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Compliance minimization of structures under uncertain loadings

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Abstract

This paper investigates compliance optimization of structures under multiple load cases. The problem can be solved through a multi-criterion optimization where the load cases associated with each and every loading configuration are treated as components of a multi-objective function vector. Alternatively, the multi-objective optimization problem can be re-formulated using a minimax strategy that does not require simultaneous consideration of all the load cases as components of a multi-objective function vector. Instead, it is shown that, for compliance optimization purposes, it is sufficient to consider only those loads which define the convex hull of the applicable load set, thereby drastically reducing the number of load cases involved in the design procedure. The efficiency of the technique proposed is illustrated through two examples consisting of variable thickness beams and plates subjected to uncertain loadings.

Keywords: compliance minimization, uncertain loading, minimax strategy

1 Introduction

It is generally accepted that optimization techniques result in tremendous design improvements in the aerospace, mechanical, naval and civil engineering. However, difficulties associated with a large number of potential load cases may lead to designs optimized to withstand only one or a few load cases. This inevitably implies in a vulnerability of the optimal design to load cases that were not included in the optimization procedure. Unfortunately, typical aerospace and mechanical structural components are subjected not only to a few load cases but hundreds or even thousands of load cases what renders severe the problem of vulnerability.

For instance, aerospace structures must be lightweight and meet a number of design criteria such as minimum buckling load, maximum stress, maximum displacement, etc. The load cases applied to an aircraft wing lead to deformation of all its components. A global finite element model of an aircraft wing contains more than 200 load cases. One static analysis of the complete wing simultaneously contemplates all load cases.

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Notation

$\mathbf{A}$	symmetric, positive-definite matrix
$\mathbf{C}$	compliance
f_i	load component of load vector $\boldsymbol{f}$
$\overline{f}$	equivalent uniformly distributed loading
$\overline{f}_i$	scaling factor
$\mathbf{f}$	load vector
F_i	loading component net force
$\mathcal{F}$	admissible load space
h	equivalent thickness
h_{base}	thickness of base structure
h_i	thickness at node i
$\mathbf{h}$	vector of design variables (thicknesses)
$\mathcal{H}$	feasible design space
$\mathbf{K}$	stiffness matrix
l	number of load cases
L	beam length
n	number of nodes
R_i	proportionality parameter
ϕ	auxiliary objective function
Ω	arbitrary set
Ω_{hull}	hull of Ω

However, in sizing optimization alone, the consideration of all load cases leads to a huge number of maximum stress constraints whose treatment is a fabulous challenge even for the most sophisticated algorithms and computers available nowadays. This practical difficulty suggests that innovative methods and techniques must be proposed that can simultaneously handle all load cases without excessively burdening the optimization process. New approaches must be proposed to encompass the entire load space in the optimization or to quickly indicate which load cases (if any) do not need to be considered for optimization purposes.

It is reasonable to assume that a structural component is subjected to an enormous multiplicity of load cases during its operation. Hence, one approach even more advantageous than the multiple load case approach is the uncertain load approach since it is more comprehensive in the sense that it considers, strictly speaking, an infinite number of load cases. Since there are hundreds or even thousands of load cases typically involved in practical structural design one can admit that these load cases belong to a well-defined load space and then propose a strategy that optimizes the structure against the entire load space instead of a finite number of load cases. The proposed strategy would then be more conservative because it would also subject the structure to loadings that were initially unspecified, i.e., that were not originally eligible load cases.

Optimization techniques of the nature described above are usually probabilistic since they describe the load space through probability density distributions where the elements of the load space have distinct probabilities of occurrence. A difficulty associated with that approach is the selection of the loading configurations that have the highest probability of occurrence and what probability distribution is that [6].

The technique developed in this paper is based on convex modelling [1] where a load space is defined and all the elements of that load space have equal probability of occurrence. The optimization technique in this case does not provide designs based on probability distributions but designs based on extremal properties that depend on the load space chosen. The outcome of the technique is an optimal design for which one loading or several loadings of the load space are the most dangerous or harmful to the structure. On the other hand, it is guaranteed that all the other loadings contained in the load space are conservative in the sense that they are less harmful to the optimal design.

