The asymptotic solutions for boundary value problem to a convective diffusion equation with chemical reaction near a cylinder.

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Abstract

The work deals with a boundary value problem for a quasilinear partial elliptical equation. The equation describes a stationary process of convective diffusion near a cylinder and takes into account the value of a chemical reaction for large Peclet numbers and for large constant of chemical reaction. The quantity the rate constant of the chemical reaction and Peclet number is assumed to have a constant value. The leading term of the asymptotics of the solution is constructed in the boundary layer as the solution for the quasilinear ordinary differential equation. In this paper, we construct asymptotic expansion of solutions for a quasilinear partial elliptical equation in the boundary layer near the cylinder.

Keywords
convective diffusion equation, the method of matched asymptotic expansions, the diffusion boundary layer, the saddle point, the stream function, quasilinear parabolic degenerate equation, the stability condition for difference scheme.

1 Introduction

The stationary convective diffusion equation in the presence of a bulk chemical reaction is given by (e.g., see [1, 2])

$$\Delta U = Pe(\overline{V},\nabla)\cdot U + k_v F(U), \tag{1.1}$$

$$U = 1 \text{ at } r = 1\,;\; U \to 0 \text{ when } r \to \infty, \tag{1.2}$$

where

$$\overline{V} = (V_\gamma, V_\theta, 0)\,,\; V_\gamma = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, V_\theta = -\frac{\partial\psi}{\partial r}, \tag{1.3}$$

$$\psi(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta \tag{1.4}$$

is the stream function [3], r and θ are polar coordinates, Δ is the Laplace operator, Pe is the Peclet number, and k_V is parameter depending on the chemical reaction rate. The angle θ is measured relative to the free-stream direction.

Problems analogous to (1.1) and (1.2), and a broader class of problems, were considered in [1,2], [4-6, 9, 10]. In the absence of chemical reaction, problem (1.1) and (1.2) was analyzed in [4, 5] by the method of matched asymptotic expansions [7, 8]. It is well known (see, for example, [2, Chapter 5, (6.1)-(6.3)]) that, in the limit cases $Pe >> 1$, $k_V = const$; $Pe = const$ and $k_V >> Pe$, the solution to problem (1.1) and (1.2) is simplified.

In the case when the volume chemical reaction of the first order (F(u) = u) the asymptotics of solution in all space outside the drop was constructed in [9]. In study, the number $\mu_0 = k_V / Pe$ is assumed to have a constant value.

It is assummed that F(C) is continuous and

$$F\colon R1 \to R1,\ F(0) = 0,\ F'(0)=0,\ 0 < F''(C), \tag{1.5}$$

and the asymptotic is

$$F(u) = u^2 + F_3 u^3 + F_4 u^4 + F_5 u^5 + O(u^6) \text{ for } u \to 0. \tag{1.6}$$

2 The diffusion boundary layer.

In this report the quantity $\mu = k_V / Pe$ is assumed to have a constant value. In this case, all terms in Eq. (1) are similar in order of magnitude in the neighborhoods of saddle points. The small parameter $\varepsilon = (Pe)^{-1/2}$ is introduced for convenience, and Eq. (1) is rewritten as

$$\varepsilon^2 \Delta u - \frac{1}{r}\left(\frac{\partial u}{\partial r}\frac{\partial \psi}{\partial\theta} - \frac{\partial u}{\partial\theta}\frac{\partial\psi}{\partial r}\right) - \mu_0 F(u) = 0. \tag{2.1}$$

When $\varepsilon = 0$ the Eq. (1) has the suddle points $O_1(1,\pi)$ and $O_2(1,0)$ and the equation is equivalent the dynamical system.

