Latin American Journal of
Solids and Structures
www.lajss.org

Dynamic response to moving masses of rectangular plates with general boundary conditions and resting on variable winkler foundation

T. O. Awodola*, S. T. Oni

Department of Mathematical Sciences
Federal University of Technology, Akure, Nigeria.

*Author email: oluthomas71@yahoo.com

Received 18 Jan 2012
In revised form 14 Sep 2012

Abstract

The dynamic response to moving masses of rectangular plates with general classical boundary conditions and resting on variable Winkler elastic foundation is investigated in this work. The governing fourth order partial differential equation is solved using a technique based on separation of variables, the modified method of Struble and the integral transformations. Numerical results in plotted curves are then presented. The results show that as the value of the rotatory inertia correction factor Ro increases, the response amplitudes of the plate decrease and that, for fixed value of Ro, the displacements of the plate decrease as the foundation modulus Fo increases for the variants of the classical boundary conditions considered. The results also show that for fixed Ro and Fo, the transverse deflections of the rectangular plates under the actions of moving masses are higher than those when only the force effects of the moving load are considered. For the rectangular plate, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem for all variants of classical boundary conditions, that is, resonance is reached earlier in moving mass problem than in moving force problem. When Fo and Ro increase, the critical speed increases, hence, risk is reduced.

Keywords

Winkler foundation, Rotatory Inertia, Foundation modulus, Moving Force, Moving Mass, Resonance, Critical speed.

1 INTRODUCTION

Structures such as bridges, roadways, decking slabs, girders and belt drive (carrying machine chain) are constantly acted upon by moving masses and, hence, the problem of analyzing the dynamic response of elastic structures under the action of moving masses continues to motivate a variety of investigations [1-6]. In most analytical studies in Engineering and Mathematical Physics, structural members are commonly modeled as a beam or as a plate.

The effects of moving loads on solid bodies are dual [1]. On one hand is the gravitational effect of the moving load while on the other hand is the inertia effect of the mass of the load on the vibrating solid bodies. When the inertia effect of the moving load is considered, the governing differential equation of motion becomes complex and cumbersome and no longer has constant coefficients. In particular, the coefficients become variable and singular. If the inertia effect of the moving load is neglected, the problem is termed moving force problem and when it is retained, it is termed moving mass problem.

Aside the problem arising from the inclusion of the inertia terms in moving mass problems, difficulties often arise from the type of specified end-conditions. There are four classical boundary conditions that are commonly of practical interest to an applied Mathematician or an Engineer. These are Pinned end conditions (Simply supported end conditions), Fixed / Clamped end conditions, Free end conditions and Sliding end conditions [1, 7].

The analysis of beam and plate on Winkler foundation when the foundation modulus is constant is very common in literature. The work of Timoshenko [8] gave impetus to research work in this area of study. He used energy methods to obtain solutions in series form for simply supported finite beams on elastic foundations subjected to time dependent point loads moving with uniform velocity across the beam. Steele [9] also investigated the response of a finite, simply supported Bernoulli-Euler beam to a unit force moving at a uniform velocity. He analyzed the effects of this moving force on beams with and without an elastic foundation. Using a considerably simpler vector formulation with a Laplace rather than Fourier transformation, Steele [10] presented a review of the transient response of the Bernoulli-Euler beam and the Timoshenko beam on elastic foundation due to moving loads

Several researchers have also made tremendous efforts in the study of dynamics of structures under moving loads [11, 12, 13, 14, 15, 16, 17, 18]. Recently, Oni and Awodola [19] considered the dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. More recently, Oni and Awodola [20] investigated the dynamic behaviour under moving concentrated masses of simply supported rectangular plates resting on variable Winkler elastic foundation.

In most of the investigations in literature on vibration of rectangular plate under moving loads and resting on elastic foundations, work has been restricted to cases when the elastic foundations are regarded as being constant. The more complicated case, when the elastic foundation varies along the span of the structure has been neglected, where this is considered, work has been restricted to the simplest form of the problem when the structure is simply supported. This paper is therefore concerned with the problem of assessing the dynamic response to moving concentrated masses of rectangular plates with general classical boundary conditions and resting on variable Winkler elastic foundations.

2 GOVERNING EQUATION

The problem of the dynamic response to moving concentrated masses of rectangular plate with general classical boundary condition and resting on Winkler foundation with stiffness variation is considered. Consider a rectangular plate carrying an arbitrary number (say N) of concentrated masses M_i moving with constant velocities c_i, i = 1, 2, 3,...,N along a straight line parallel to the xaxis issuing from point y = s on the y-axis. The equation governing the dynamic transverse displacement W(x,y,t) of the rectangular plate when it is resting on a variable Winkler foundation and traversed by several moving concentrated masses is the fourth order partial differential equation given by [20]

$$\begin{aligned} D\nabla^4 W(x,y,t) + \mu\frac{\partial^2 W(x,y,t)}{\partial t^2} &= \mu R_0\left[\frac{\partial^4}{\partial t^2\,\partial x^2} + \frac{\partial^4}{\partial t^2\,\partial y^2}\right]W(x,y,t) \\ -F_0\left[4x - 3x^2 + x^3\right]W(x,y,t) &+ \sum_{i=1}^{N}\Big[M_i g\delta(x - c_i t)\delta(y - s) \\ -M_i\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\,\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right) & W(x,y,t)\delta(x - c_i t)\delta(y - s)\Big] \end{aligned} \tag{1}$$

where D is the bending rigidity of the plate, *m* is mass per unit area of the plate, x is the position co-ordinate in x − direction, y is position co-ordinate in y − direction, t is the time, R_0 is the rotatory inertia correction factor, ∇^2 is the two-dimensional Laplacian operator, F_0 is the foundation modulus and δ (.) is the Dirac-Delta function.

At this juncture, the boundary condition is arbitrary and the initial condition, without any loss of generality, is taken as

$$W(x,y,t) = 0 = \frac{\partial W(x,y,t)}{\partial t} \tag{2}$$

3 ANALYTICAL APPROXIMATE SOLUTION

Evidently, an exact closed form solution of the above fourth order partial differential equation (1) does not exist. Consequently, an approximate solution is sought. Thus, the technique based on separation of variable described in [11] is employed. This versatile technique requires that the solution of equation (1) takes the form

$$W(x,y,t) = \sum_{n=1}^{\infty}\varphi_n(x,y)T_n(t) \tag{3}$$

where ϕ_n are the known eigen functions of the plate with the same boundary conditions and have the form of [20]

$$\nabla^4\varphi_n - \omega_n^4\varphi_n = 0 \tag{4}$$

where

$$\omega_n^4 = \frac{\Omega_n^2 \mu}{D} \tag{5}$$

Ω_n, n = 1, 2, 3, ... , are the natural frequencies of the dynamical system and $T_n(t)$ are amplitude functions which have to be calculated.