2 The Compliance Minimization Strategy

The objective of this section is to investigate the linear problem of minimum strain energy stored in a given structure when it is subjected to uncertain loads. The strain energy, also referred to as compliance, is an indicative measure of the structure flexibility. It gives the designer an idea of how much the structure deforms, its levels of stress and strain and can be useful in estimating natural frequencies through Rayleigh's quotient [7, 8, 11, 12]

The minimax strategy is employed to handle the load uncertainties. Technically, it is desired to minimize the compliance of structures subjected to uncertain loads. The uncertain loads are assumed to belong to a meaningful load space such that any element of that space is denoted by $\mathbf{f}$. When $\mathbf{f}$ is fixed, i.e., one particular loading is selected, the compliance minimization problem can be stated as in equation (1)

$$\min_{\mathbf{h}\in\mathcal{H}} C(\mathbf{h},\mathbf{f}) \tag{1}$$

where $\mathcal{H}$ is the design space where all constraints on $\mathbf{h}$ are satisfied. The design variables $\mathbf{h}$ can be regarded as usual design variables (thickness distribution, fiber angle orientation, number of plies, position of stiffeners, etc). However, $\mathbf{f}$ is uncertain and, therefore, it should not be fixed. Following the multi-objective optimization approach one can state the problem as in equation (2) where now there are l load cases, each one corresponding to a component of the multi-objective function vector [3].

$$\min_{\mathbf{h}\in\mathcal{H}} \left\{ \begin{array}{c} C_1(\mathbf{h},\mathbf{f}_1) \\ C_2(\mathbf{h},\mathbf{f}_2) \\ \dots \\ C_{l-1}(\mathbf{h},\mathbf{f}_{l-1}) \\ C_l(\mathbf{h},\mathbf{f}_l) \end{array} \right\} \tag{2}$$

As stated in equation (2) the problem must be solved through a multi-objective function optimization approach. However, it is possible to re-formulate the problem and cast it into a minimax formulation:

$$\min_{\mathbf{h}\in\mathcal{H}} \max_{\mathbf{f}\in\mathcal{F}} C(\mathbf{h},\mathbf{f}) \tag{3}$$

where $\mathcal{F}$ is the load space that necessarily contains $\mathbf{f}_1$, $\mathbf{f}_2$, ..., $\mathbf{f}_l$. The question of whether the "min" and the "max" operators can be switched in equation (3) depends ultimately on the existence of a saddle point [5]. Nonetheless, existence of a saddle point cannot be assumed beforehand and, hence, in principle, the order of the operators must not be switched.

The optimization problem stated in equation (3) is bilevel. Its solution yields simultaneously the best design described in terms of $\mathbf{h}$ and the worst loading configuration described in terms of $\mathbf{f}$. Equation (3) can be written in a slightly modified version.

$$\min_{\mathbf{h}\in\mathcal{H}} \max_{\mathbf{f}\in\mathcal{F}} C(\mathbf{h},\mathbf{f}) = \min_{\mathbf{h}\in\mathcal{H}} \phi(\mathbf{h}) \quad , \quad \phi(\mathbf{h}) = \max_{\mathbf{f}\in\mathcal{F}} C(\mathbf{h},\mathbf{f}) \tag{4}$$

The problem given in equation (4) consists in maximizing the objective function ϕ with respect to the uncertain loads $\mathbf{f}$ and, subsequently, minimizing it with respect to $\mathbf{h}$. This formulation guarantees that, for the optimal design, any variation of $\mathbf{f}$ within the admissible load space $\mathcal{F}$ necessarily results in a better (lower) compliance. This is because of the "max" part of the bilevel optimization.

Basic concepts relating to convex sets, convex hulls and convex combinations will be briefly described and two propositions required to fully understand the coming optimization strategy will be presented.

The convex hull Ω_{hull} of a given set Ω is the smallest subset of Ω such that each and every element in Ω can be written as a convex combination of the elements in Ω_{hull}. The conclusion drawn from this definition is that the convex hull of any convex polyhedron is the collection of all its vertices. Figure 1 depicts a convex set whose boundary is a convex polygon with its vertices emphasized. The convex hull of this set corresponds to the vertices of the polygonal line.

Proposition 1. If $\mathbf{A}$ is a symmetric and positive-definite matrix, then the quadratic form $C(\mathbf{f}) = \mathbf{f}^T\mathbf{A}\mathbf{f}$ is a convex function in $\mathbf{f}$.

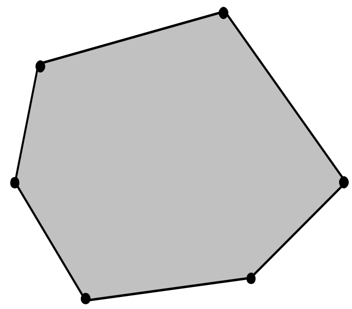

Figure 1: Convex hull of a convex set

Proposition 2. Given n real numbers a_1, a_2, ..., a_n the maximum value of $f = \sum_{i=1}^{n} \xi_i a_i^2$ where $\sum_{i=1}^{n} \xi_i = 1$ and $\xi_i \geq 0$ for $i \in [1, ..., n]$ is $f_M = a_M^2$, a_M^2 being the maximum of all a_i^2 for $i \in [1, ..., n]$.

