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Thermal flexural analysis of cross-ply laminated plates using trigonometric shear deformation theory

Abstract

Thermal stresses and displacements for orthotropic, two-layer antisymmetric, and three-layer symmetric square cross-ply laminated plates subjected to nonlinear thermal load through the thickness of laminated plates are presented by using trigonometric shear deformation theory. The in-plane displacement field uses sinusoidal function in terms of thickness co-ordinate to include the shear deformation effect. The theory satisfies the shear stress free boundary conditions on the top and bottom surfaces of the plate. The present theory obviates the need of shear correction factor. Governing equations and boundary conditions of the theory are obtained using the principle of virtual work. The validity of present theory is verified by comparing the results with those of classical plate theory and first order shear deformation theory and higher order shear deformation theory.

Keywords

Cross-ply laminated plates; orthotropic material; Trigonometric shear deformation theory; thermal stresses; non linear thermal loading

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1 INTRODUCTION

Composite materials are widely used, particularly in aerospace engineering. By virtue of their high strength to weight ratios and because of their mechanical properties in various directions, they can be tailored as per requirements. Further they combine a number of unique properties, including corrosion resistance, high damping, temperature resistance and low thermal coefficient of expansion. These unique properties have resulted in the expanded use of the advance composite materials in structures subjected to severe thermal environment. These structures are usually referred to as high temperature structures. Examples are provided by structures used in high speed aircraft, spacecraft etc. The high velocities of such structures give rise to aerodynamic heating, which produces intense thermal stresses that reduces the strength of aircraft structure. Coefficients of thermal expansion in the direction of fibers are usually much smaller than those in the transverse direction. This results in high stresses at the interfaces. In order to describe the correct thermal response of laminated plates including shear deformation effects refined theories are required.

Thermal stress analysis of isotropic plates is given by Boley and Weiner [1] and thermal stresses of laminated plates subjected to linear thermal load across the thickness of the plate with classical plate theory are given by, Jones [2] and Reddy [3]. Classical plate theory does not take into account the transverse shear effects, which are more pronounced in laminated plates. Rolfes, Noor and Sparr [4] applied first order shear deformation theory for the analysis of laminated plates which takes in to account the transverse shear stresses in the laminates. First order shear deformation theory does not satisfy the transverse shear stress free boundary condition as the transverse shear strains are assumed to be constant in thickness direction. The global higher order theory taking into account both transverse shear and normal stresses has been applied by Matsunaga [5] to analyze the laminated plates subjected to linear thermal loading. Fares, *et al.* [6] discussed thermal effect on transverse displacement of cross-ply laminated plate subjected nonlinear thermal load using refined first-order theory. Wu, *et al.* [7] discussed a global-local higher theory considering transverse normal deformation to predict the thermal response of laminated plate subjected to linear thermal load. Using shear flexible element thermal stresses in laminated plates subjected to linear thermal load is discussed by Ganapathi *et al.*[8]. Semi-analytical model for composite plates subjected to linear thermal load has been developed by Kant, *et al.*[9]. Three dimensional thermal analysis of composite laminated plates subjected to linear thermal load is discussed by Reddy and Savoia [10]. The global-local higher order theory is derived by Zhen and Wanji [11, 12] for laminated plates subjected to linear thermal load. Rohwer, Rolfes, and Sparr [13] discussed higher order theories for thermal stresses in layered plates subjected to linear thermal load. A new efficient higher order zigzag theory is presented for laminated plates under linear thermal loading by Kapuria and Achary [14]. Thermal flexural analysis of symmetric laminated plates subjected to linear thermal load is presented by Ali, *et al.* [15] by using displacement-based higher order theory. For the evaluation of displacements and stresses in functionally graded plates subjected to thermal and mechanical loadings, a two-dimensional higher-order deformation theory is developed by Matsunaga [16]. Analytical solution for bending of cross-ply laminated plates under thermo-mechanical single sinusoidal loading is presented by Zenkour [17] using unified shear deformation plate theory. Fares and Zenkour [18] developed mixed variational formula for the thermal bending and thermo-mechanical bending under linear thermal load. Ghugal and Kulkarni [19] presented thermal stresses in cross-ply laminated plates subjected to linear thermal load through the thickness of plate using refined shear deformation theory.

However, from a review of the above literature it is found that displacements and stresses are evaluated under linear thermal load without considering the effect of non linear variation of thermal load across the thickness of plate. It is found that, the complete set of results of thermal stresses and displacements of laminated plates subjected to nonlinear thermal load through the thickness of laminated plate is not available in the literature. The objective of this paper is to present an equivalent single layer shear deformation theory for evaluation of displacements and stresses of cross-ply laminated plates subjected to non-linear thermal load across the thickness of plate.

2 THEORETICAL FORMULATION

Consider a rectangular cross-ply laminated plate of length a, width b, and total thickness h composed of orthotropic layers. The material of each layer is assumed to have one plane of material property symmetry parallel to x-y plane. The coordinate system is such that the mid-plane of the plate coincides with x-y plane, and z axis is normal to the middle plane. The upper surface of the plate $(z=-h/2)$ is subjected to a thermal load $T(x,y,z)$. The region of the plate in (0-x, y, z) right handed Cartesian coordinate system is

$$0 \le x \le a; \quad 0 \le y \le b; \quad -\frac{h}{2} \le z \le \frac{h}{2} \tag{1}$$

2.1 The displacement field

The displacement field at a point located at (x, y, z) in the plate is of the form [20]:

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y) - z\frac{\partial w(x,y)}{\partial x} + \frac{h}{\pi}\sin\frac{\pi z}{h}\varphi(x,y) \\ v(x,y,z,t) &= v_0(x,y) - z\frac{\partial w(x,y)}{\partial y} + \frac{h}{\pi}\sin\frac{\pi z}{h}\psi(x,y) \\ w(x,y,z,t) &= w(x,y) \end{aligned} \tag{2}$$

Here (u,v,w) are the axial displacements along x, y and z directions respectively, and are functions of the spatial co-ordinates; (u_0,v_0,w_0) are the displacements of a point on the mid-plane, and φ and ψ are the rotations about the y and x axes in xz and yz planes due to bending. The generalized displacements (u_0,v_0,w,φ,ψ) are functions of the (x,y) co-ordinates. Trigonometric shear deformation theory, represents richer kinematics of the theory and does not require shear correction factor, whereas the classical laminate plate theory and first order shear deformation theory adequately describe the kinematic behaviour of most laminates. Present theory can yield more accurate displacements and stresses for thin and thick laminates.