In order to solve the equation (1), it is rewritten as

$$\begin{aligned}&\frac{D}{\mu}\nabla^4 W(x,y,t) + \frac{\partial^2 W(x,y,t)}{\partial t^2} = R_0\left[\frac{\partial^4}{\partial t^2\,\partial x^2} + \frac{\partial^4}{\partial t^2\,\partial y^2}\right]W(x,y,t)\\&-\frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]W(x,y,t) + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s)\right.\\&\left.-\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\,\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x - c_i t)\delta(y - s)\right]\end{aligned} \tag{6}$$

Rewriting the right hand side of equation (6) in the form of a series, we have

$$\begin{aligned}&R_0\left[\frac{\partial^4}{\partial t^2\,\partial x^2} + \frac{\partial^4}{\partial t^2\,\partial y^2}\right]W(x,y,t) - \frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]W(x,y,t) + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s)\right.\\&\left.-\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\,\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x - c_i t)\delta(y - s)\right] = \sum_{n=1}^{\infty}\varphi_n(x,y)B_n(t)\end{aligned} \tag{7}$$

When equation (3) is used in equation (7) we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big\{R_0\left[\varphi_{n,xx}(x,y)T_{n,tt}(t) + \varphi_{n,yy}(x,y)T_{n,tt}(t)\right] - \frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]\varphi_n(x,y)T_n(t)\\&+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s) - \frac{M_i}{\mu}\Big(\varphi_n(x,y)T_{n,tt}(t) + 2c_i\varphi_{n,x}(x,y)T_{n,t}(t)\\&+c_i^2\varphi_{n,xx}(x,y)T_n(t)\ \Big)\delta(x - c_i t)\delta(y - s)\ \Big]\ \Big\} = \sum_{n=1}^{\infty}\varphi_n(x,y)B_n(t)\end{aligned} \tag{8}$$

where

$$\begin{aligned}&\varphi_{n,x}(x,y)\ \textit{implies}\ \frac{\partial\varphi_n(x,y)}{\partial x},\quad \varphi_{n,xx}(x,y)\ \textit{implies}\ \frac{\partial^2\varphi_n(x,y)}{\partial x^2},\\&\varphi_{n,y}(x,y)\ \textit{implies}\ \frac{\partial\varphi_n(x,y)}{\partial y},\quad \varphi_{n,yy}(x,y)\ \textit{implies}\ \frac{\partial^2\varphi_n(x,y)}{\partial y^2},\\&T_{n,t}(t)\ \textit{implies}\ \frac{dT_n(t)}{dt}\ \ \textit{and}\ \ T_{n,tt}(t)\ \textit{implies}\ \frac{d^2T_n(t)}{dt^2}\end{aligned} \tag{9}$$

Integrating equation (8) on area A of the plate, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\int_A \Big\{R_0\big[\varphi_{n,xx}(x,y)\varphi_p(x,y)T_{n,tt}(t)+\varphi_{n,yy}(x,y)\varphi_p(x,y)T_{n,tt}(t)\big]\\&-\frac{F_0}{\mu}\big[4x-3x^2+x^3\big]\varphi_n(x,y)\varphi_p(x,y)T_n(t)+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{\mu}\varphi_p(x,y)\delta(x-c_it)\delta(y-s)\\&-\frac{M_i}{\mu}\Big(\varphi_n(x,y)\varphi_p(x,y)T_{n,tt}(t)+2c_i\varphi_{n,x}(x,y)\varphi_p(x,y)T_{n,t}(t)\\&+c_i^2\varphi_{n,xx}(x,y)\varphi_p(x,y)T_n(t)\ \Big)\delta(x-c_it)\delta(y-s)\ \Big]\Big\}dA=\sum_{n=1}^{\infty}\int_A \varphi_n(x,y)\varphi_p(x,y)B_n(t)\,dA\end{aligned}\tag{10}$$

Considering the orthogonality of $\phi_n(x,y)$, we have that

$$\begin{aligned}B_n(t)=\frac{1}{P^*}&\sum_{n=1}^{\infty}\int_A \Big\{R_0\big[\varphi_{n,xx}(x,y)\varphi_p(x,y)T_{n,tt}(t)+\varphi_{n,yy}(x,y)\varphi_p(x,y)T_{n,tt}(t)\big]\\&-\frac{F_0}{\mu}\big[4x-3x^2+x^3\big]\varphi_n(x,y)\varphi_p(x,y)T_n(t)+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{\mu}\varphi_p(x,y)\delta(x-c_it)\delta(y-s)\\&-\frac{M_i}{\mu}\Big(\varphi_n(x,y)\varphi_p(x,y)T_{n,tt}(t)+2c_i\varphi_{n,x}(x,y)\varphi_p(x,y)T_{n,t}(t)\\&+c_i^2\varphi_{n,xx}(x,y)\varphi_p(x,y)T_n(t)\ \Big)\delta(x-c_it)\delta(y-s)\ \Big]\Big\}dA\end{aligned}\tag{11}$$

where $P^*=\int_A \varphi_p^2\,dA$

Using (11) and taking into account (3) and (4), equation (6) can be written as

$$\begin{aligned}&\varphi_n(x,y)\left[\frac{D\omega_n^4}{\mu}T_n(t)+T_{n,tt}(t)\right]=\frac{\varphi_n(x,y)}{P^*}\sum_{q=1}^{\infty}\int_A \Big\{R_0\big[\varphi_{q,xx}(x,y)\varphi_p(x,y)T_{q,tt}(t)\\&+\varphi_{q,yy}(x,y)\varphi_p(x,y)T_{q,tt}(t)\big]-\frac{F_0}{\mu}\big[4x-3x^2+x^3\big]\varphi_q(x,y)\varphi_p(x,y)T_q(t)\\&+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{\mu}\varphi_p(x,y)\delta(x-c_it)\delta(y-s)-\frac{M_i}{\mu}\Big(\varphi_q(x,y)\varphi_p(x,y)T_{q,tt}(t)\\&+2c_i\varphi_{q,x}(x,y)\varphi_p(x,y)T_{q,t}(t)+c_i^2\varphi_{q,xx}(x,y)\varphi_p(x,y)T_q(t)\ \Big)\delta(x-c_it)\delta(y-s)\ \Big]\Big\}dA\end{aligned}\tag{12}$$

Equation (12) implies that

$$
\begin{aligned}
&T_{n,tt}(t) + \frac{D\omega_n^4}{\mu} T_n(t) = \frac{1}{P^*} \sum_{q=1}^{\infty} \int_A \Big\{ R_0 \big[\varphi_{q,xx}(x,y)\varphi_p(x,y) T_{q,tt}(t) \\
&+ \varphi_{q,yy}(x,y)\varphi_p(x,y) T_{q,tt}(t) \big] - \frac{F_0}{\mu} \big[4x - 3x^2 + x^3 \big] \varphi_q(x,y)\varphi_p(x,y) T_q(t) \\
&+ \sum_{i=1}^{N} \Big[\frac{M_i g}{\mu} \varphi_p(x,y)\delta(x - c_i t)\delta(y - s) - \frac{M_i}{\mu} \big(\varphi_q(x,y)\varphi_p(x,y) T_{q,tt}(t) \\
&+ 2c_i \varphi_{q,x}(x,y)\varphi_p(x,y) T_{q,t}(t) + c_i^2 \varphi_{q,xx}(x,y)\varphi_p(x,y) T_q(t) \big) \delta(x - c_i t)\delta(y - s) \Big] \Big\} dA
\end{aligned}
\tag{13}
$$

Equation (13) is a set of coupled second order ordinary differential equations.