Combining both propositions it is concluded that, given l load cases $\mathbf{f}_1$, $\mathbf{f}_2$, ..., $\mathbf{f}_l$ and all the load cases resulting from their convex combinations, the maximum of $C(\mathbf{f}) = \mathbf{f}^T\mathbf{A}\mathbf{f}$ (a convex function) is necessarily associated with the load case M that yields the maximum $C(\mathbf{f}_M) = \mathbf{f}_M^T\mathbf{A}\mathbf{f}_M$.

The minimax problem stated in equation (4) must be solved in order to obtain optimal designs that can satisfactorily withstand any load within the admissible load space $\mathcal{F}$. The inner optimization in equation (4) requires computation of function $\phi(\mathbf{h})$ what is apparently a laborious task. However, when a static linear finite element structural analysis is considered, the expression for the compliance is simply $C(\mathbf{f}) = \mathbf{f}^T\mathbf{K}^{-1}\mathbf{f}$ where $\mathbf{f}$ are the load vectors and $\mathbf{K}$ is the stiffness matrix that is always symmetric and positive-definite. Hence, the two propositions previously presented guarantee that the entire load space $\mathcal{F}$ must not be searched but only its convex hull, i.e., the points that belong the to the convex hull of the load space are individually assessed and the maximum compliance associated is the value of $\phi(\mathbf{h})$. The situation can be better understood through visualization of the compliance surface and the admissible load space as in Fig. 2.

Only two load components f_i and f_j are illustrated in Fig. 2 for visualization purposes. The compliance surface passes through the origin of the reference system to recall the fact that $C = 0$ only when $\mathbf{f} = \mathbf{0}$. The points on the $f_i f_j$ plane are eligible load cases and the dashed polygonal line connects the points belonging to the convex hull of the load space. According to the previous discussion the worst compliance must be associated with one of the points in the convex hull, hence, only those points must be assessed in order to find the worst compliance of the entire admissible load space.

The compliance surface must be symmetric with respect to the f_iC plane and to the f_jC plane but its exact shape depends on the design variables $\mathbf{h}$. The optimization search proposed in equation (4) will select the best $\mathbf{h}$ such that the worst compliance C is minimized.

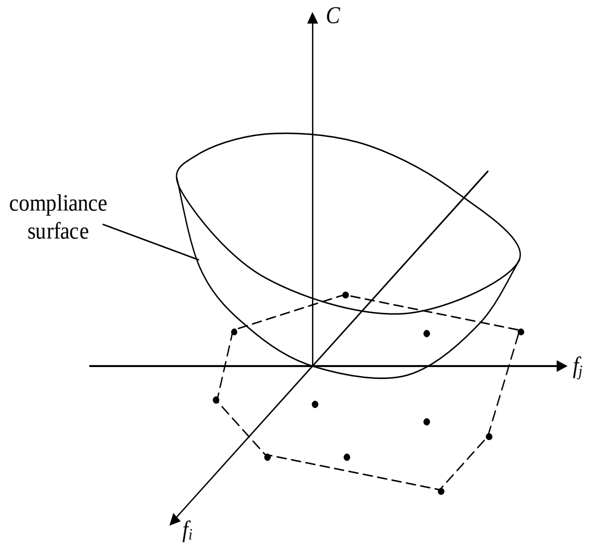

Figure 2: Compliance surface and admissible load space

3 Uncertain Loading Representation

The uncertain loading representation discussed in this section is suitable for a one-dimensional problem such as a beam and for a two-dimensional problem such as a plate, both acted upon by distributed transverse loadings.

The one-dimensional admissible load space can be represented by a collection of piecewise linear basis functions defined at specific locations along the beam. An arbitrary transverse piecewise linear loading applied to the beam is shown in Fig. 3a. Notice that the definition of more of basis functions expands the admissible load space. The piecewise linear basis functions shown in Fig. 3b are defined at five points: 1, 2, 3, 4 and 5. A particular set f_1, f_2, f_3, f_4 and f_5 (the heights of the triangles in Fig. 3b) reflects the loading in Fig. 3a. Points 1, 2, 3 ,4 and 5 may not coincide with nodes of the finite element mesh. However, coincidence as in Fig. 3a is desirable in order to facilitate numerical procedures.

