H(k,\xi,\zeta)\left(1 + q\frac{1+\nu}{K_s}\left(\frac{h}{L}\right)^2 E(k,\xi)\right)} - (T_m - T_0)\frac{G(k,\xi,\zeta)}{H(k,\xi,\zeta)} \tag{35}$$

Here, the functions $E(k,\xi)$, $F(k,\xi)$, and $G(k,\xi,\zeta)$ are defined in Eq. (32) and function $H(k,\xi,\zeta)$ is defined as given below

$$H(k,\xi,\zeta) = \frac{1}{2} + \frac{\xi+\zeta}{k+2} + \frac{\xi\zeta}{2k+2} \tag{36}$$

6.3 Nonlinear temperature distribution (NLTD)

Assume an FGM beam where the temperature in ceramic-rich and metal-rich surfaces are T_c and T_m, respectively. The governing equation for the steady-state one-dimensional heat conduction equation, in the absence of heat generation, becomes

$$\frac{d}{dz}\left(K(z)\frac{dT}{dz}\right) = 0$$
$$T(\frac{h}{2}) = T_c, T(-\frac{h}{2}) = T_m \tag{37}$$

where $K(z)$ is given by Eq. (2). Solving this equation via polynomial series yields the temperature distribution across the beam thickness as

$$T = T_m + \frac{(T_c - T_m)}{D}\left[\sum_{i=0}^{N}\frac{(-1)^i}{ik+1}\left(\frac{K_{cm}}{K_m}\right)^i\left(\frac{1}{2}+\frac{z}{h}\right)^{ik+1}\right] \tag{38}$$

with

$$D = \sum_{i=0}^{N}\frac{(-1)^i}{ik+1}\left(\frac{K_{cm}}{K_m}\right)^i \tag{39}$$

Here N is the number of terms which should be taken into account to assure the convergence of the series. Evaluating N_x^T and solving for ΔT gives the critical bucking value of the temperature difference as

$$\Delta T_{cr}^{NLTD} = \frac{\frac{p}{\alpha_m}\left(\frac{h}{L}\right)^2 F(k,\xi)}{I(k,\xi,\zeta,\gamma)\left(1+q\frac{1+\nu}{K_s}\left(\frac{h}{L}\right)^2 E(k,\xi)\right)} - (T_m - T_0)\frac{G(k,\xi,\zeta)}{I(k,\xi,\zeta,\gamma)} \tag{40}$$

In this relation, $\gamma = \dfrac{K_{cm}}{K_m}$ and the function $I(k,\xi,\zeta,\gamma)$ is defined as

$$I(k,\xi,\zeta,\gamma) = \frac{1}{D}\sum_{i=0}^{N}\frac{(-\gamma)^i}{ik+1}\left[\frac{1}{ik+2}+\frac{\xi+\zeta}{ik+k+2}+\frac{\xi\zeta}{ik+2k+2}\right] \tag{41}$$

It should be pointed out that in each case of thermal loading, an iterative process should be implemented to calculate the critical buckling temperature difference. To this end, properties are evaluated at reference temperature and the critical buckling temperature difference is calculated. Properties of the constituents are then evaluated at the current temperature and again critical buckling temperature difference is obtained. This process should be continued to obtain the convergent critical buckling temperature difference.

7 RESULTS AND DISCUSSION

Consider a ceramic-metal functionally graded beam. The combination of materials consist of Silicon-Nitride as ceramic and stainless steel as metal. The elasticity modulus, the thermal expansion coefficient, and the thermal conductivity coefficient for these constituents are highly dependent to the temperature and their properties may be evaluated in any temperature based on Toloukian model. Each property of the constituents follow the next dependency to the temperature

$$P(T) = P_0\left(1 + P_{-1}T^{-1} + P_1T + P_2T^2 + P_3T^3\right) \tag{42}$$

In this equation T is measured in Kelvin. The constants P_i are unique for the constituents and for the constituents of this study are given in Table 3. Poisson's ratio for simplicity is chosen as 0.28.