Expressing the Dirac-Delta function in the Fourier cosine series as

$$
\delta(x - c_i t) = \frac{1}{L_X} \left[1 + 2\sum_{j=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} \cos\frac{j\pi x}{L_X} \right] \tag{14}
$$

and

$$
\delta(y - s) = \frac{1}{L_Y} \left[1 + 2\sum_{k=1}^{\infty} \cos\frac{k\pi s}{L_Y} \cos\frac{k\pi y}{L_Y} \right] \tag{15}
$$

equation (13) then becomes

$$
\begin{aligned}
&\frac{d^2 T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*} \sum_{q=1}^{\infty} \Bigg\{ R_0 P_1^* \frac{d^2 T_q(t)}{dt^2} - \frac{F_0}{\mu} P_2^* T_q(t) \\
&- \sum_{i=1}^{N} \frac{M_i}{L_X L_Y \mu} \Bigg[2 \Bigg(\frac{P_3^*}{2} + \sum_{k=1}^{\infty} \cos\frac{k\pi s}{L_Y} P_3^{**}(k) + \sum_{j=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} P_3^{***}(j) \\
&+ 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} \cos\frac{k\pi s}{L_Y} P_3^{****}(j,k) \Bigg) \frac{d^2 T_q(t)}{dt^2} + 4c_i \Bigg(\frac{P_4^*}{2} + \sum_{k=1}^{\infty} \cos\frac{k\pi s}{L_Y} P_4^{**}(k) \\
&+ \sum_{j=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} \cos\frac{k\pi s}{L_Y} P_4^{****}(j,k) \Bigg) \frac{dT_q(t)}{dt} \\
&+ 2c_i^2 \Bigg(\frac{P_5^*}{2} + \sum_{k=1}^{\infty} \cos\frac{k\pi s}{L_Y} P_5^{**}(k) + \sum_{j=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} P_5^{***}(j) \\
&+ 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \cos\frac{j\pi c_i t}{L_X} \cos\frac{k\pi s}{L_Y} P_5^{****}(j,k) \Bigg) T_q(t) \Bigg] \Bigg\} = \sum_{i=1}^{N} \frac{M_i g}{P^* \mu} \varphi_p(c_i t, s)
\end{aligned}
\tag{16}
$$

where

$$\alpha_n^2 = \frac{D\omega_n^4}{\mu},$$

$$P_1^* = \int_0^{L_X}\int_0^{L_Y}\left[\varphi_{n,xx}(x,y)+\varphi_{n,yy}(x,y)\right]\varphi_p(x,y)\,dy\,dx,$$

$$P_2^* = \int_0^{L_X}\int_0^{L_Y}\left[4x-3x^2+x^3\right]\varphi_n(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_3^* = \int_0^{L_X}\int_0^{L_Y}\varphi_n(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_3^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\varphi_n(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_3^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\varphi_n(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_3^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\varphi_n(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_4^* = \int_0^{L_X}\int_0^{L_Y}\varphi_{n,x}(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_4^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\varphi_{n,x}(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_4^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\varphi_{n,x}(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_4^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\varphi_{n,x}(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_5^* = \int_0^{L_X}\int_0^{L_Y}\varphi_{n,xx}(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_5^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\varphi_{n,xx}(x,y)\varphi_p(x,y)\,dy\,dx,$$

$$P_5^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\varphi_{n,xx}(x,y)\varphi_p(x,y)\,dy\,dx$$

$$\text{and}\quad P_5^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\varphi_{n,xx}(x,y)\varphi_p(x,y)\,dy\,dx,$$

Equation (16) is the transformed equation governing the problem of the rectangular plate on a variable Winkler elastic foundation. This differential equation holds for all variants of the classical boundary conditions.

In what follows, $\phi_n(x,y)$ are assumed to be the products of the beam functions $\psi_{ni}(x)$ and $\psi_{nj}(y)$ [20]. That is

$$\varphi_n(x,y) = \psi_{ni}(x)\psi_{nj}(y) \tag{17}$$

These beam functions can be defined respectively, as

$$\psi_{ni}(x) = \sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X} \tag{18}$$

and

$$\psi_{nj}(y) = \sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y} \tag{19}$$

where A_{ni}, A_{nj}, B_{ni}, B_{nj}, C_{ni} and C_{nj} are constants determined by the boundary conditions. $\mathbf{\Omega}_{ni}$ and $\mathbf{\Omega}_{nj}$ are called the mode frequencies.

In order to solve equation (16) we shall consider only one mass M traveling with uniform velocity c along the line y = s. The solution for any arbitrary number of moving masses can be obtained by superposition of the individual solution since the governing differential equation is linear. Thus for the single mass M_1 equation (16) reduces to

$$\begin{aligned}
&\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{R_0P_1^*\frac{d^2T_q(t)}{dt^2} - \frac{F_0}{\mu}P_2^*T_q(t) - \Gamma\Bigg[2\Bigg(\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k) \\
&+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\Bigg)\frac{d^2T_q(t)}{dt^2} + 4c\Bigg(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k) \\
&+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\Bigg)\frac{dT_q(t)}{dt} + 2c^2\Bigg(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k) \\
&+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\Bigg)T_q(t)\Bigg]\Bigg\} = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s)
\end{aligned} \tag{20}$$

where

$$\Gamma = \frac{M}{L_XL_Y\mu} \tag{21}$$

Equation (20) is the fundamental equation of our problem when the rectangular plate has arbitrary end support conditions. In what follows, we shall discuss two special cases of the equation (20) namely; the moving force and the moving mass problems.

3.1 Case I: rectangular plate traversed by a moving force

By setting $\Gamma = 0$ in equation (20), an approximate model of the differential equation describing the response of a rectangular plate resting on a variable Winkler elastic foundation and traversed by a moving force would be obtained.

Thus, setting $\Gamma = 0$ in equation (20), we have

$$\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{P_1^* R_0}{P^*}\sum_{q=1}^{\infty}\frac{d^2T_q(t)}{dt^2} + \frac{P_2^* F_0}{\mu P^*}\sum_{q=1}^{\infty}T_q(t) = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{22}$$

Evidently, an exact analytical solution to this equation is not possible. Consequently, the approximate analytical solution technique, which is a modification of the asymptotic method of Struble discussed in [20] shall be used.