The normal and shear strains are obtained within the framework of linear theory of elasticity. The infinitesimal strains associated with the displacement field (2) are as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{3}$$

The stress-strain relationship for the k^{th} layer in a laminated plate under thermal loading can be written as

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\}_{(k)} = \left[\begin{array}{ccc} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{array} \right]_{(k)} \left\{ \begin{array}{c} \varepsilon_x - \alpha_x T \\ \varepsilon_y - \alpha_y T \\ \gamma_{xy} \end{array} \right\}_{(k)} ,$$

$$\left\{ \begin{array}{c} \tau_{yz} \\ \tau_{xz} \end{array} \right\}_{(k)} = \left[\begin{array}{cc} \bar{Q}_{44} & 0 \\ 0 & \bar{Q}_{55} \end{array} \right]_{(k)} \left\{ \begin{array}{c} \gamma_{yz} \\ \gamma_{xz} \end{array} \right\}_{(k)} \tag{4}$$

where lamina reduced stiffnesses $\bar{Q}_{ij}^{(k)}$ are as follows

$$\bar{Q}_{11}^{(k)} = \frac{E_1^{(k)}}{1-\mu_{12}^{(k)}\mu_{21}^{(k)}},\ \bar{Q}_{22}^{(k)} = \frac{E_2^{(k)}}{1-\mu_{12}^{(k)}\mu_{21}^{(k)}},\ \bar{Q}_{12}^{(k)} = \frac{\mu_{12}^{(k)}E_1^{(k)}}{1-\mu_{12}^{(k)}\mu_{21}^{(k)}},$$

$$\bar{Q}_{66}^{(k)} = G_{12}^{(k)}\ ,\bar{Q}_{44}^{(k)} = G_{23}^{(k)},\ \bar{Q}_{55}^{(k)} = G_{13}^{(k)} \tag{5}$$

where E_i are Young's moduli; μ_{ij} are Poisson's ratios and G_{ij} are shear moduli, α_x and α_y are the coefficients of linear thermal expansion in x and y directions respectively and thermal load across the thickness is assumed to be

$$T(x,y,z) = T_1(x,y) + \frac{z}{h}T_2(x,y) + \frac{\psi(z)}{h}T_3(x,y) \tag{6}$$

where T_1, T_2 and T_3 are thermal loads and $\psi(z) = \frac{h}{\pi}\sin\frac{\pi z}{h}$. The nonlinear term associated with thermal load T_3 is the trigonometric function in terms of thickness coordinate.

2.2 Governing Equations and Boundary Conditions

Using the expressions for strains, stresses, and principle of virtual work, variational consistent differential equations and boundary conditions for the plate under consideration are obtained. The principal of virtual work when applied to the plate leads to:

$$\int_{-h/2}^{h/2}\int_0^b\int_0^a \left(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy}\right)\, dx\, dy\, dz = 0 \tag{7}$$

where the symbol δ denotes variational operator. In Eq. (7) mechanical load is taken as zero since the plate is subjected to pure nonlinear thermal load. Employing the Green's theorem in above equation successively and collecting the coefficients of $\delta u_0, \delta v_0, \delta w, \delta\phi, \delta\psi$ we can obtain the governing equations as follows:

$$\begin{aligned}\delta u_0:\ & -A_{11}\frac{\partial^2 u_0}{\partial x^2}-A_{66}\frac{\partial^2 u_0}{\partial y^2}-(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial y\partial x}+B_{11}\frac{\partial^3 w}{\partial x^3}+(B_{12}+2B_{66})\frac{\partial^3 w}{\partial y^2\partial x}\\ & -E_{11}\frac{\partial^2\varphi}{\partial x^2}-E_{66}\frac{\partial^2\varphi}{\partial y^2}-(E_{12}+E_{66})\frac{\partial^2\psi}{\partial y\partial x}+(L_{11}+L_{12})\frac{\partial T_1}{\partial x}+(P_{11}+P_{12})\frac{\partial T_2}{\partial x}\\ & +(R_{11}+R_{12})\frac{\partial T_3}{\partial x}=0\end{aligned} \tag{8}$$

$$\begin{aligned}\delta v_0:\ & -A_{22}\frac{\partial^2 v_0}{\partial y^2}-A_{66}\frac{\partial^2 v_0}{\partial x^2}-(A_{12}+A_{66})\frac{\partial^2 u_0}{\partial y\partial x}+B_{22}\frac{\partial^3 w}{\partial y^3}+(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x^2\partial y}\\ & -E_{22}\frac{\partial^2\psi}{\partial y^2}-E_{66}\frac{\partial^2\psi}{\partial x^2}-(E_{12}+E_{66})\frac{\partial^2\varphi}{\partial y\partial x}+(L_{12}+L_{22})\frac{\partial T_1}{\partial y}+(P_{12}+P_{22})\frac{\partial T_2}{\partial y}\\ & +(R_{12}+R_{22})\frac{\partial T_3}{\partial y}=0\end{aligned} \tag{9}$$

$$\begin{aligned}\delta w:\ & -B_{11}\frac{\partial^3 u_0}{\partial x^3}-(B_{12}+2B_{66})\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^3 v_0}{\partial y\partial x^2}\right)-B_{22}\frac{\partial^3 v_0}{\partial y^3}\\ & +D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}\\ & -F_{11}\frac{\partial^3\varphi}{\partial x^3}-F_{22}\frac{\partial^3\psi}{\partial y^3}-(F_{12}+2F_{66})\left(\frac{\partial^3\varphi}{\partial x\partial y^2}+\frac{\partial^3\psi}{\partial x^2\partial y}\right)\\ & +(S_{11}+S_{12})\frac{\partial^2 T_1}{\partial x^2}+(T_{11}+T_{12})\frac{\partial^2 T_2}{\partial x^2}+(U_{11}+U_{12})\frac{\partial^2 T_3}{\partial x^2}\\ & +(S_{12}+S_{22})\frac{\partial^2 T_1}{\partial y^2}+(T_{12}+T_{22})\frac{\partial^2 T_2}{\partial y^2}+(U_{12}+U_{22})\frac{\partial^2 T_3}{\partial y^2}=0\end{aligned} \tag{10}$$

$$\begin{aligned}\delta\varphi:\ & -E_{11}\frac{\partial^2 u_0}{\partial x^2}-E_{66}\frac{\partial^2 u_0}{\partial y^2}-(E_{12}+E_{66})\frac{\partial^2 v_0}{\partial y\partial x}+F_{11}\frac{\partial^3 w}{\partial x^3}+(F_{12}+2F_{66})\frac{\partial^3 w}{\partial x\partial y^2}\\ & -H_{11}\frac{\partial^2\varphi}{\partial x^2}-H_{66}\frac{\partial^2\varphi}{\partial y^2}+C_{55}\varphi-(H_{12}+H_{66})\frac{\partial^2\psi}{\partial y\partial x}+(V_{11}+V_{12})\frac{\partial T_1}{\partial x}+(W_{11}+W_{12})\frac{\partial T_2}{\partial x}\\ & +(X_{11}+X_{12})\frac{\partial T_3}{\partial x}=0\end{aligned} \tag{11}$$

$$\delta\psi : -E_{22}\frac{\partial^2 v_0}{\partial y^2} - E_{66}\frac{\partial^2 v_0}{\partial x^2} - (E_{12}+E_{66})\frac{\partial^2 u_0}{\partial x\partial y} + F_{22}\frac{\partial^3 w}{\partial y^3} + (F_{12}+2F_{66})\frac{\partial^3 w}{\partial x^2\partial y}$$
$$-H_{66}\frac{\partial^2 \psi}{\partial x^2} - H_{22}\frac{\partial^2 \psi}{\partial y^2} + C_{44}\psi - (H_{12}+H_{66})\frac{\partial^2 \varphi}{\partial x\partial y} + (V_{12}+V_{22})\frac{\partial T_1}{\partial y} + (W_{12}+W_{22})\frac{\partial T_2}{\partial y} \quad (12)$$
$$+(X_{12}+X_{22})\frac{\partial T_3}{\partial y} = 0$$