Table 3 Introduced coefficients of Eq. (42)

	E_c	E_m	α_c	α_m	K_c	K_m
P_0	$348.43e+9$	$201.04e+9$	$5.8723e-6$	$12.33e-6$	13.723	15.379
P_1	$-3.070e-4$	$3.079e-4$	$9.095e-4$	$8.086e-4$	$-1.032e-3$	$-1.264e-3$
P_2	$2.16e-7$	$-6.534e-7$	0	0	$5.466e-7$	$2.092e-6$
P_3	$-8.946e-11$	0	0	0	$-7.876e-11$	$-7.223e-10$

To validate the results, the effect of shear is plotted in Figure 3, for an isotropic homogeneous beam with temperature independent material properties. For this purpose, the results are compared between the Euler and Timoshenko beam theories. The beam is under the uniform temperature rise loading. Non-dimensional critical buckling temperature is defined by $\lambda_{cr} = \alpha_m \Delta T_{cr}^{UTR} (L/h)^2$. It is apparent that the critical buckling temperature for beams with L/h ratio more than 50 is identical between the two theories. But, for L/h ratio less than 50, the difference between the two theories become larger, and it will become more different for L/h values less than 20. The same graph is reported in [8] based on the numerical shooting method.

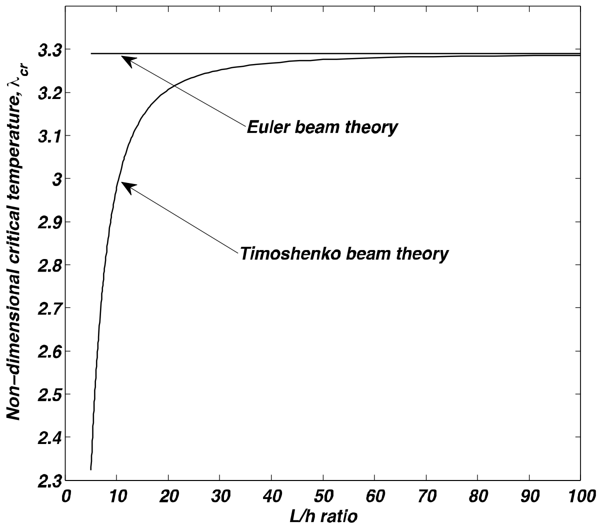

Figure 3 Effect of transverse shear on critical buckling temperature difference

In Figure 4, the critical buckling temperature difference of an FGM beam under the uniform temperature rise loading is depicted. Both edges are clamped. TD case indicates that properties are temperature dependent, whereas TID indicates that properties are evaluated at reference temperature. As seen, as the power law index increases, the critical buckling temperature decreases permanently. When it is compared to the TD case, TID case overestimates the buckling temperatures. Difference between TID and TD cases is more pronounced at higher temperatures

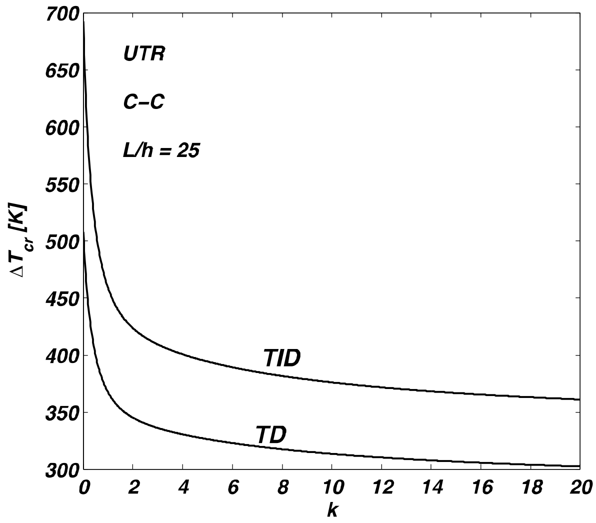

Figure 4 Effects of power law index and temperature dependency on ΔT_{cr}

In Figure 5, two other cases of thermal loadings are compared with respect to each other. As seen in both of these cases, also, an increase in power law index results in lower buckling temperature. LTD case as an approximate solution of the NLTD case underestimates the critical buckling temperatures except for the case of reduction of an FGM beam to the associated homogeneous cases. This is expected since in these cases, the exact solution of the heat conduction equation is also linear.