The loading distribution is actually described by the heights of triangles f_1, f_2, f_3, f_4, f_5. If the height of triangle i is denoted by f_i it can be written as in equation (5)

$$f_i = R_i \overline{f}_i \tag{5}$$

where $\overline{f}_i$ is a scaling factor and $0 \leq R_i \leq 1$ is the proportionality parameter that provides a measure of the contribution of each piecewise linear function to the applied loading. $\overline{f}_i$ is selected such that the areas of all triangles in Fig. 3c are equal to some value $\overline{f}L$ as in equation (6) where m is the number of basis functions and L is the beam length.

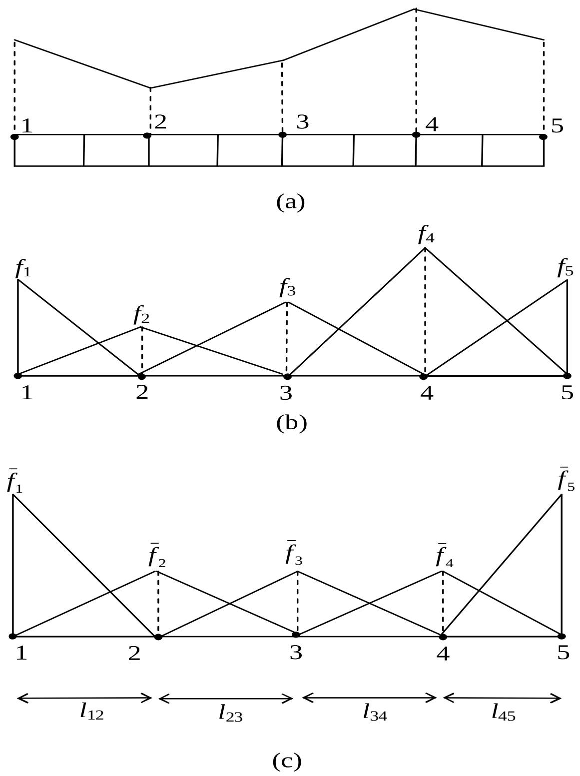

Figure 3: Discretization of the 1D loading distribution

$$\overline{f}_1 = \frac{2\overline{f}L}{l_{12}} \qquad\qquad \overline{f}_m = \frac{2\overline{f}L}{l_{m-1,m}}$$

$$\overline{f}_i = \frac{2\overline{f}L}{l_{i-1,i} + l_{i,i+1}} \quad \text{for} \quad 1 < i < m \tag{6}$$

$\overline{f}_i$ provides, therefore, a normalization to the piecewise linear functions such that the contribution of each of them in terms of net force is equal to $\overline{f}L$, $\overline{f}$ being the equivalent uniformly distributed loading. The relative contribution is given by the proportionality parameters R_i. Moreover, in order to maintain a constant force $\overline{f}L$ the R_i's hold the relationship given by equation (7).

$$\sum_{i=1}^{n} R_i = 1 \tag{7}$$

The distributed loadings represented by piecewise linear basis functions may be applied either as a convex combination such as in equation (7) or individually. In the former case the resulting applied load is the summation of the contribution of the individual basis loadings. Definition of a net force is convenient in order to better describe individual contributions of the basis loadings. For the distribution i the resulting net force F_i is given by:

$$F_i = R_i \overline{f} L \tag{8}$$

where R_i is the proportionality parameter previously introduced that describe the shape of the loading distribution. The minimax technique employed gives the worst loading distribution in terms of R_i. $\overline{f}$ is positive but otherwise arbitrary; it is made equal to $1/L$ for the beam numerical simulations.

The contribution of each individual basis loading is measured by its associated parameters R_i. These parameters vary freely during the optimization, the only constraints imposed being that they are positive and add up to one as stated in equation (7).

The two-dimensional situation is quite similar to the one-dimensional case. However, the transverse loading is no longer described by piecewise linear functions. Alternatively, the resultant transverse loading is a collection of piecewise basis functions defined at specified positions and whose shapes consist of bilinear functions of the form $N(\xi, \eta) = (1-\xi)(1-\eta)/4$ where ξ and η are appropriate local coordinates. Figure 4 shows the shapes of three possible basis functions $\overline{f}_i$, $\overline{f}_j$ and $\overline{f}_k$ that play the same role as that of the $\overline{f}_1$,...,$\overline{f}_5$ basis functions depicted in Fig. 3c.