The associated boundary conditions are of the form:

1) Along the edges $x=0$ and $x=a$, following are the boundary conditions

$$\delta u_0 : A_{11}\frac{\partial u_0}{\partial x} + A_{12}\frac{\partial v_0}{\partial y} - B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} + E_{11}\frac{\partial \varphi}{\partial x} + E_{12}\frac{\partial \psi}{\partial y}$$
$$-(L_{11}+L_{12})T_1 - (P_{11}+P_{12})T_2 - (R_{11}+R_{12})T_3 = N_x = 0 \text{ or } u_0 \text{ is prescribed.} \quad (13)$$

$$\delta v_0 : A_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2B_{66}\frac{\partial^2 w}{\partial x\partial y} + E_{66}\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}\right) = N_{xy} = 0 \text{ or } v_0 \text{ is prescribed.} \quad (14)$$

$$\partial w : B_{11}\frac{\partial^2 u_0}{\partial x^2} + (B_{12}+2B_{66})\frac{\partial^2 v_0}{\partial x\partial y} + 2B_{66}\frac{\partial^2 u_0}{\partial y^2} - D_{11}\frac{\partial^3 w}{\partial x^3} - (D_{12}+4D_{66})\frac{\partial^3 w}{\partial y^2\partial x}$$
$$+F_{11}\frac{\partial^2 \varphi}{\partial x^2} + (F_{12}+2F_{66})\frac{\partial^2 \psi}{\partial x\partial y} + 2F_{66}\frac{\partial^2 \varphi}{\partial y^2} - (S_{11}+S_{12})\frac{\partial T_1}{\partial x} - (T_{11}+T_{12})\frac{\partial T_2}{\partial x} \quad (15)$$
$$-(U_{11}+U_{12})\frac{\partial T_3}{\partial x} = V_x = 0 \text{ or } w \text{ is prescribed.}$$

$$\frac{\partial \delta w}{\partial x} : -B_{11}\frac{\partial u_0}{\partial x} - B_{12}\frac{\partial v_0}{\partial y} + D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} - F_{11}\frac{\partial \varphi}{\partial x} - F_{12}\frac{\partial \psi}{\partial y}$$
$$+(S_{11}+S_{12})T_1 + (T_{11}+T_{12})T_2 + (U_{11}+U_{12})T_3 = M_x = 0 \text{ or } \frac{\partial w}{\partial x} \text{ is prescribed.} \quad (16)$$

$$\delta\varphi : E_{11}\frac{\partial u_0}{\partial x} + E_{12}\frac{\partial v_0}{\partial y} - F_{11}\frac{\partial^2 w}{\partial x^2} - F_{12}\frac{\partial^2 w}{\partial y^2} + H_{11}\frac{\partial \varphi}{\partial x} + H_{12}\frac{\partial \psi}{\partial y}$$
$$-(V_{11}+V_{12})T_1 - (W_{11}+W_{12})T_2 - (X_{11}+X_{12})T_3 = M_x^s = 0 \text{ or } \varphi \text{ is prescribed.} \quad (17)$$

$$\delta\psi : E_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2F_{66}\frac{\partial^2 w}{\partial x\partial y} + H_{66}\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}\right) = M_{xy}^s = 0 \text{ or } \psi \text{ is prescribed.} \quad (18)$$

2) Along the edges $y=0$ and $y=b$, following are the boundary conditions

$$\delta u_0 : A_{66}(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x})-2B_{66}\frac{\partial^2 w}{\partial x \partial y}+E_{66}(\frac{\partial \varphi}{\partial y}+\frac{\partial \psi}{\partial x})=N_{xy}=0 \text{ or } u_0 \text{ is prescribed.} \tag{19}$$

$$\begin{aligned}\delta v_0 : & A_{12}\frac{\partial u_0}{\partial x}+A_{22}\frac{\partial v_0}{\partial y}-B_{12}\frac{\partial^2 w}{\partial x^2}-B_{22}\frac{\partial^2 w}{\partial y^2}+E_{12}\frac{\partial \varphi}{\partial x}+E_{22}\frac{\partial \psi}{\partial y} \\ & -(L_{12}+L_{22})T_1-(P_{12}+P_{22})T_2-(R_{12}+R_{22})T_3=N_y=0 \quad \text{or } v_0 \text{ is prescribed.}\end{aligned} \tag{20}$$

$$\begin{aligned}\delta w : & (B_{12}+2B_{66})\frac{\partial^2 u_0}{\partial x \partial y}+B_{22}\frac{\partial^2 v_0}{\partial y^2}+2B_{66}\frac{\partial^2 v_0}{\partial x^2}-D_{22}\frac{\partial^3 w}{\partial y^3}-(D_{12}+4D_{66})\frac{\partial^3 w}{\partial x^2 \partial y} \\ & +(F_{12}+2F_{66})\frac{\partial^2 \varphi}{\partial x \partial y}+F_{22}\frac{\partial^2 \psi}{\partial y^2}+2F_{66}\frac{\partial^2 \psi}{\partial x^2}-(S_{12}+S_{22})\frac{\partial T_1}{\partial y}-(T_{12}+T_{22})\frac{\partial T_2}{\partial y} \\ & -(U_{12}+U_{22})\frac{\partial T_3}{\partial y}=V_y=0 \quad \text{or } w \text{ is prescribed.}\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial \delta w}{\partial y} : & -\left(B_{12}\frac{\partial u_0}{\partial x}+B_{22}\frac{\partial v_0}{\partial y}\right)+D_{12}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}-F_{12}\frac{\partial \varphi}{\partial x}-F_{22}\frac{\partial \psi}{\partial y} \\ & +(S_{12}+S_{22})T_1+(T_{12}+T_{22})T_2+(U_{12}+U_{22})T_3=M_y=0 \quad \text{or } \frac{\partial w}{\partial y} \text{ is prescribed.}\end{aligned} \tag{22}$$

$$\delta \phi : E_{66}(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x})-2F_{66}\frac{\partial^2 w}{\partial x \partial y}+H_{66}(\frac{\partial \phi}{\partial y}+\frac{\partial \psi}{\partial x})=M^s_{xy}=0 \text{ or } \phi \text{ is prescribed.} \tag{23}$$

$$\begin{aligned}\delta \psi : & E_{12}\frac{\partial u_0}{\partial x}+E_{22}\frac{\partial v_0}{\partial y}-F_{12}\frac{\partial^2 w}{\partial x^2}-F_{22}\frac{\partial^2 w}{\partial y^2}+H_{12}\frac{\partial \phi}{\partial x}+H_{22}\frac{\partial \psi}{\partial y} \\ & -(V_{12}+V_{22})T_1-(W_{12}+W_{22})T_2-(X_{12}+X_{22})T_3=M^s_y=0 \text{ or } \psi \text{ is prescribed.}\end{aligned} \tag{24}$$