The asymptotic expansions (AE) of the solution in the diffusion boundary layer was considered in a earlier study [11]. This solution was continued up to the front stagnation point $O_1(1,\pi)$ (up to the line $\theta = \pi$). The natural variables in the diffusion boundary layer are $t=\varepsilon^{-1}(r-1),\theta$. The AE of the solution $u(t,\theta,\varepsilon)$ is sought as

$$u(t,\theta,\varepsilon)=u_0(t,\theta)+\varepsilon u_1(t,\theta)+\dots \tag{2.2}$$

From (2.1), (2.2) and (1.1) – (1,4), in variables t,θ , determining $u_0(t,\theta)$ in the domain $0<\theta<2\pi,\ 0<t$, we obtain the problem

$$\frac{\partial^2 u_0}{\partial r^2}-2t\cos\theta\frac{\partial u_0}{\partial r}+2\sin\theta\frac{\partial u_0}{\partial\theta}-\mu F(u_0)=0\,, \tag{2.3}$$

$$u_0(0,\theta)=1;\quad u_0(t,\theta)\to 0 \text{ as } t\to\infty. \tag{2.4}$$

The asymptotics of the solution to the problem (2.3), (2.4) function $u_0(t,\theta)$ as $\theta\to\pi$ is [11]

$$u_{0,0}(t)+O((\pi-\theta)^2\exp(-\delta t^2)),$$

where $u_{0,0}(t)=O(\exp(-\delta t^2)),\ \ \delta>0$.

3 The asymptotics $u_0(t,\theta)$ as $\theta\to 0$.

The asymptotics of the function $u_0(t,\theta)$ as $\theta\to 0$ is sought in the view

$$V_0(t)+O(\theta^2), \tag{3.1}$$

where the function $V_0(t)$ is constructed [12] for small μ as the solution for the problem

$$LV_0-\mu F(V_0)=V_0''(t)-tV_0'(t)-\mu F(V_0(t))=0 \tag{3.2}$$

$$V_0(0)=1,\ V_0(t)=O(1) \text{ as } t\to\infty\,. \tag{3.3}$$

Theorem 1. Let F(u) satisfies conditions (1.5), (1.6) and $\mu=const$, then at $t\to\infty$ the solution of the equation (3.2) asymptotics holds

$$V_0(t)=\frac{c_{01}}{\mu\ln t+C}+\frac{c_{02}}{(\mu\ln t+C)^2}+\frac{c_{03}}{(\mu\ln t+C)^3}+\dots+\ln(\mu\ln t+C)\left(\frac{c_{12}}{(\mu\ln t+C)^2}+\frac{c_{13}}{(\mu\ln t+C)^3}+\dots\right)+$$
$$+\ln^2(\mu\ln t+C)\left(\frac{c_{23}}{(\mu\ln t+C)^2}+\dots\right)+\dots \tag{3.4}$$

where

$$c_{0,1}=1\,,\ c_{0,2}-const\,,\ c_{1,2}=-c_{0,1}^3F_3\,,\ c_{2,3}=c_{1,2}^2(3-2c_{0,1})^{-1}\,,$$

$$c_{1,3}=\frac{1}{3-2c_{0,1}}(3F_3c_{0,1}^2c_{1,2}+2c_{2,3}+2c_{0,2}c_{1,2})\,,\ c_{0,3}=\frac{1}{3-2c_{0,1}}(c_{1,3}+c_{0,1}^4F_4+c_{0,2}^2+3F_3c_{0,1}^2c_{0,2})\,,\ \dots\,.$$

The idea of the proof is similar to works [13, 14]. Let us search the function $V_0(t)$ in the form of the sum

$$V_0(t)=V_n(t)+w(t)\,, \tag{3.5}$$

where

$$V_n(t)=\sum_{i=0}^{n-1}\ln^i(\mu\ln(t)+C)\sum_{k=i+1}^{n}\frac{c_{i,k}}{(\mu\ln(t)+C)^k}\,.$$

Substituting sum (3.5) into equation (3.2), we obtain the problem

$$Lw-\mu(F(w+V_n)-F(V_n))=H_{n-1}(t)\,, \tag{3.6}$$

$$w(t)\to 0,\ w'(t)\to 0, \text{ for } t\to\infty\,, \tag{3.7}$$

where $H_n(t)=O\left((\ln t)^{-n-1+\delta}\right)$, δ -sufficiently small.