To solve equation (22), first, we neglect the rotatory inertial term and rearrange the equation to take the form

$$\frac{d^2T_n(t)}{dt^2} + \left[\alpha_n^2 + \Gamma^* P_2^*\right]T_n(t) + \Gamma^* P_2^* \sum_{\substack{q=1 \\ q\neq n}}^{\infty} T_q(t) = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{23}$$

where

$$\Gamma^* = \frac{F_0}{\mu P^*} \tag{24}$$

Consider a parameter $\lambda < 1$ for any arbitrary ratio Γ^* defined as

$$\lambda = \frac{\Gamma^*}{1+\Gamma^*} \tag{25}$$

so that

$$\Gamma^* = \lambda + o(\lambda^2) \tag{26}$$

Substituting equation (26) into the homogenous part of equation (23) yields

$$\frac{d^2T_n(t)}{dt^2} + \left[\alpha_n^2 + \lambda P_2^*\right]T_n(t) + \lambda P_2^* \sum_{\substack{q=1 \\ q\neq n}}^{\infty} T_q(t) = 0 \tag{27}$$

When λ is set to zero in equation (27), a situation corresponding to the case in which the effect of the foundation is regarded as negligible is obtained. In such a case the solution is of the form

$$T_n(t) = C_f \cos(\alpha_n t - \beta) \tag{28}$$

where C_f, α_n and β are constants.

Since $\lambda < 1$ for any arbitrary mass ratio Γ^*, Struble's technique requires that the asymptotic solution of the homogenous part of equation (23) be of the form

$$T_n(t) = A_n(t)\cos\left[\alpha_n t - \Phi_n(t)\right] + \lambda T_1(t) + o(\lambda^2) \tag{29}$$

where $A_n(t)$ and $\Phi_n(t)$ are slowly varying functions of time or equivalently

$$\begin{aligned} \frac{dA_n(t)}{dt} &\rightarrow o(\lambda); & \frac{d^2A_n(t)}{dt^2} &\rightarrow 0(\lambda^2) \\ \frac{d\Phi_n(t)}{dt} &\rightarrow o(\lambda); & \frac{d^2\Phi_n(t)}{dt^2} &\rightarrow 0(\lambda^2) \end{aligned} \tag{30}$$

where $\rightarrow$ implies " is of "

Thus, equation (27) can be replaced with

$$\frac{d^2T_n(t)}{dt^2} + \gamma_s^2 T_n(t) = 0 \tag{31}$$

where

$$\gamma_s = \alpha_n + \frac{\lambda P_2^*}{2\alpha_n} \tag{32}$$

represents the modified frequency due to the effect of the foundation. It is observed that when $\lambda = 0$, we recover the frequency of the moving force problem when the effect of the foundation is neglected.

Using equation (3.31), equation (22) can be written as

$$\frac{d^2T_n(t)}{dt^2} + \gamma_s^2 T_n(t) - \frac{P_1^* R_0}{P^*}\sum_{q=1}^{\infty}\frac{d^2T_q(t)}{dt^2} = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{33}$$

The homogenous part of equation (33) is rearranged to take the form

$$\frac{d^2T_n(t)}{dt^2} + \frac{\gamma_s^2}{1-\lambda_0 P_1^*}T_n(t) - \frac{\lambda_0 P_1^*}{1-\lambda_0 P_1^*}\sum_{\substack{q=1 \\ q\neq n}}^{\infty}\frac{d^2T_q(t)}{dt^2} = 0 \tag{34}$$

where $\lambda_0 = \dfrac{R_0}{P^*}$

Now consider the parameter $\varepsilon_0 < 1$ for any arbitrary mass ratio λ_0 defined as

$$\varepsilon_0 = \frac{\lambda_0}{1+\lambda_0} \tag{35}$$

It can be shown that

$$\lambda_0 = \varepsilon_0 + o(\varepsilon_0^2) \tag{36}$$

Following the same argument, equation (34) can be replaced with

$$\frac{d^2 T_n(t)}{dt^2} + \gamma_{sf}^2 T_n(t) = 0 \tag{37}$$

where

$$\gamma_{sf} = \gamma_s \left[1 + \frac{\varepsilon_0 P_1^*}{2} \right] \tag{38}$$

is the modified frequency corresponding to the frequency of the free system due to the presence of the rotatory inertia. It is observed that when $\varepsilon_0 = 0$, we recover the frequency of the moving force problem when the rotatory inertia effect is neglected.

In order to solve the non-homogenous equation (33), the differential operator which acts on $T_n(t)$ is replaced by the equivalent free system operator defined by the modified frequency γ_{sf}. Thus

$$\frac{d^2 T_n(t)}{dt^2} + \gamma_{sf}^2 T_n(t) = K_0 \Psi_{pi}(ct)\Psi_{pj}(s) \tag{39}$$

where

$$K_0 = \frac{Mg}{P^* \mu} \tag{40}$$

Therefore, the moving force problem is reduced to the non-homogeneous ordinary differential equation given as

$$\frac{d^2 T_n(t)}{dt^2} + \gamma_{sf}^2 T_n(t) = K_0 \Psi_{pj}(s)\left[\sin\alpha_{pi}t + A_{pi}\cos\alpha_{pi}t + B_{pi}\sinh\alpha_{pi}t + C_{pi}\cosh\alpha_{pi}t \right] \tag{41}$$

where $\alpha_{pi} = \dfrac{\Omega_{pi} c}{L_X}$

When equation (41) is solved in conjunction with the initial conditions, one obtains expression for $T_n(t)$. Thus in view of equation (3), one obtains

$$\begin{aligned}W(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty} & \frac{K_0\Psi_{pj}(s)}{\gamma_{sf}[\gamma_{sf}^4-\alpha_{pi}^4]}\Big\{[\gamma_{sf}^2-\alpha_{pi}^2][C_{pi}\,\gamma_{sf}(\cosh\alpha_{pi}t-\cos\gamma_{sf}t) \\ & + B_{pi}(\gamma_{sf}\sinh\alpha_{pi}t-\alpha_{pi}\sin\gamma_{sf}t)]+[\gamma_{sf}^2+\alpha_{pi}^2][A_{pi}\gamma_{sf}(\cos\alpha_{pi}t-\cos\gamma_{sf}t) \\ & -(\alpha_{pi}\sin\gamma_{sf}t-\gamma_{sf}\sin\alpha_{pi}t)]\ \Big\}[\sin\frac{\Omega_{ni}x}{L_X}+A_{ni}\cos\frac{\Omega_{ni}x}{L_X}+B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} \\ & +C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}][\sin\frac{\Omega_{nj}y}{L_Y}+A_{nj}\cos\frac{\Omega_{nj}y}{L_Y}+B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y}+C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}]\end{aligned} \tag{42}$$

Equation (42) represents the transverse displacement response to a moving force of a rectangular plate resting on variable Winkler elastic foundation and having arbitrary edge supports.

3.2 Case II: rectangular plate traversed by a moving mass

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus $\Gamma \neq 0$ and one is required to solve the entire equation (20) when no term of the coupled differential equation is neglected. This is termed the moving mass problem.