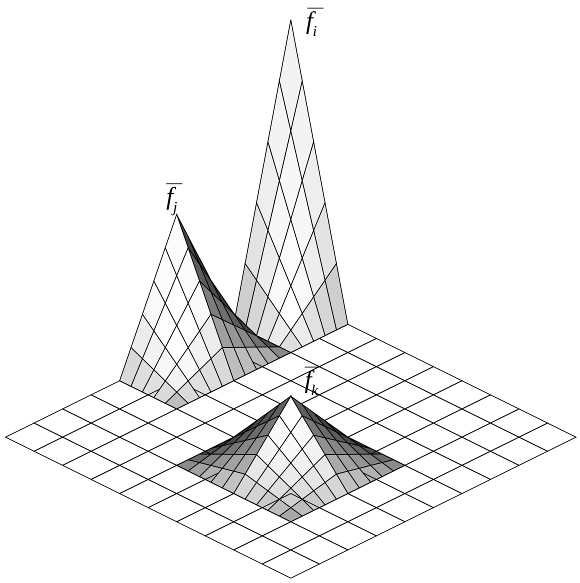

Figure 4: Discretization of the 2D loading distribution

Notice that the basis functions shown in Fig. 4 do not have all the same values for $\overline{f}_i$, $\overline{f}_j$

or $\overline{f}_k$. This is because a constant resultant transverse force must be maintained for each basis functions. In fact, for the loadings depicted in Fig. 4, $4\overline{f}_i = 2\overline{f}_j = \overline{f}_k$ since those basis functions are defined over different areas.

4 The Beam Example

The beam optimized has a constant volume such that, in addition to the constraint introduced by equation (7), another constraint exists: assuming elements of equal length, the following relation must be valid:

$$\frac{h_1}{2} + \sum_{i=2}^{n-1} h_i + \frac{h_n}{2} = (n-1)h \tag{9}$$

where n is the number of nodes and h is the corresponding beam thickness if the thickness is constant along the entire beam.

The definitions of the constant volume constraint and the convex combination in equation (7) provide grounds to pose a bounded optimization problem as in equation (4) where $\mathbf{h}$ is the vector of beam thicknesses and the load vector $\mathbf{f}$ corresponds to the vector of uncertain loads $\mathbf{f} = (R_1, R_2, ..., R_l)$ when l loading components are considered. Solution of the problem stated in equation (4) provides, simultaneously, the optimal design and the worst loading combination in terms of $\mathbf{f}$. The optimization problem stated in equation (4) is bilevel and its numerical solution is often laborious. However, the fact that the compliance is a convex function with respect to $\mathbf{f}$ allows for a tremendous simplification as previously explained.

In equation (4) it is seen that the function $\phi(\mathbf{h})$ must be maximized with respect to the admissible load space $\mathcal{F}$. However, since compliance is shown to be a convex function in $\mathcal{F}$, in order to maximize $\phi(\mathbf{h})$ it suffices to evaluate C at the points that define the convex hull of the admissible load set. The bilevel optimization problem shown in equation (3) is reduced to a traditional optimization problem were the objective function must be computed at a number of points that define the convex hull of the admissible load set. Initially 100,000 designs are randomly generated, tested against the points that define the convex hull of the admissible load space, and the best one is taken to be the starting point for a Powell's search [13]. The optimization search is assumed to have converged when the relative difference between consecutive values of ϕ is less than 0.1%.

As an example an Euler-Bernoulli beam with different number of nodes and different number of basis functions is optimized for minimum compliance. The beam material is aluminum with Young modulus of 70 GPa and the beam length is 0.3 m. Moreover, the beam is simply supported at both ends. The beam has a rectangular cross-section with constant width but its height (the beam thickness) is variable. Figure 5 depicts the beam and a base beam used to avoid cross-sections with zero thickness that might be obtained by the numerical procedure but are physically unrealistic. The base beam has a constant thickness throughout while the design variables are the thicknesses that go on top of the base beam.

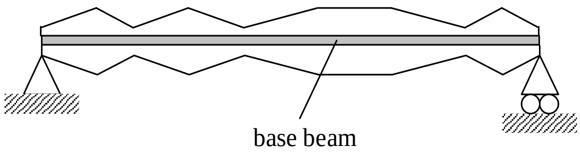

Figure 5: Variable thickness beam

A discretization similar to that of Fig. 3 is used with a varying number of nodes but the finite element mesh has always equal length elements. The equivalent uniformly distributed loading $\overline{f}$ present in Eqs. (6) and (8) is chosen as to give a unit resultant force $\overline{f}L = 1$ N.

Depending on the mesh refinement different number of load cases may be used. Table 1 shows the minimum compliances for a beam with base thickness of $h_{\text{base}} = 0.5$ mm obtained when $n = 9$ and $h = 1.5$ mm in equation (9). Columnwise it can be seen that more nodes imply more flexibility to sustain the uncertain load cases. If the number of nodes is fixed more load cases always leads to higher compliances. This is because the load space is broader what increases the chances of causing damage to the structure.