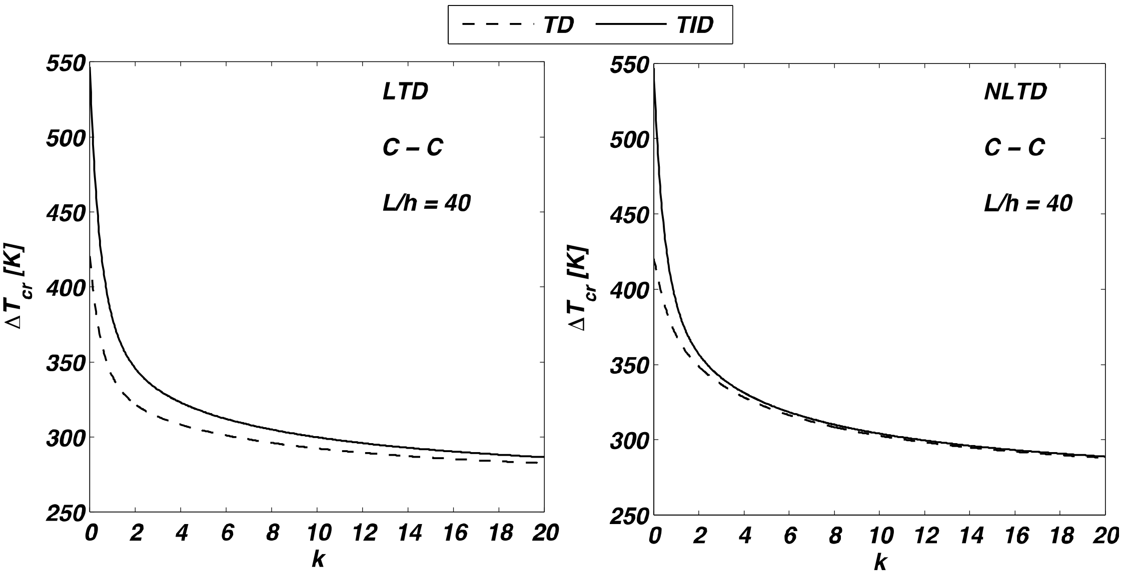

Figure 5 Effects of power law index on ΔT_{cr} of FGM beams under LTD and NLTD cases

The influence of boundary conditions on buckling temperature difference is plotted in Figure 6. The uniform temperature rise case of loading is assumed and properties are assumed to be TD. The case of a homogeneous beam is chosen. As expected, the higher buckling temperature belongs to a beam with both edges clamped and the lower one is associated to a beam with one side simply supported and the other one roller. The critical buckling temperature of $\boldsymbol{S}-\boldsymbol{S}$ and $\boldsymbol{C}-\boldsymbol{R}$ cases are the same.

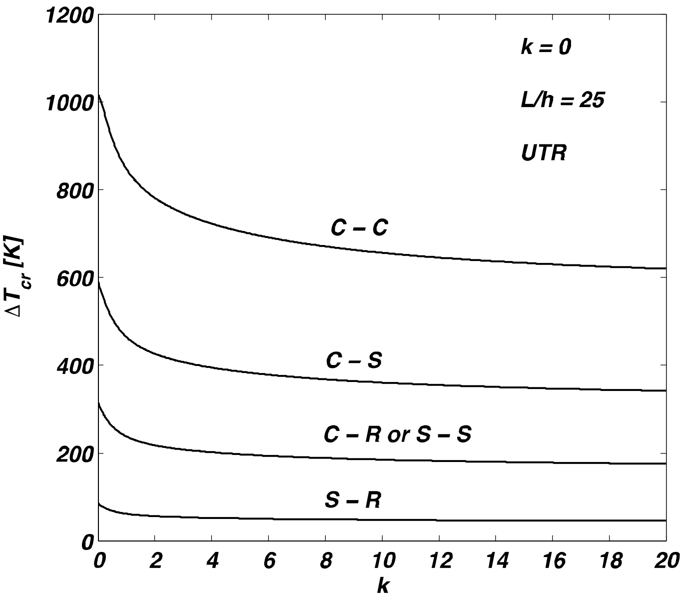

Figure 6 Boundary conditions effect on $\Delta \boldsymbol{T}_{cr}$

The effect of uniform temperature rise field on the axial buckling load of $\boldsymbol{C}-\boldsymbol{R}$ and $\boldsymbol{C}-\boldsymbol{C}$ beams is demonstrated in Figure 7. The obtained buckling loads are normalized by the equation $\boldsymbol{n}_{cr}^{T}=\dfrac{12\boldsymbol{P}_{cr}\boldsymbol{L}^2}{\boldsymbol{E}_c^{ref}\boldsymbol{b}\boldsymbol{h}^3}$, where $\boldsymbol{E}_c^{ref}$ is the ceramic elasticity module at reference temperature. As expected, an increase in the power law index results in the lower buckling load. This is expected since as power law index decreases, FGM beam tends to a ceramic beam which is stiffer than metal. With the increase of temperature rise parameter, buckling load decreases since the constituents lose their stiffnesses.