3) At corners $(x=0,y=0),(x=0,y=b),(x=a,y=0)$ and $(x=a,y=b)$ the following condition hold:

$$B_{66}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)-2D_{66}\frac{\partial^2 w}{\partial x \partial y}+F_{66}\left(\frac{\partial \varphi}{\partial y}+\frac{\partial \psi}{\partial x}\right)=M_{xy}=0 \text{ or } w \text{ is prescribed.} \tag{25}$$

where laminate stiffness coefficients A_{ij} and B_{ij} ... etc, appeared in above equations are defined in terms of reduced stiffness coefficients $\bar{Q}_{ij}^{(k)}$ for the layers k =1, 2,,..., n as follows:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z, z^2) dz, \quad (i, j = 1,2,6) \tag{26a}$$

$$(E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \frac{h}{\pi} \sin \frac{\pi z}{h} \left(1, z, \frac{h}{\pi} \sin \frac{\pi z}{h} \right) dz \tag{26b}$$

$$(L_{ij}, P_{ij}, R_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \alpha_i^{(k)} \bar{Q}_{ij}^{(k)} \left(1, \frac{z}{h}, \frac{\psi(z)}{h} \right), \quad (i = x, y) \tag{26c}$$

$$(S_{ij}, T_{ij}, U_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \alpha_i^{(k)} \bar{Q}_{ij}^{(k)} \left(z, \frac{z^2}{h}, \frac{\psi(z) z}{h} \right), \quad (i = x, y) \tag{26d}$$

$$(V_{ij}, W_{ij}, X_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \alpha_i^{(k)} \bar{Q}_{ij}^{(k)} \frac{h}{\pi} \sin \frac{\pi z}{h} \left(1, \frac{z}{h}, \frac{\psi(z)}{h} \right), \quad (i = x, y) \tag{26e}$$

$$C_{ij} = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \cos^2 \frac{\pi z}{h} dz, \quad (i, j = 4,5) \tag{26f}$$

For symmetric cross-ply laminated plate stiffness coefficients $B_{ij}, E_{ij}, P_{ij}, R_{ij}, S_{ij}, V_{ij} = 0$.

3 ILLUSTRATIVE EXAMPLE

To assess the performance of present theory under combined linear and nonlinear thermal load, orthotropic, two-layer antisymmetric and three layer symmetric laminated plates are considered herein.

Example

Simply supported square orthotropic, two-layer antisymmetric, and three-layer symmetric laminated plates subjected to temperature field $T(x,y,z) = T_1(x,y) + \frac{z}{h} T_2(x,y) + \frac{\psi(z)}{h} T_3(x,y)$ through the thickness of plate are considered with following lamina material properties:

$$\frac{E_1}{E_2} = 25, G_{12} = G_{13} = 0.5 E_2, G_{23} = 0.2 E_2, \mu_{12} = 0.25, \frac{\alpha_y}{\alpha_x} = 3$$

α_x is coefficient of thermal expansion in the direction of fiber and α_y is coefficient of thermal expansion in transverse direction.

3.1 The solution scheme

Here we concern with the close form solutions of simply supported square and rectangular plates. The boundary conditions for simply supported edges are

$$v_0 = w = \psi = N_x = M_x = M_x^s = 0 \ \text{ at } x = 0 \text{ and } x = a$$
$$u_0 = w = \varphi = N_y = M_y = M_y^s = 0 \ \text{ at } y = 0 \text{ and } y = b \tag{27}$$

The following is the solution form for $u_0(x,y), v_0(x,y), w(x,y), \varphi(x,y), \psi(x,y)$ that satisfies above boundary conditions exactly;

$$u_0(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{0mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{28a}$$

$$v_0(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{0mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{28b}$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{28c}$$

$$\varphi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varphi_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{28d}$$

$$\psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{28e}$$

Thermal load is expanded in double Fourier sine series as follows:

$$T_1(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{1mn}\, sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b}$$

$$T_2(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{2mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{28f}$$

$$T_3(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{3mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

For single sinusoidal thermal load $(m=n=1)$, series coefficients lead to $T_{1mn}=T_{2mn}=T_{3mn}=T_0$, where, the maximum intensity of thermal load is T_0. Substitution of solution form given by equations (28a)-(28f) into governing equations (8)-(12) results into a system of the algebraic equations which can be written into a matrix form as follows:

$$[K]\{\delta\}=\{f\} \tag{29}$$

where $[K]$ is the symmetric stiffness matrix, $\{\delta\}=\{u_{0mn}, v_{0mn}, w_{mn}, \varphi_{mn}, \psi_{mn}\}^T$ and $\{f\}$ is the generalized force vector.

From solution of these equations unknown coefficients $\{\delta\}$ can be obtained readily. Substituting these coefficients into equations (28a)-(28f), generalized displacements and rotations can be obtained and subsequently inplane stresses and transverse stresses can be obtained. Although the transverse shear stress components can be calculated from the constitutive relations, these stresses may not satisfy the continuity conditions at the interface between layers. Hence transverse shear stresses in orthotropic, symmetric and antisymmetric cross-ply laminated plates are obtained by using three dimensional stress equilibrium equations of elasticity. These equations are as follows.

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=0 \tag{30a}$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}=0 \tag{30b}$$

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}=0 \tag{30c}$$

Substitute the expressions of inplane normal stress (σ_x) and inplane shear stress (τ_{xy}) in the equation (30a) and inplane normal stress σ_y and inplane shear stress (τ_{xy}) in the equation (30b) and integrate them with respect to thickness coordinate z in a layerwise manner. The integration constants are obtained by imposing the stress boundary conditions of k^{th} layer on the upper and lower surfaces of the k^{th} layer. Using this procedure final expressions for transverse shear stresses are obtained to evaluate these stresses through the thickness of laminated plate.

4 RESULTS

In this paper, displacements and stresses are determined for square orthotropic, antisymmetric and symmetric laminated plates subjected to non-linear thermal load across the thickness of plate. Results are presented in the following normalized forms for the purpose of discussion.

Normalized displacements $(\bar{u},\bar{v},\bar{w})$ and thermal stresses $(\bar{\sigma}_x,\bar{\sigma}_y,\bar{\tau}_{xy},\bar{\tau}_{zx},\bar{\tau}_{zy})$ for orthotropic plate:

$$\bar{u}=u\left(0,\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 a^2},\bar{v}=v\left(\frac{a}{2},0,-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 a^2},\bar{w}=w\left(\frac{a}{2},\frac{b}{2},0\right)\frac{10\times h}{\alpha_1 T_0 b^2}$$

$$\bar{\sigma}_x=\sigma_x\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\sigma}_y=\sigma_y\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\tau}_{xy}=\tau_{xy}\left(0,0,-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},$$

$$\bar{\tau}_{xz}=\tau_{xz}\left(0,\frac{b}{2},0\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\tau}_{yz}=\tau_{yz}\left(\frac{a}{2},0,0\right)\frac{1}{\alpha_1 T_0 E_2 a^2}$$

Normalized displacements and thermal stresses for two-layer antisymmetric laminated plates:

$$\bar{u}=u\left(0,\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 a^2},\bar{v}=v\left(\frac{a}{2},0,-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 a^2},\bar{w}=w\left(\frac{a}{2},\frac{b}{2},0\right)\frac{10\times h}{\alpha_1 T_0 b^2},$$

$$\bar{\sigma}_x=\sigma_x\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\sigma}_y=\sigma_y\left(\frac{a}{2},\frac{b}{2},+\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\tau}_{xy}=\tau_{xy}\left(0,0,-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},$$

$$\bar{\tau}_{xz}=\tau_{xz}\left(0,\frac{b}{2},0\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\tau}_{yz}=\tau_{yz}\left(\frac{a}{2},0,0\right)\frac{1}{\alpha_1 T_0 E_2 a^2}$$