Thus, equation (20) can be rewritten in the form

$$\begin{aligned}&\left[1+\frac{2\varepsilon}{P^*}\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\right]\frac{d^2T_n(t)}{dt^2} \\ &+\frac{4\varepsilon c}{P^*}\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_n(t)}{dt} \\ &+\left[\gamma_{sf}^2+\frac{2\varepsilon c^2}{P^*}\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)\right]T_n(t) \\ &+\frac{\varepsilon}{P^*}\sum_{\substack{q=1\\ q\neq n}}\left[2\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2}\right. \\ &4c\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt} \\ &\left.+2c^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \frac{\varepsilon g L_X L_Y}{P^*}\Psi_{pi}(ct)\Psi_{pj}(s)\end{aligned} \tag{43}$$

where $\varepsilon = \dfrac{M}{L_X L_Y \mu}$

we rearrange equation (43) to take the form

$$\frac{d^2T_n(t)}{dt^2} + \frac{\mu_0 R_2(t)}{1+\mu_0 R_1(t)}\frac{dT_n(t)}{dt} + \frac{\gamma_{sf}^2 + \mu_0 R_3(t)}{1+\mu_0 R_1(t)}T_n(t) + \frac{\mu_0}{1+\mu_0 R_1(t)}\sum_{\substack{q=1\\ q\neq n}}^{\infty}\Big[R_1(t)\frac{d^2T_q(t)}{dt^2} + R_2(t)\frac{dT_q(t)}{dt}$$
$$+R_3(t)T_q(t)\Big] = \frac{\mu_0 g L_X L_Y}{[1+\mu_0 R_1(t)]P^*}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{44}$$

where ε has been written as a function of the mass ratio μ_o,

$$R_1(t) = \frac{2}{P^*}\left[\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right]$$

$$R_2(t) = \frac{2c}{P^*}\left[\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right]$$

$$R_3(t) = \frac{2c^2}{P^*}\left[\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right]$$

Considering the homogeneous part of the equation (44) and going through the same arguments and analysis as in the previous case, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is

$$\beta_{sf} = \gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right] \tag{45}$$

retaining terms to $o(\mu_o)$ only.

Thus, to solve the non-homogeneous equation (44), the differential operator which acts on $T_n(t)$ and $T_q(t)$ is replaced by the equivalent free system operator defined by the modified frequency β_{sf}. Therefore, taking into account equations (18) and (19), we have

$$\frac{d^2T_n(t)}{dt^2} + \beta_{sf}^2 T_n(t) = G_0\Psi_{pj}(s)\sin\alpha_{pi}t + A_{pi}\cos\alpha_{pi}t + B_{pi}\sinh\alpha_{pi}t + C_{pi}\cosh\alpha_{pi}t \tag{46}$$

where

$$G_0 = \frac{\mu_0 g L_X L_Y}{P^*} \tag{47}$$

It is noticed that equation (46) is analogous to equation (41) with β_{sf} and G_0 replacing γ_{sf} and K_0 respectively. Therefore, when equation (46) is solved in conjunction with the initial conditions, one obtains expression for $T_n(t)$ and in view of equation (3), one obtains

$$\begin{aligned} W(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty} & \frac{G_0 \Psi_{pj}(s)}{\beta_{sf}[\beta_{sf}^4 - \alpha_{pi}^4]} \Big\{ [\beta_{sf}^2 - \alpha_{pi}^2][C_{pi}\,\beta_{sf}(\cosh\alpha_{pi}t - \cos\beta_{sf}t) \\ & + B_{pi}(\beta_{sf}\sinh\alpha_{pi}t - \alpha_{pi}\sin\beta_{sf}t)] + [\beta_{sf}^2 + \alpha_{pi}^2][A_{pi}\beta_{sf}(\cos\alpha_{pi}t - \cos\beta_{sf}t) \\ & - (\alpha_{pi}\sin\beta_{sf}t - \beta_{sf}\sin\alpha_{pi}t)] \Big\} [\sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} \\ & + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}][\sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}] \end{aligned} \tag{48}$$

Equation (48) is the transverse displacement response to a moving mass of a rectangular plate resting on variable Winkler elastic foundation and having arbitrary edge supports. The constants A_{ni}, A_{pi}, A_{nj}, A_{pj}, B_{ni}, B_{pi}, B_{nj}, B_{pj}, C_{ni}, C_{pi}, C_{nj} and C_{pj} are to be determined from the choice of the end support condition.

4 ANALYSIS OF THE SOLUTION

Next, the phenomenon of resonance is examined. Equation (42) clearly shows that the rectangular plate on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\gamma_{sf} = \frac{\Omega_{pi} c}{L_X} \tag{49}$$

while equation (48) shows that the same plate under the action of a moving mass experiences resonance effect whenever

$$\beta_{sf} = \frac{\Omega_{pi} c}{L_X} \tag{50}$$

where

$$\beta_{sf} = \gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right] \tag{51}$$

Equations (50) and (51) imply that

$$\beta_{sf} = \gamma_{sf}\left[1-\frac{\mu_0}{2}\left(R_1-\frac{R_3}{\gamma_{sf}^2}\right)\right] = \frac{\Omega_{pi}c}{L_X} \tag{52}$$

Consequently from equations (49) and (52), for the same natural frequency, the critical speed (and the natural frequency) for the system of a rectangular plate traversed by a moving mass is smaller than that of the same system traversed by a moving force, for all variants of classical boundary conditions. Thus, for the same natural frequency of the plate, the resonance is reached earlier when we consider the moving mass system than when we consider the moving force system.

5 ILLUSTRATIVE EXAMPLES

In this section, we shall illustrate the foregoing analysis by two practical examples. Particularly we shall consider classical boundary conditions such as clamped end conditions and simple-clamped end conditions.

5.1 Rectangular plate clamped at all edges

For a rectangular plate clamped at all its edges, the boundary conditions are given by

$$W(0,y,t)=0, \qquad W(L_X,y,t)=0\,, \qquad W(x,0,t)=0, \qquad W(x,L_Y,t)=0 \tag{53}$$

$$\frac{\partial W(0,y,t)}{\partial x}=0, \quad \frac{\partial W(L_X,y,t)}{\partial x}=0\,, \quad \frac{\partial W(x,0,t)}{\partial y}=0, \quad \frac{\partial W(x,L_Y,t)}{\partial y}=0 \tag{54}$$

Thus for the normal modes

$$\Psi_{ni}(0)=0, \qquad \Psi_{ni}(L_X)=0\,,\ \Psi_{nj}(0)=0, \qquad \Psi_{nj}(L_Y)=0 \tag{55}$$

$$\frac{\partial \Psi_{ni}(0)}{\partial x}=0, \quad \frac{\partial \Psi_{ni}(L_X)}{\partial x}=0\,, \quad \frac{\partial \Psi_{nj}(0)}{\partial y}=0, \quad \frac{\partial \Psi_{nj}(L_Y)}{\partial y}=0 \tag{56}$$