Table 1: Minimum beam compliances (μJ)

number of	number of load cases				
nodes	2	3	5	7	9
3	3.716	5.486	-	-	-
5	3.599	5.291	6.752	-	-
7	3.569	5.253	-	6.988	-
9	3.559	5.241	6.712	-	7.069

In Table 1 the base beam thickness h_{base} and h were fixed. However, variation of both of them certainly affects the optimal design. If one maintains the sum $h + h_{\text{base}}$ constant a greater h means more freedom to the structure so it can adapt to withstand broader load spaces while a greater base beam thickness implies that the thickness distribution cannot vary as much what renders the structure vulnerable to loading variations. Table 2 shows the effects of the balance between h and h_{base} in the optimal compliance when 9 nodes are considered along with 5 load cases. It is clear from Table 2 that fixing a design configuration is a bad strategy when there are multiple potential load cases.

Making $h + h_{\text{base}}$ constant is equivalent to admitting that a fixed amount of structural mass is available for the optimization. When h=0 the thickness distribution is perfectly uniform and there is no room for redistribution of the available material in order to better withstand the loading space. Figure 6 shows the thickness distribution of the optimal designs.

Table 2: Effect of thickness variability on beam

h_{base}(mm)	h(mm)	compliance(μJ)
0.0	2.0	6.702
0.5	1.5	6.712
1.0	1.0	7.022
1.5	0.5	7.673
2.0	0.0	10.77

5 The Plate Example

Similarly to the beam example, in this section the plate consists in a base plate on top of which a variable thickness distribution is assumed for optimization purposes. Figure 7 shows the structure under investigation while the applicable uncertain loadings are depicted in Fig. 4. The base plate can be seen in black.

The thickness distribution is given in terms of the thicknesses at the nodal positions of the finite element mesh and, using the same bilinear interpolation functions as in the loading basis functions definition, the thickness within a point of an element can be computed. Then, numerical integration is used to obtain the element stiffness matrices taking into account the thickness variability. Classical thin plate assumptions [10] are considered valid in the study. The approximated energy functional computed to obtain the finite element equations is based on the R16 element [2] that has six degrees of freedom per node and employs the Hermitian polynomials as interpolation functions.

The plate considered is squared with 0.4 m side and has Young modulus of 70 GPa and Poisson ratio of 0.3. The mesh employed has 16 elements (4×4) and is simply supported along the four edges. The mesh has 25 nodes and, hence, 25 design variables (thicknesses) defined at these nodes. Figure 8 shows the mesh and the node numbering.

Increasingly larger load spaces are considered. First, only four basis functions are defined at nodes 1, 5, 21, and 25 (load space 1). The load space is expanded in a second moment by defining basis functions at nodes 1, 3, 5, 11, 13, 15, 21, 23 and 25 (load space 2). Finally, the most comprehensive load space consists of 25 basis functions defined at all nodes (load space 3). The load spaces just defined must not be confused with the traditional idea of load case, i.e., a fixed loading distribution. On the contrary, load spaces 1, 2 and 3 correspond to an infinite number of load cases spanned by the appropriate basis functions. Since the boundary conditions and the basis functions are symmetric about the plate center lines, the optimal designs must also be symmetric about those lines. Moreover, because the plate is square, symmetry must also exist about the plate diagonals. Hence, the optimal thicknesses h_i must satisfy the following relationships: $h_1 = h_5 = h_{21} = h_{25} = h_a$, $h_2 = h_4 = h_6 = h_{10} = h_{16} = h_{20} = h_{22} = h_{24} = h_b$, $h_3 = h_{11} = h_{15} = h_{23} = h_c$, $h_7 = h_9 = h_{17} = h_{19} = h_d$, $h_8 = h_{12} = h_{14} = h_{18} = h_e$, and

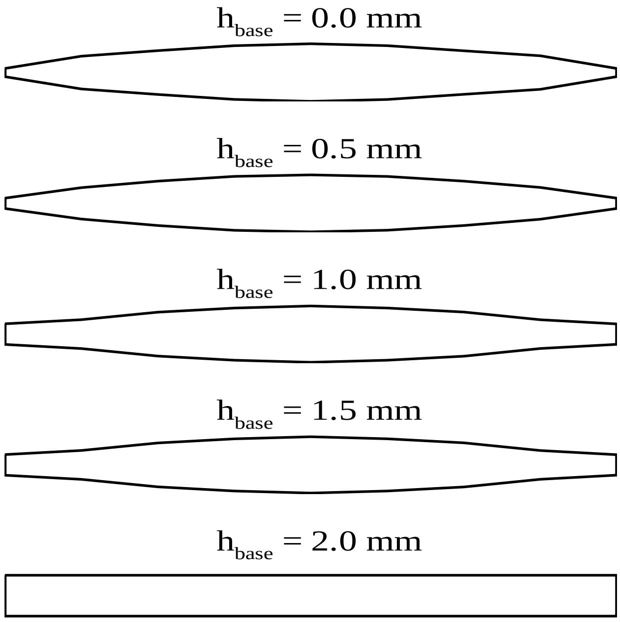

Figure 6: Optimal beams

$h_{13} = h_f$.