Normalized displacements and thermal stresses for three-layer symmetric laminated plates:

$$\bar{u}=u\left(0,\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 a^2},\bar{v}=v\left(\frac{a}{2},0,-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 a^2},\bar{w}=w\left(\frac{a}{2},\frac{b}{2},0\right)\frac{10\times h}{\alpha_1 T_0 b^2},$$

$$\bar{\sigma}_x=\sigma_x\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\sigma}_y=\sigma_y\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\tau}_{xy}=\tau_{xy}\left(0,0,-\frac{h}{2}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},$$

$$\bar{\tau}_{xz}=\tau_{xz}\left(0,\frac{b}{2},-\frac{h}{6}\right)\frac{1}{\alpha_1 T_0 E_2 a^2},\bar{\tau}_{yz}=\tau_{yz}\left(\frac{a}{2},0,-\frac{h}{6}\right)\frac{1}{\alpha_1 T_0 E_2 a^2}$$

Results obtained for normalized displacements and stresses are presented in Tables 1 and 2 and in Figures 1 through 12.

Table 1 Normalized displacements and stresses for square orthotropic, two-layer antisymmetric and three layer symmetric cross-ply laminated plates subjected to nonlinear thermal load $(T_1 = 0, T_2 = T_3 = 1)$ for aspect ratio 4.

Plate	Theory	$\bar{u}$	$\bar{v}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{EE}$	$\bar{\tau}_{zy}^{EE}$
0^0	Present	0.2844	0.3434	1.8919	-1.5424	1.3607	0.9862	0.0066	-0.1191
	HSDT	0.2858	0.3141	1.8504	-1.6251	1.4518	0.9423	0.0453	-0.1504
	FSDT	0.2814	0.3563	1.9279	-1.3164	1.3224	1.0018	0.0143	-0.1186
	CPT	0.2874	0.2874	1.8293	-1.7270	1.5349	0.9027	0.0755	-0.1798
$0^0/90^0$	Present	0.2914	0.3321	1.9460	-2.0811	2.0811	0.9794	-0.1246	-0.1246
	HSDT	0.2934	0.3329	2.0156	-2.2418	2.2418	0.9839	-0.1250	-0.1268
	FSDT	0.2926	0.3325	1.9899	-2.1765	2.1765	0.9820	-0.1268	-0.1262
	CPT	0.2926	0.3325	1.9899	-2.1765	2.1765	0.9820	-0.1262	-0.1262
0/90/0	Present	0.2855	0.3269	1.9405	-1.6163	1.4118	0.9620	0.0384	-0.1212
	HSDT	0.2857	0.3139	1.8803	-1.6150	1.4527	0.9417	0.0672	-0.1317
	FSDT	0.2810	0.3388	1.9463	-1.2646	1.3781	0.9734	0.0425	-0.1157
	CPT	0.2873	0.2873	1.8292	-1.7249	1.5350	0.9027	0.1111	-0.1559

Table 2 Normalized displacements and stresses for square orthotropic, two-layer antisymmetric and three layer symmetric cross-ply laminated plates subjected to nonlinear thermal load $(T_1 = 0, T_2 = T_3 = 1)$ for aspect ratio 10.

Plate	Theory	$\bar{u}$	$\bar{v}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{EE}$	$\bar{\tau}_{zy}^{EE}$
0^0	Present	0.2867	0.3004	1.8464	-1.6898	1.4943	0.9223	0.0237	-0.0663
	HSDT	0.2871	0.2874	1.8496	-1.7120	1.5018	0.9189	0.0254	-0.0674
	FSDT	0.2860	0.3032	1.8520	-1.6323	1.4859	0.9256	0.0246	-0.0663
	CPT	0.2874	0.2874	1.8293	-1.7270	1.5349	0.9027	0.0302	-0.0719
$0^0/90^0$	Present	0.2924	0.3325	1.9827	-2.1609	2.1609	0.9816	-0.0504	-0.0504
	HSDT	0.2930	0.3327	2.0007	-2.2040	2.2040	0.9828	-0.0503	-0.0506
	FSDT	0.2926	0.3325	1.9899	-2.1765	2.1765	0.9820	-0.0505	-0.0505
	CPT	0.2926	0.3325	1.9899	-2.1765	2.1765	0.9820	-0.0505	-0.0505
0/90/0	Present	0.2869	0.2976	1.8599	-1.7015	1.5030	0.9189	0.0369	-0.0588
	HSDT	0.2873	0.2977	1.8654	-1.7285	1.5025	0.9188	0.0381	-0.0588
	FSDT	0.2857	0.3001	1.8583	-1.6103	1.4960	0.9203	0.0376	-0.0584
	CPT	0.2873	0.2873	1.8292	-1.7249	1.5350	0.9027	0.0445	-0.0623

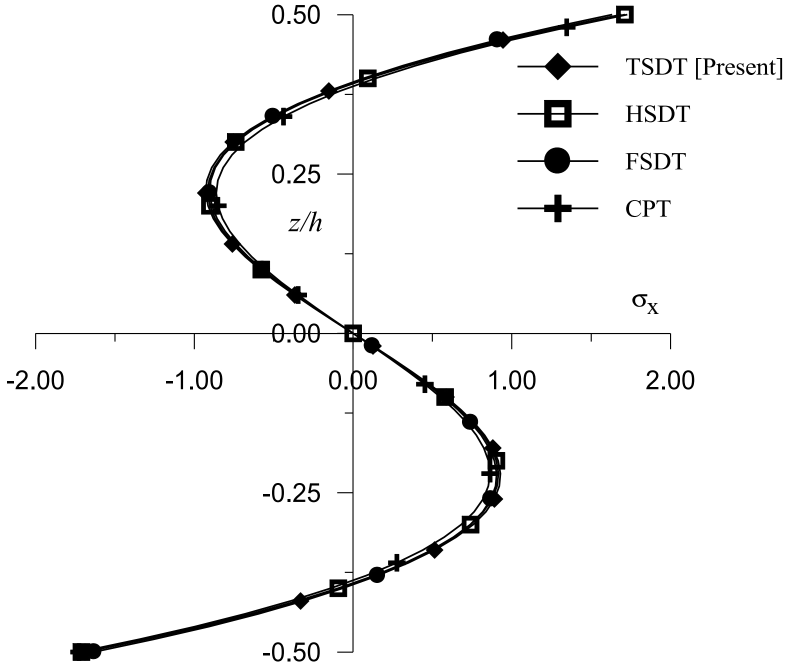

Figure 1 Variation of normalized inplane normal stress $\bar{\sigma}_x$ through the thickness of orthotropic plate for aspect ratio 10