Let's consider the problem

$$w''-t\,w'-\mu F'(V_n)w=h(t,V_n,w), \tag{3.8}$$

$$w(t)\to 0,\ w'(t)\to 0, \text{ for } t\to\infty\,, \tag{3.9}$$

where problem (3.6), (3.7) is equivalent to a problem (3.8), (3.9) and

$$h(t,V_n,w)=g(t,V_n,w)+H_{n-1}(t),\ \ g(t,V_n,w)=\mu(F(w+V_n)-F(V_n)-F'(V_n)w)\,,\ g(t,V_n,w)=O\left(w^2\right)\,. \tag{3.10}$$

For construction the solution $w(x)$ of the problem (3.8), (3.9) we obtain integral equation

$$w(t)=-\int_t^{\infty}W^{-1}(s)\big(\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)\big)h(s,V_n,w)ds, \tag{3.11}$$

where $\varphi_1(t),\varphi_2(t)$ are linearly independent solutions to the linear homogeneous equation:

$$w''-t\,w'-\mu F'(V_n)w=0, \tag{3.12}$$

$W(t)=\exp(t^2/2)$ is the Wronskian.

We have asymptotics for $\varphi_1(t)$, $\varphi_2(t)$, using the results of the works [15, 16]

$$\varphi_1(t)=(\mu\ln t+C)^{-2}\left(1+O\left((\ln t)^{-1+\delta}\right)\right)\text{ for } t\to\infty \tag{3.13}$$

$$\varphi_2=e^{\frac{t^2}{2}}t^{-1}(\mu\ln t+C)^2\left(1+O\left((\ln t)^{-1+\delta}\right)\right)\text{ for } t\to\infty \tag{3.14}$$

where $0<\delta$ - is small. Such solutions $\varphi_1(t),\varphi_2(t)$ of the equation (3.12) exists.

For $\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)$ we find estimate

$$\begin{aligned}\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)&=e^{\frac{s^2}{2}}s^{-1}(\mu\ln s+C)^2(\mu\ln t+C)^{-2}\left(1+O\left((\ln t)^{-1+\delta}\right)+O\left((\ln s)^{-1+\delta}\right)\right)-\\&-e^{\frac{t^2}{2}}t^{-1}(\mu\ln t+C)^2(\mu\ln s+C)^{-2}\left(1+O\left((\ln s)^{-1+\delta}\right)+O\left((\ln t)^{-1+\delta}\right)\right).\end{aligned} \tag{3.15}$$

We proceed by applying the method of successive approximations.

$$w_{n+1}(t)=-\int_t^{\infty}W^{-1}(s)\big(\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)\big)h(s,V_n,w_n)ds \tag{3.16}$$

We choose $w_0\equiv 0$,

$$w_1(t)=-\int_t^{\infty}W^{-1}(s)\big(\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)\big)H_{n-1}(s)ds\ . \tag{3.17}$$

From (3.6), (3.13) – (3.17) it is find estimate

$$|w_1|\le M(\ln t)^{-n+\delta}\ , \tag{3.18}$$

then by formulas (3.10), (3.13) – (3.18) we have

$$\begin{aligned}|w_2(t)-w_1(t)|&\le\left|-\int_t^{\infty}W^{-1}(s)\big(\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)\big)g(V_n,w_1)ds\right|\le\\&\le-\int_t^{\infty}W^{-1}(s)\big(\varphi_1(t)\varphi_2(s)-\varphi_1(s)\varphi_2(t)\big)g'(V_n,\overline{w}_1)w_1ds\le\mu MK(\ln t)^{-2n+1+2\delta}\le\frac{M}{2}(\ln t)^{-n+\delta},\ t>>1,\end{aligned} \tag{3.19}$$

$$|g(V_n,w_2)-g(V_n,w_1)|\le g'(\overline{w},V_n)|w_2-w_1|\le\mu MK(\ln t)^{-2n+2\delta}.$$

From (3.19) we obtain $|w_3(t)-w_2(t)|\le\frac{M}{2^2}(\ln t)^{-n+\delta}$, $\delta>0$ and $\forall n\ge 3$ $|w_{n+1}(t)-w_n(t)|<\frac{M}{2^n}(\ln t)^{-n+\delta}$.