Table 3 presents the optimal designs for the situations where $h = 0.75$ mm and $h_{\text{base}} = 0.25$ mm. The compliances marked with a superscript '*' refer to the highest compliance for that load case. Notice that load spaces 1 and 2 possess only one particular basis function for which the compliance is the greatest. However, in load space 3, the value of 91.6 μJ occurs for two distinct basis functions. This shows that the optimal designs obtained by the present optimization strategy equally withstand different load cases. Generally stated, as the load space is enlarged more load cases will correspond to the worst load case for the optimal design.

Table 3: Plate optimal designs

thickness	load space			compliance	load space		
	1	2	3		1	2	3
h_a(mm)	5.37	2.89	0.26	$C_a(\mu\text{J})$	26.4*	22.1	80.4
h_b(mm)	0.25	0.25	0.28	$C_b(\mu\text{J})$	-	-	61.7
h_c(mm)	0.25	0.25	0.82	$C_c(\mu\text{J})$	-	38.2	26.1
h_d(mm)	0.26	0.25	0.67	$C_d(\mu\text{J})$	-	-	91.6*
h_e(mm)	1.73	2.24	2.06	$C_e(\mu\text{J})$	-	-	74.5
h_f(mm)	1.17	1.65	2.06	$C_f(\mu\text{J})$	-	58.1*	91.6*

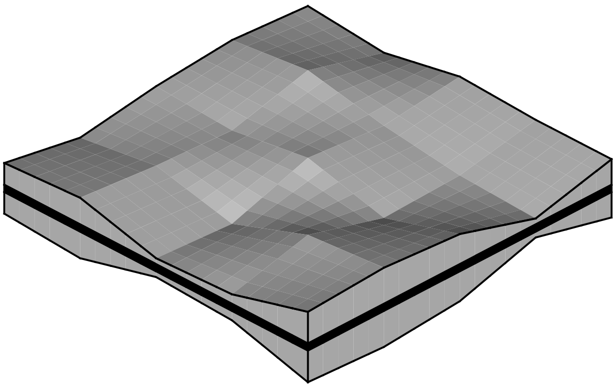

Figure 7: Variable thickness plate

Making $h + h_{\text{base}}$=1.0 mm to ensure constant plate mass and varying h_{base} one can assess the effect of the base plate thickness on the optimal designs. Table 4 shows the optimal designs and compliances when load space 2 previously defined is applied to the square plate. Notice that when h_{base}=0.0 mm the thicknesses h_b and h_d are zero. This result is physically unrealistic unless a plate with holes is an acceptable design. $C_a = C_f = 47.7$ μJ in this case but C_c is not too far from that value. On the other hand, as h_{base} gets larger the values of C_a, C_c and C_f become more and more spaced. This has to do with the adaptability of the optimal design to efficiently sustain a variety of load cases (or larger load spaces). When h_{base} is large there is little room left for the optimal design to arrange its stiffness in such a way that the entire load space is adequately supported.

Table 4: Effect of thickness variability on plate

h(mm)	1.00	0.75	0.50	0.25	0.00
h_{base}(mm)	0.00	0.25	0.50	0.75	1.00
h_a(mm)	1.16	2.89	0.50	0.75	1.00
h_b(mm)	0.00	0.25	0.50	0.75	1.00
h_c(mm)	0.75	0.25	0.50	0.75	1.00
h_d(mm)	0.00	0.25	0.50	0.75	1.00
h_e(mm)	2.89	2.24	2.15	1.45	1.00
h_f(mm)	1.79	1.65	1.86	1.97	1.00
C_a(μJ)	47.7*	22.2	42.7	30.1	24.6
C_c(μJ)	34.7	38.1	43.0	44.1	48.9
C_f(μJ)	47.7*	58.1*	64.8*	79.1*	110.7*