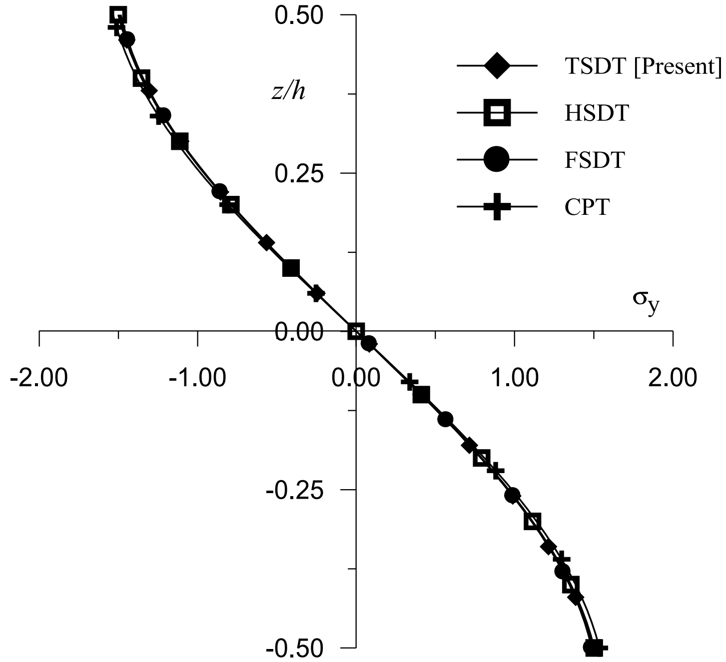

Figure 2 Variation of normalized inplane normal stress $\bar{\sigma}_y$ through the thickness of orthotropic plate for aspect ratio 10

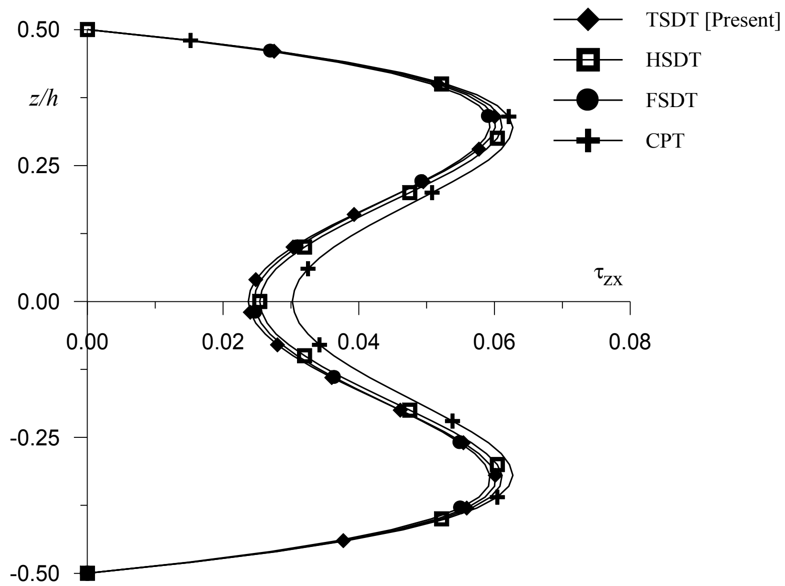

Figure 3 Variation of normalized transverse shear stress $\overline{\tau}_{zx}$ through the thickness of orthotropic plate for aspect ratio 10 and obtained by equilibrium equations

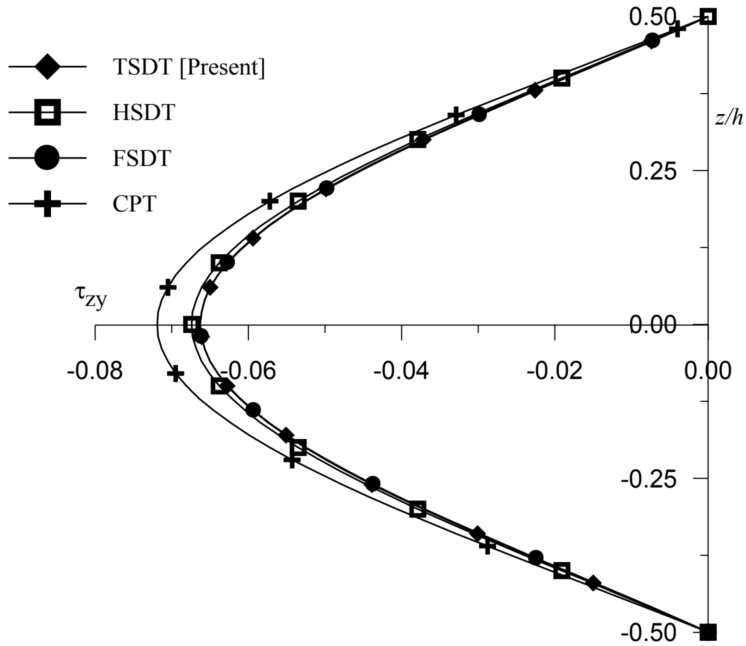

Figure 4 Variation of normalized transverse shear stress $\overline{\tau}_{zy}$ through the thickness of orthotropic plate for aspect ratio 10 and obtained by equilibrium equations

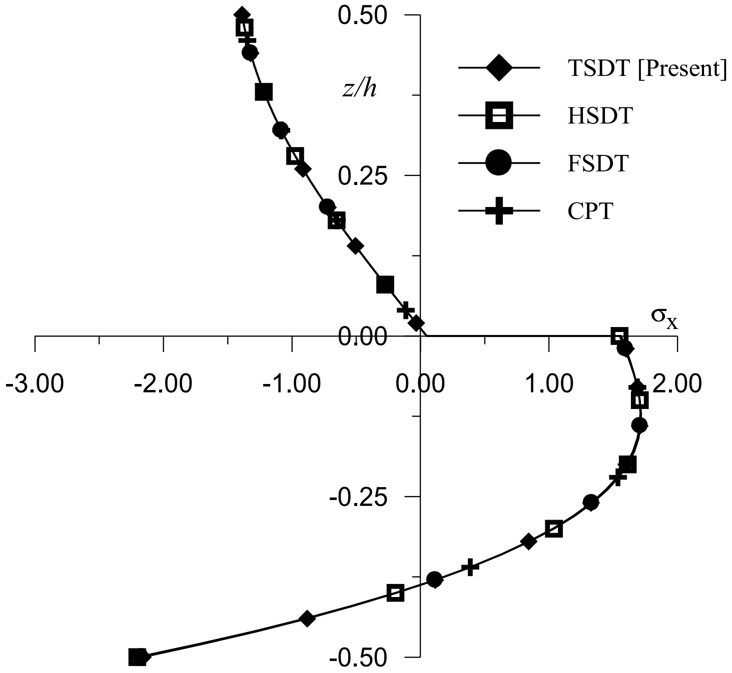

Figure 5 Variation of normalized inplane normal stress $\bar{\sigma}_x$ through the thickness of two-layer laminated plate for aspect ratio 10