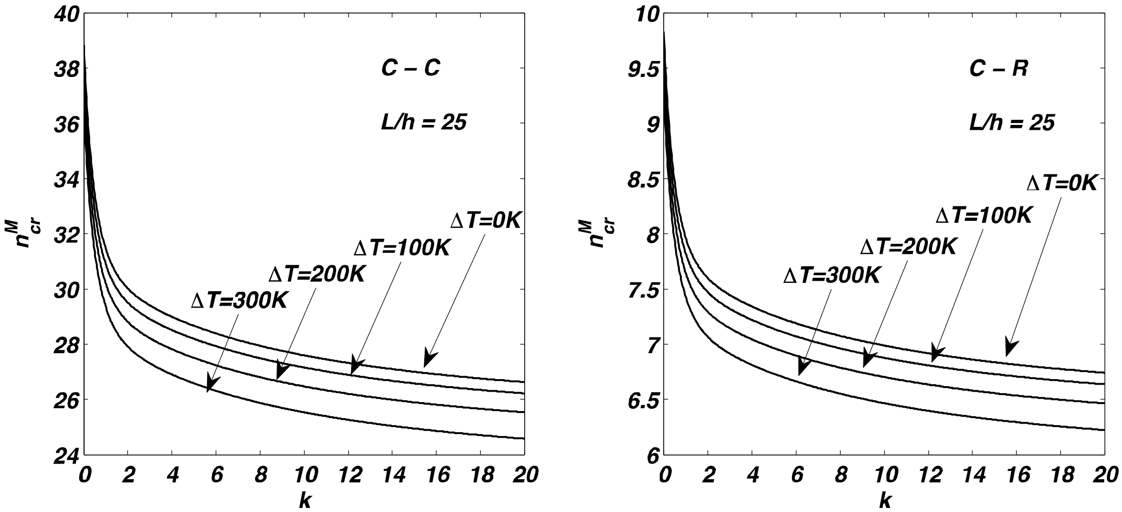

Figure 7 Effect of thermal environment on mechanical buckling of FGM beams

8 CONCLUSION

In the present article, the equilibrium and stability equations for the FGM beams with various types of boundary conditions are obtained. The derivation is based on the Timoshenko beam theory, with the assumption of power law composition for the constituent material. The buckling analysis under three types of thermal loadings is presented. Also, the mechanical buckling analysis under thermal loads is studied. Closed form solutions are derived for the critical temperature/load. It is concluded that:

1) The $\boldsymbol{C-C}$ and $\boldsymbol{C-R}$ functionally graded beams exhibit the bifurcation type buckling while the $\boldsymbol{S-S}$, $\boldsymbol{C-S}$ and $\boldsymbol{S-R}$ FGM beams commence to deflect with the initiation of thermal loading.

2) In each case of thermal loading, the critical buckling temperature for FGM beams is lower than fully ceramic beam but greater than fully metallic beam.

3) According to the Euler and Timoshenko beam theories, the critical buckling temperature of isotropic homogeneous beam is independent of elasticity modulus; but for an FGM beam the elasticity modulus of the constituent materials have significant effect on critical buckling temperature.

4) The critical buckling temperature of $C-R$ and $S-S$ homogeneous beams are identical for studied cases of thermal loading, while $S-S$ and $C-R$ FGM beams reveal different behaviors when are subjected to in-plane thermal loading.

5) The Euler beam theory over-predicts the critical temperature of thick beams, especially for h/L greater than 0.05.

6) Temperature dependency of the constituents has significant effect on critical buckling temperature difference. The value of ΔT_{cr} is overestimated when the properties are assumed to be independent of temperature.

7) In each case of thermal loading, the Timoshenko beam theory predicts lower values for critical buckling temperature in comparison with the Euler beam theory.

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