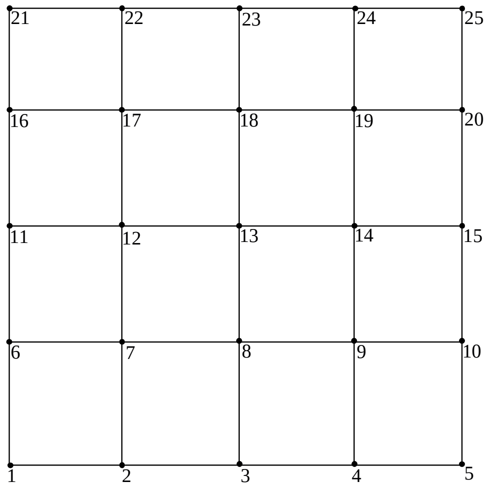

Figure 8: Square plate mesh and node numbering

6 Conclusions

A new technique has been proposed for compliance minimization of structural components subjected to many loading combinations. In real applications it is not unusual to have structures that experience many different load cases during operation. In particular, structural components for aerospace applications must withstand severe loads while operating in hostile environmental conditions. For example, a vertical control surface (rudder) of an aircraft is subjected to approximately 100 important load cases, including but not limited to maneuver loads, gust loads and landing loads. When an airplane rudder is optimized for minimum weight, maximum strain constraints are frequently active; either the spars or the panels could fail due to large stresses. Therefore, minimizing compliance becomes an important factor in the design of efficient, lightweight control surfaces.

The difficulty involved is then to decide how to perform the compliance optimization when many load cases are present. The traditional approach consists in selecting one load case which is deemed to be the most dangerous and, applying a reasonable margin of safety, optimize for that single loading configuration. The drawback of this strategy is notorious: the structure stiffness is arranged in such a way that only one loading configuration is efficiently supported. This makes the optimized component extremely vulnerable to loading configurations not considered in the optimization procedure.

The proportionality parameters introduced in equation (5) must be in the interval [0,1] and add up to one. However, there is no other constraint relating them what can be interpreted as a uniform probability of occurrence of all basis functions. If a particular loading distribution is to be favored in the optimization then it is possible to redefine the parameters R_i and write it as a combination of a specified component $\overline{R}_i$ and a non-specified component ΔR_i where

$R_i = \overline{R}_i + \Delta R_i$. The component $\overline{R}_i$ is a known constant and reflects knowledge (or certainty) in the loading distribution while the ΔR_i's must obey $\sum \Delta R_i = 1 - \sum \overline{R}_i$. The optimization strategy in this case goes essentially unaltered but the convex hull of the load space must now be obtained considering the specified component of R_i.

It was considered that proportionality parameter R_i is positive in the simulations carried out. However, when the transverse loadings are admitted to be either positive or negative, R_i may also assume negative values. Even when R_i is negative the optimization strategy must not be modified. This can be reasoned observing Fig. 2 where a representative compliance surface is shown. Since there is symmetry about the planes f_iC and f_jC this means that when the region of the load space where $R_i > 0$ and $R_j > 0$ is considered the regions where $R_i > 0$ and $R_j < 0$, $R_i < 0$ and $R_j > 0$, and $R_i < 0$ and $R_j < 0$ are automatically covered. Mathematically, equation (7) can also be written as $\sum |R_i| = 1$.

An alternate possibility to that presented here would be to consider a multi-criterion optimization procedure where the compliances associated with each and every loading configuration would be treated as components of a multi-objective function vector. This approach leads naturally to the concept of Pareto sets and objective spaces [3]. However, numerical evaluation of Pareto sets is a laborious task which involves pitfalls. Moreover, once the set of optimal points is obtained, a last decision must be made in order to select one which will become effectively the optimal point [4].

The present work shows that it is unnecessary to consider all the load cases applied to a structural component and yet, one can do better than picking only one loading configuration. It was shown that, for compliance optimization purposes, it is sufficient to consider only those loads which define the convex hull of the applicable load space, thereby drastically reducing the number of load cases involved in the design. Therefore, it is extremely important to find a way to automatically identify the convex hull of a set of loading vectors. Consider that there are l load cases represented by vectors $\mathbf{f}_1$, $\mathbf{f}_2$,..., $\mathbf{f}_l$. In order to identify the convex hull of those vectors one can proceed as follows: for $j = 1$,..., l check whether $\mathbf{f}_j$ can be written as a convex combination of the remaining $l-1$ vectors $\mathbf{f}_1$, $\mathbf{f}_2$,..., $\mathbf{f}_{j-1}$, $\mathbf{f}_{j+1}$,..., $\mathbf{f}_l$. If such a representation is possible then $\mathbf{f}_j$ does not belong to the convex hull. The checking just proposed can be efficiently done through step 1 of a simplex algorithm [9] whereby an initial feasible solution is obtained, provided it exists.

An additional advantage of the method presented is that the resulting optimal designs are insensitive to perturbations in the loads. Quantitatively, this means that if any loading configuration within the admissible space is applied to the optimal designs, it is guaranteed that the compliance will decrease. The optimal design is the physical realization of a compromise between minimum compliance load and the ability to sustain a variety of load cases.

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