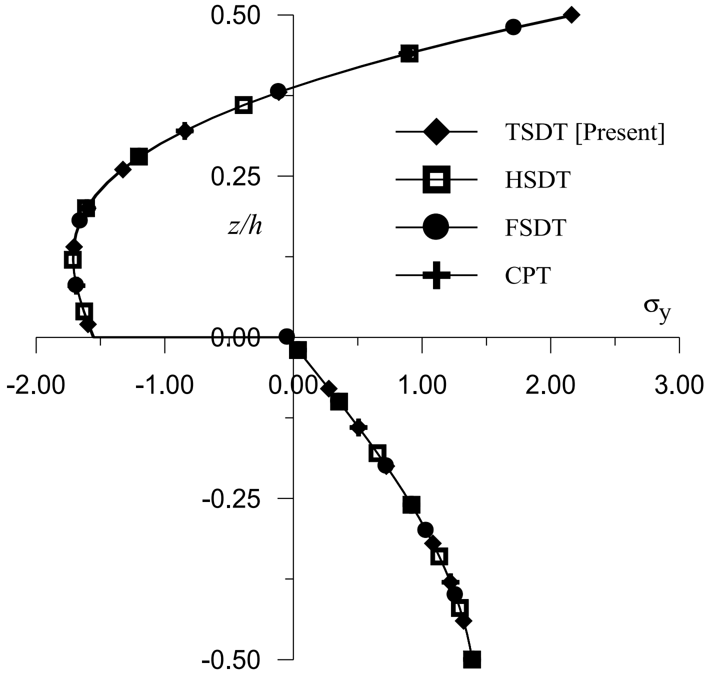

Figure 6 Variation of normalized inplane normal stress $\bar{\sigma}_y$ through the thickness of two-layer laminated plate for aspect ratio 10

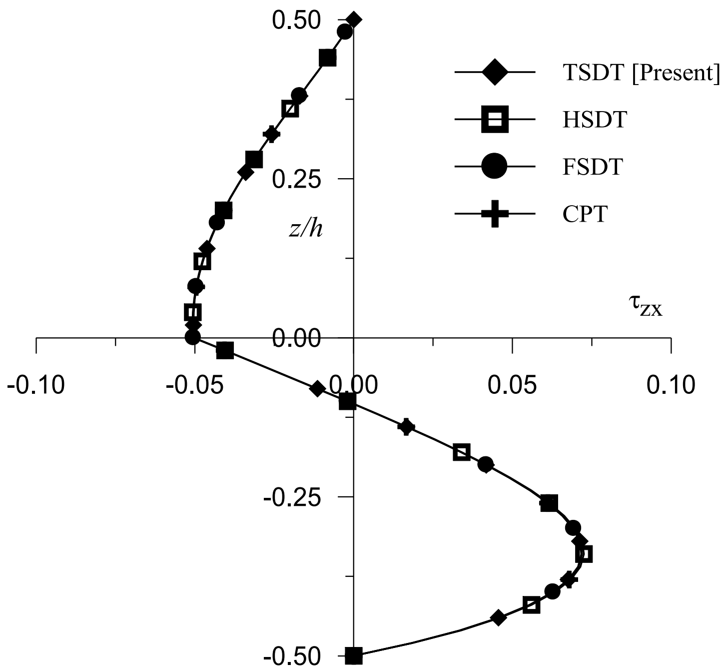

Figure 7 Variation of normalized transverse shear stress $\overline{\tau}_{zx}$ through the thickness of two-layer laminated plate for aspect ratio 10

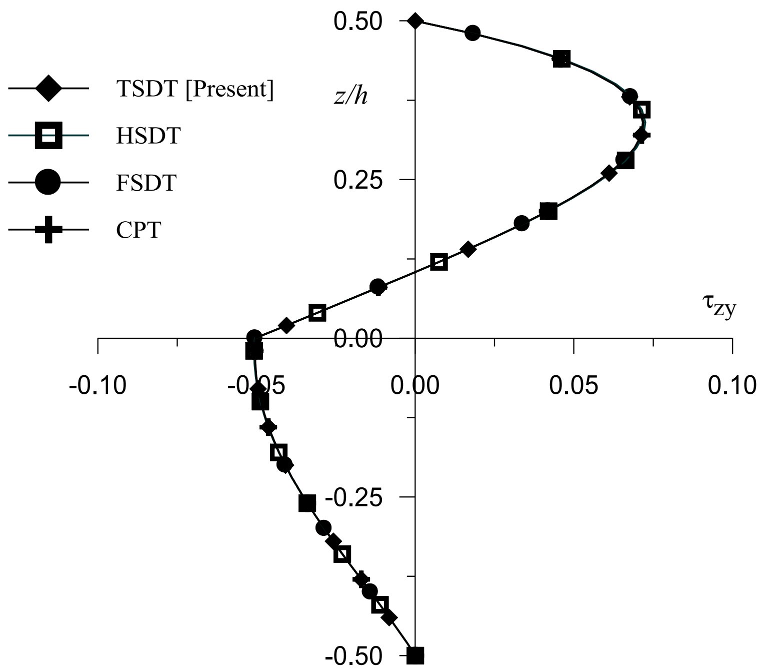

Figure 8 Variation of normalized transverse shear stress $\overline{\tau}_{zy}$ through the thickness of two-layer laminated plate for aspect ratio 10

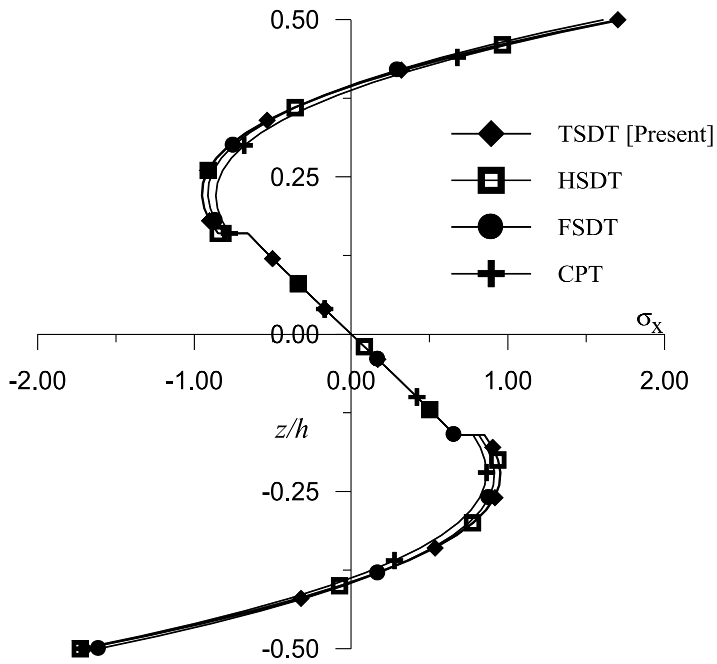

Figure 9 Variation of normalized inplane normal stress $\bar{\sigma}_x$ through the thickness of three-layer laminated plate for aspect ratio 10

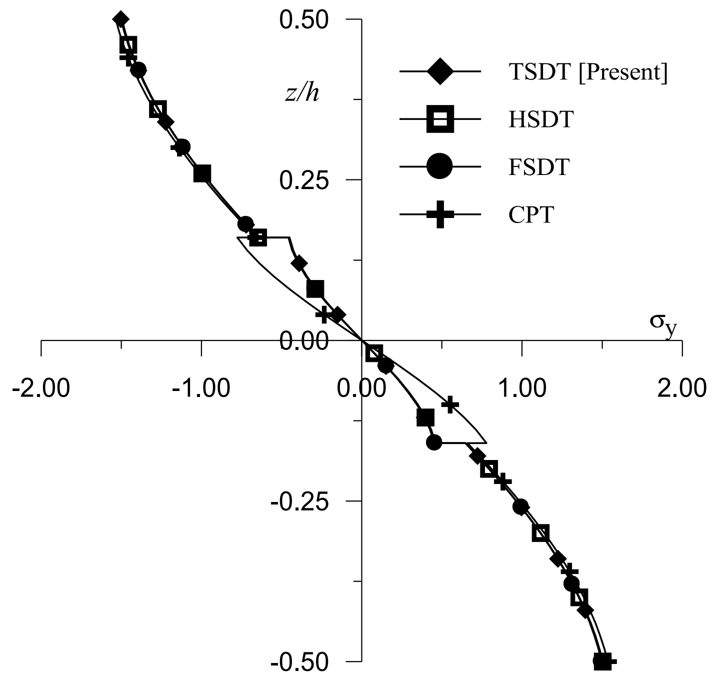

Figure 10 Variation of normalized inplane normal stress $\bar{\sigma}_y$ through the thickness of three-layer laminated plate for aspect ratio 10