For simplicity, our initial conditions are of the form

$$W(x,y,0)=0=\frac{\partial W(x,y,0)}{\partial t} \tag{57}$$

Using the boundary conditions and the initial conditions it can be shown that

$$
\begin{aligned}
A_{ni} &= \frac{\sinh\Omega_{ni} - \sin\Omega_{ni}}{\cos\Omega_{ni} - \cosh\Omega_{ni}} = \frac{\cos\Omega_{ni} - \cosh\Omega_{ni}}{\sin\Omega_{ni} + \sinh\Omega_{ni}}, \quad \Rightarrow \quad A_{pi} = \frac{\sinh\Omega_{pi} - \sin\Omega_{pi}}{\cos\Omega_{pi} - \cosh\Omega_{pi}} \\
A_{nj} &= \frac{\sinh\Omega_{nj} - \sin\Omega_{nj}}{\cos\Omega_{nj} - \cosh\Omega_{nj}} = \frac{\cos\Omega_{nj} - \cosh\Omega_{nj}}{\sin\Omega_{nj} + \sinh\Omega_{nj}}, \quad \Rightarrow \quad A_{pj} = \frac{\sinh\Omega_{pj} - \sin\Omega_{pj}}{\cos\Omega_{pj} - \cosh\Omega_{pj}}
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
B_{ni} &= -1, \Rightarrow B_{pi} = -1 \qquad & C_{ni} &= -A_{ni} \Rightarrow C_{pi} = -A_{pi} \\
B_{nj} &= -1 \Rightarrow B_{pj} = -1, \qquad & C_{nj} &= -A_{nj} \Rightarrow C_{pj} = -A_{pj}
\end{aligned} \tag{59}
$$

and from (58), one obtains

$$
\cos\Omega_{ni}\cosh\Omega_{ni} = 1 \tag{60}
$$

which is termed the frequency equation for the dynamical problem, such that [2]

$$
\Omega_1 = 4.73004,\ \Omega_2 = 7.85320,\ \Omega_3 = 10.99561 \tag{61}
$$

Using (58), (59) and (61) in equations (42) and (48) one obtains the displacement response respectively to a moving force and a moving mass of a rectangular plate resting on a variable Winkler elastic foundation and clamped at all its edges.

5.2 Rectangular plate simply supported at edges x = 0, x = L_x and clamped at edges y = 0, y = L_y

For a rectangular plate clamped at edges y = 0, y = L_Y with simple supports at edges x = 0, x = L_X, the boundary conditions at such opposite edges are

$$
W(0,y,t) = 0, \qquad W(L_X,y,t) = 0\,, \quad W(x,0,t) = 0, \qquad W(x,L_Y,t) = 0 \tag{62}
$$

$$
\frac{\partial^2 W(0,y,t)}{\partial x^2} = 0, \quad \frac{\partial^2 W(L_X,y,t)}{\partial x^2} = 0\,, \quad \frac{\partial W(x,0,t)}{\partial y} = 0, \quad \frac{\partial W(x,L_Y,t)}{\partial y} = 0 \tag{63}
$$

and for the normal modes

$$
\Psi_{ni}(0) = 0, \qquad \Psi_{ni}(L_X) = 0\,, \quad \Psi_{nj}(0) = 0, \qquad \Psi_{nj}(L_Y) = 0 \tag{64}
$$

$$
\frac{\partial^2 \Psi_{ni}(0)}{\partial x^2} = 0, \quad \frac{\partial^2 \Psi_{ni}(L_X)}{\partial x^2} = 0\,, \quad \frac{\partial \Psi_{nj}(0)}{\partial y} = 0, \quad \frac{\partial \Psi_{nj}(L_Y)}{\partial y} = 0 \tag{65}
$$

Using the boundary conditions, the following values of the constants and the frequency equation are obtained for the clamped edges.

$$A_{nj} = \frac{\sinh\Omega_{nj} - \sin\Omega_{nj}}{\cos\Omega_{nj} - \cosh\Omega_{nj}}, \quad \Rightarrow \quad A_{pj} = \frac{\sinh\Omega_{pj} - \sin\Omega_{pj}}{\cos\Omega_{pj} - \cosh\Omega_{pj}} \tag{66}$$

$$B_{nj} = -1 \Rightarrow B_{pj} = -1, \; C_{nj} = -A_{nj} \Rightarrow C_{pj} = -A_{pj} \tag{67}$$

The frequency equation of the clamped edges is given by the following determinant equation

$$\begin{vmatrix} (\sinh\Omega_{nj} - \sin\Omega_{nj}) & (\cos\Omega_{nj} - \cosh\Omega_{nj}) \\ (\cos\Omega_{nj} - \cosh\Omega_{nj}) & (\sin\Omega_{nj} + \sinh\Omega_{nj}) \end{vmatrix} = 0 \tag{68}$$

which when simplified yields

$$\cos\Omega_{nj}\cosh\Omega_{nj} = 1 \tag{69}$$

For the simple edges, it is readily shown that

$$A_{ni} = 0,\ B_{ni} = 0,\ C_{ni} = 0, \text{ and } \boldsymbol{\Omega}_{ni} = n_i\boldsymbol{\pi} \tag{70}$$

Similarly,

$$A_{pi} = 0,\ B_{pi} = 0,\ C_{pi} = 0, \text{ and } \boldsymbol{\Omega}_{pi} = p_i\boldsymbol{\pi} \tag{71}$$

Using (66), (67), (69) and (70) in equations (42) and (48) one obtains the displacement response respectively to a moving force and a moving mass of a simple-clamped rectangular plate resting on a variable Winkler elastic foundation.

6 NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

In order to carry out the calculations of practical interests in dynamics of structures and Engineering design for the illustrative examples, a rectangular plate of length $L_Y = 0.914$m and breadth $L_X = 0.457$m is considered. It is assumed that the mass travels at the constant velocity 0.8123m/s. Furthermore, values for E, S and Γ are chosen to be $3.109\text{x}10^9$kg/m^2, 0.4m and 0.2 respectively. For various values of the foundation moduli F_0 and the rotatory inertia correction factor R_0, the deflections of the plate for all the illustrative examples are calculated and plotted against time t.

Figures 6.1 and 6.2 display the effect of foundation modulus (F_0) on the transverse deflection of the clamped rectangular plate in both cases of moving force and moving mass respectively. The graphs show that the response amplitude decreases as the value of the foundation modulus increases.

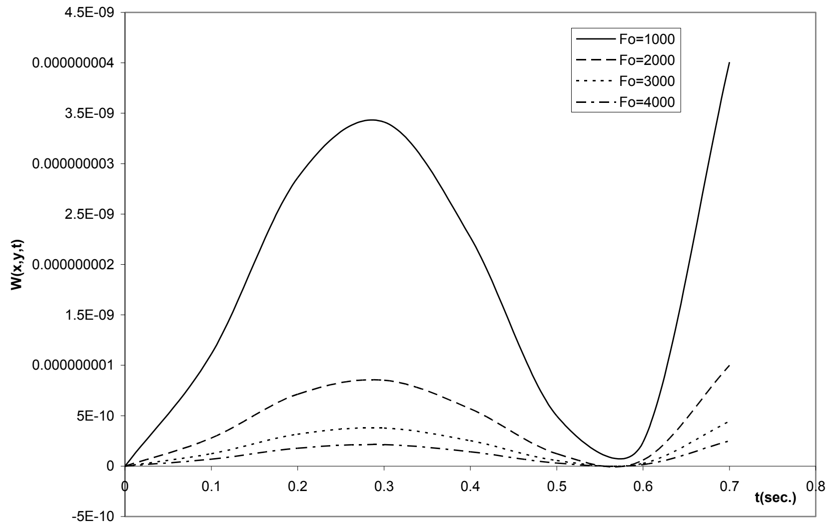

Figure 6.1 Displacement of moving force for clamped rectangular plate on variable Winkler foundation for various values of foundation modulus Fo.