There exist $M>0$ that for solution of the equation (3.11) inequality is hold

$$|w(t)|\le 2M(\ln t)^{-n+\delta}.$$

4 Numerical solution and finding the constant C .

We rewrite Eq. (3.2) in the form of the system

$$\begin{cases} v_0'(t)=z(t)\\ z'(t)=t\,z(t)+\mu\cdot F(v_0(t)).\end{cases} \tag{4.1}$$

Consider system (4.1) on the interval [0, X_0], Following [17, 18], we first discuss the stability conditions for the explicit Euler scheme

$$\begin{cases} v_{n+1}=v_n+hz_n\\ z_{n+1}=z_n+h[x_nz_n+\mu F(v_n)]\end{cases} \tag{4.2}$$

Replacing $F(v_n)$ by the sum $F(v_0)+F'(v_0)(v_n-v_0)$ and assuming that t_0,v_0, and z_0 are known and $t_n=t_{n-1}+h$, we find a solution to difference scheme (4.2).

The stability condition for difference scheme (4.2) is fulfilled [17-19] if

$$h\cdot\mu\cdot F'(t)|<<1,h<0,|hX_0|<1\ .$$

This implies that one should take X_0 and integrate backwards (i. e., with increments $h<0$) in the interval [0, X_0]. The initial conditions at the point X_0 the form

$$v_0(X_0) = V_{0,} \quad z(X_0) = Z_0, \tag{4.3}$$

where V_0, Z_0 are found from (3.4)

$$V_0(t) = \frac{c_{01}}{\mu \ln t + C} + \frac{c_{02}}{(\mu \ln t + C)^2} + \frac{c_{03}}{(\mu \ln t + C)^3} + \ln(\mu \ln t + C)\left(\frac{c_{12}}{(\mu \ln t + C)^2} + \frac{c_{13}}{(\mu \ln t + C)^3}\right),$$

$$Z_0 = \frac{-\mu c_{01}}{t(\mu \ln t + C)^2} - \frac{\mu c_{02}}{t(\mu \ln t + C)^3} - \frac{\mu c_{03}}{t(\mu \ln t + C)^4} + \frac{\mu}{t(\mu \ln t + C)}\left(\frac{c_{12}}{(\mu \ln t + C)^2} + \frac{c_{13}}{(\mu \ln t + C)^3}\right) -$$

$$-\frac{\mu \ln(\mu \ln x + C)}{t}\left(\frac{2c_{12}}{(\mu \ln t + C)^3} + \frac{3c_{13}}{(\mu \ln t + C)^4}\right).$$

For example, let $F(u) = \ln^2(1+u)$.

The results of the numerical analysis of the problem (4.1), (4.3) for $F(u) = \ln^2(1+u)$ are (for $\mu \in [0.5,2]$, $c_{0,2} = 1$)

μ = 0.5, C_0 = 1.7216, z(0) = -0.1173; μ = 1, C_0 = 1.3943, z(0) = -1.3943;

μ = 1.5, C_0 = 1.0452, z(0) = -0.2994; μ = 2, C_0 = 0.6737, z(0) = -0.3747.

This research was supported by the Presidents Grant for Leading Scientific Schools of the Russian Federation, project no. NSh - 6249.2010.1, and by the Federal Task Program "Research and educational professional community of innovation Russia" (contract 02.740.11.0612).

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