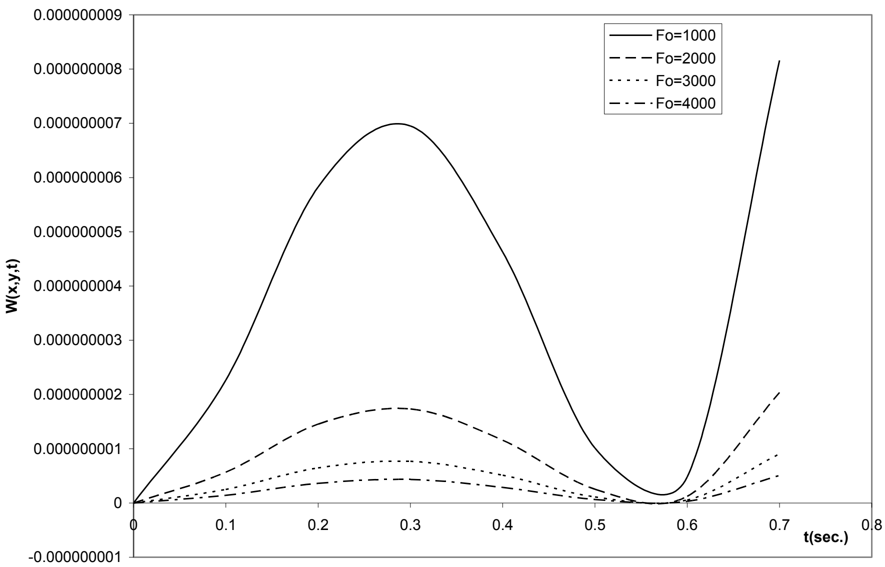

Figure 6.2 Displacement of moving mass for clamped rectangular plate on variable Winkler foundation for various values of foundation modulus Fo.

The effect of rotatory inertia correction factor (R_0) on the transverse deflection in both cases of moving force and moving mass displayed in figures 6.3 and 6.4 respectively show that an increase in the value of the rotatory inertia correction factor decreases the deflection of the simple-clamped rectangular plate resting on variable Winkler elastic foundation.

Figure 6.5 compares the displacement curves of the moving force and moving mass for a simple-clamped rectangular plate for fixed F_0 and R_0, the response amplitude of a moving mass is greater than that of a moving force problem. This result holds for other choices of classical boundary conditions.

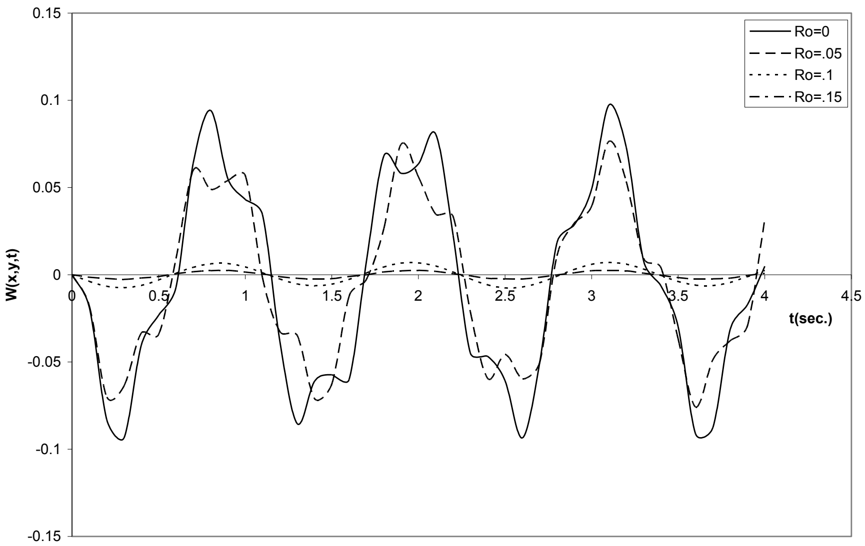

Figure 6.3 Displacement profile of moving force for simple-clamped rectangular plate on variable Winkler foundation for various values of rotatory inertia correction factor Ro.

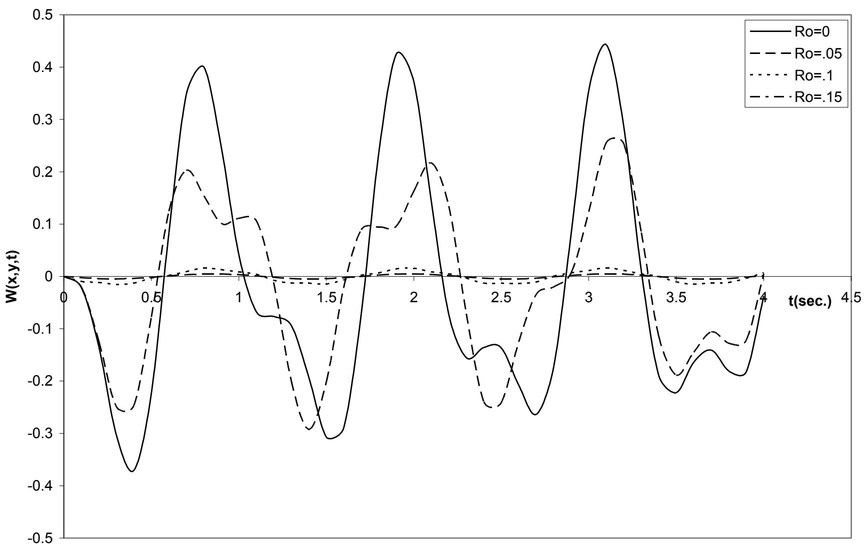

Figure 6.4 Displacement profile of moving mass for simple-clamped rectangular plate on variable Winkler foundation for various values of rotatory inertia correction factor Ro.

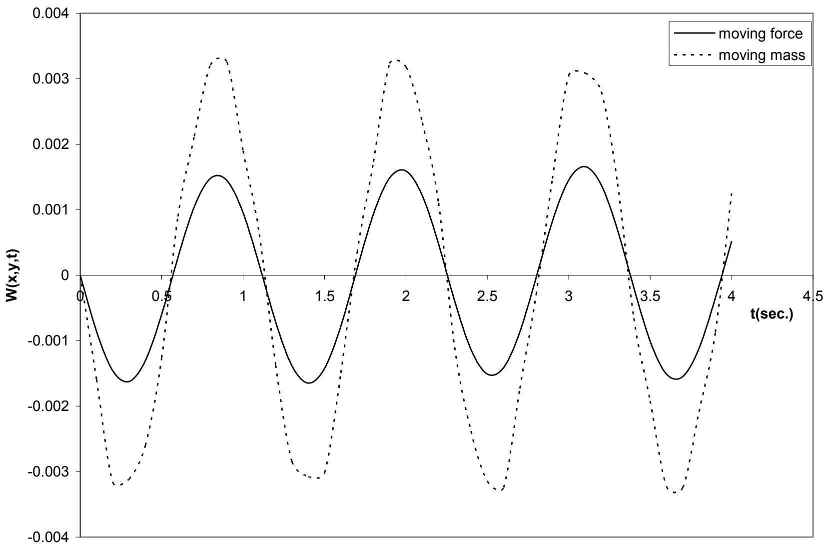

Figure 6.5 Comparison of the deflections of moving force and moving mass cases for simple-clamped rectangular plate on variable Winkler foundation with Fo=1000 and Ro=4

7 CONCLUSION

The dynamic response to moving masses of rectangular plates with general boundary conditions and resting on Winkler elastic foundation with stiffness variation is considered in this work. The fourth order partial differential equation governing the system is a non-homogenous equation with variable and singular coefficients. The method based on Separation of variables is used to transform the governing equation to a set of coupled second order ordinary differential equations. The modified Struble's technique and the method of integral transformations are employed to obtain the closed form solution of the transformed equation for both cases of moving force and moving mass problems.

From the analyses of the solutions, the resonance conditions are obtained for the problem for all variants of classical boundary conditions. The numerical analyses are carried out for both moving force and moving mass problems for two illustrative examples of classical boundary conditions; (i) clamped ends condition and (ii) simple-clamped ends condition. The analyses show that the moving force solution is not an upper bound for the accurate solution of the moving mass problem and that as the rotatory inertia correction factor increases, the response amplitudes of the plates decrease for both cases of moving force and moving mass problem. The displacements of the rectangular plates resting on variable Winkler elastic foundations decrease as the foundation modulus increases when the rotatory inertia correction factor is fixed.

Furthermore, the response amplitude for the moving mass problem is greater than that of the moving force problem for fixed values of rotatory inertia correction factor and foundation modulus, this implies that resonance is reached earlier in moving mass problem than in moving force problem of the rectangular plate resting on Winkler elastic foundation with stiffness variation. It is therefore unsafe to rely on the moving force solutions.

For the rectangular plate with general classical boundary conditions and resting on Winkler elastic foundation with stiffness variation, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem for all variants of classical boundary conditions, and as rotatory inertia correction factor and the foundation modulus increase, the critical speeds increase indicating a safer dynamical system.

Finally, the results in this work agree with what obtain in literature [21, 22]. Hence the method employed in this work is accurate and the solutions are convergent.

References

[1] Fryba, L. (1972): Vibration of solids and structures under moving loads. Groningen: Noordhoff.

[2] Oni, S. T. (1991): On the dynamic response of elastic structures to moving multi-mass system. Ph.D. thesis. University of Ilorin, Nigeria.

[3] Inglis, C. E. (1934): A mathematical treatise on vibration in railway bridges. The University press, Cambridge.

[4] Gbadeyan, J. A. and Aiyesimi, Y. M. (1990): Response of an elastic beam resting on viscoelastic foundation to a load moving at non-uniform speed. Nigerian Journal of Mathematics and Applications, Vol.3, pp 73-90.

[5] Sadiku, S. and Leipholz, H. H. E. (1981): On the dynamics of elastic systems with moving concentrated masses. Ing. Archiv. 57, 223-242.

[6] Gbadeyan, J. A. and Oni, S. T. (1995): Dynamic behaviour of beams and rectangular plates under moving loads. Journal of Sound and Vibration. 182(5), pp 677-695.

[7] Jaeger, J. C. and Strarfield, A. M. (1979): An introduction to applied Mathematics. Oxford University Press, pp 295-296.

[8] Timoshenko, S. (1921): On the correction for shear of the differential equation for transverse vibration of prismatic bars. Phil. Mag. Ser. 6, Vol. 41, pp 744-776.

[9] Steel, C. R. (1971): Beams and shells with moving loads. Int. J. Solids and Structures, Vol. 7, pp 1171-1198.

[10] Steele, C. R. (1967): The finite beam with a moving load. Journal of Applied Mechanics, Vol. 34, No. 1, pp 111-118.

[11] M. R. Shadnam, M. Mofid and J. E. Akin (2001): On the dynamic response of rectangular plate, with moving mass. Thin-Walled Structures, 39(2001), pp 797 – 806.

[12] S. T. Oni (2004): Flexural motions of a uniform beam under the actions of a concentrated mass traveling with variable velocity. Abacus, Journal of Mathematical Association of Nigeria. Vol. 31, No 2a, pp 79 – 93.

[13] Oni S. T. and Omolofe B. (2005): Dynamic analysis of a prestressed elastic beam with general boundary conditions under moving loads at varying velocities. Journal of Engineering and Engineering Technology,FUTA,Vol. 4,No1, pp55–72.

[14] Oni S. T. and Awodola T. O. (2003): Vibrations under a moving load of a non-uniform Rayleigh beam on variable elastic foundation. Journal of Nigerian Association of Mathematical Physics, Vol. 7, pp 191 – 206.

[15] Omer Civalek and Aitung Yauas (2006): Large deflection static analysis of rectangular plates on two parameter elastic foundations. International Journal of Science and Technology. Vol. 1, No. 1, pp 43 – 50.

[16] G. G. Adams (1995): Critical speeds and the response of a tensioned beam on an elastic foundation to repetitive moving loads. Int. Jour. Mech. Sci., Vol. 7, pp 773 – 781.

[17] E. Savin (2001): Dynamics amplification factor and response spectrum for the evaluation of vibrations of beams under successive moving loads. Journal of Sound and Vibrations, 248(2), pp 267 – 288.

[18] Jia-Jang Wu (2006): Vibration analysis of a portal frame under the action of a moving distributed mass using moving mass element. Int. Jour. for Numerical Methods in Engineering, 62, pp 2028 – 2052.

[19] S. T. Oni, and T. O. Awodola (2010): Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. Latin American Journal of Solids and Structures. 7, pp 3 - 20.

[20] S. T. Oni and T. O. Awodola (2011): Dynamic behaviour under moving concentrated masses of simply supported rectangular plates resting on variable Winkler elastic foundation. Latin American Journal of Solids and Structures (LAJSS), vol. 8, pp 373-392.

[21] S. T. Oni and B. Omolofe (2011): Vibration analysis of non-prismatic beam resting on elastic subgrade and under the actions of accelerating masses. Journal of the Nigerian Mathematical Society (JNMS), Vol. 30, pp. 63-109.

[22] S. T. Oni and O. K. Ogunbamike (2011): Convergence of closed form solutions of the initial-boundary value moving mass problem of rectangular plates resting on Pasternak foundations. Journal of the Nigerian Association of Mathematical Physics (J. of NAMP), Vol. 18 (May, 2011), pp. 83-90.