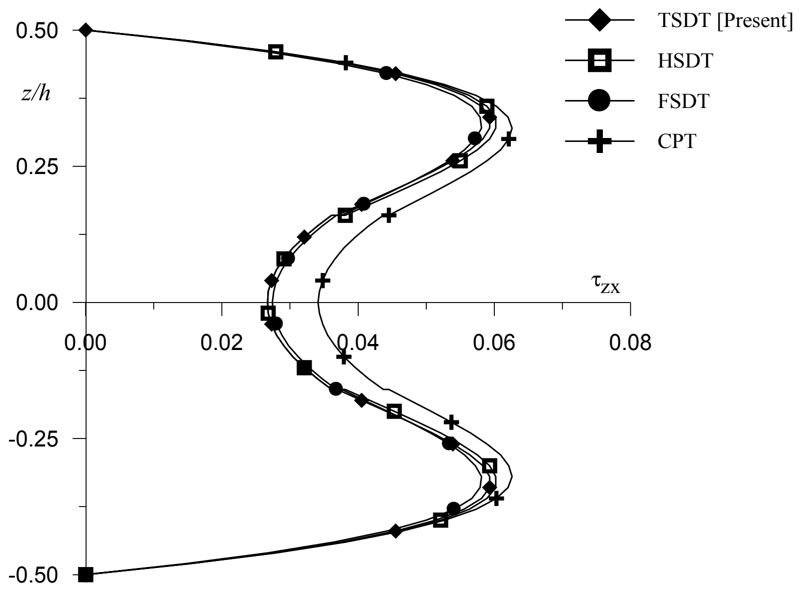

Figure 11 Variation of normalized transverse shear stress $\overline{\tau}_{zx}$ through the thickness of three-layer laminated plate for aspect ratio 10

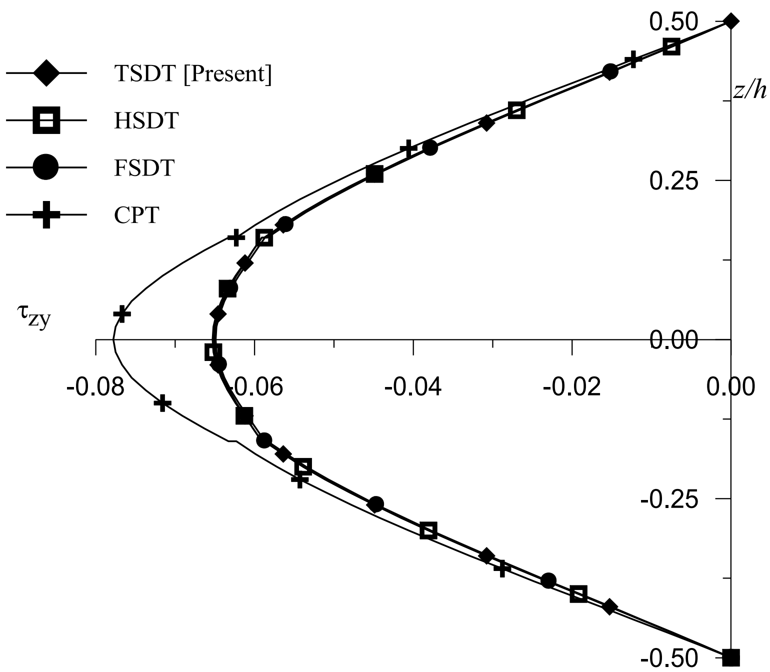

Figure 12 Variation of normalized transverse shear stress $\overline{\tau}_{zy}$ through the thickness of three-layer laminated plate for aspect ratio 10

4.1 Discussion of results

The results obtained for displacements and stresses in square orthotropic, two-layer and three-layer laminated plates under non-linear thermal load are compared and discussed with the corresponding results of classical plate theory (CPT), first order shear deformation theory (FSDT) and higher order shear deformation theory (HSDT) of Reddy [5]. It is to be noted that the complete results of displacements and stresses are specially generated using above theories for the purpose of comparison and discussion being not available in the literature for the present non-linear thermal load.

Inplane displacements $(\bar{u},\bar{v})$: Inplane displacements for orthotropic, two layer and three layer laminated plate for aspect ratio 4 and 10 are presented in Tables 1 and 10. Inplane displacements $\bar{u}$ obtained by present theory are in good agreement with HSDT and FSDT, whereas CPT over predict the inplane displacements for thick and thin plate. Inplane displacement $\bar{v}$ obtained for orthotropic plate by present theory is comparable with HSDT, whereas FSDT over predict this displacement significantly compared to that of present theory and HSDT, whereas CPT under predict the inplane displacement $\bar{v}$ for aspect ratio 4. For two layer and three layer cross-ply laminated plates, inplane displacements obtained by present theory, HSDT, FSDT and CPT are more or less identical for aspect ratio ratio 4 and 10.

Transverse displacements $\bar{w}$: The results of transverse displacements for aspect ratio 4 and 10 are presented in Tables 1 and 2. Transverse displacement obtained for orthotropic plate by present theory for aspect ratio 4 is in good agreement with higher order shear deformation theory, whereas FSDT over predict the transverse displacement for aspect ratio 4. For aspect ratio 10, the results obtained by present theory, HSDT, FSDT and CPT are more or less identical for orthotropic plate. For two layer and three layer cross-ply laminated plate, results of this displacements obtained by present theory, HSDT and FSDT are comparable for both the aspect ratios 4 and 10, whereas CPT underestimates this displacement considerably in case of three layer cross-ply laminated plate.

Inplane normal and shear stresses $(\bar{\sigma}_x,\bar{\sigma}_y,\bar{\tau}_{xy})$: Results of these stresses are presented in Tables 1 and 2 for aspect ratios 4 and 10. Inplane normal stress $\bar{\sigma}_x$ obtained for orthotropic plate and symmetric laminated plate by present theory is comparable with HSDT, whereas FSDT under predicts the normal stress $\bar{\sigma}_x$ and CPT yields much higher value for aspect ratio 4. For aspect ratio 10, results obtained by present theory are comparable with each other. The through thickness variation of normal stress $\bar{\sigma}_x$ for orthotropic plate and symmetric laminated plate are shown in Figures.1 and 9 indicating the severe effect of non-linear thermal load for aspect 10. Inplane normal stress $\bar{\sigma}_y$ obtained for orthotropic and symmetric laminated plate by present theory is comparable with HSDT and FSDT, whereas CPT over predicts the same for aspect ratio 4 and 10. The through thickness variation of $\bar{\sigma}_y$ for orthotropic plate is shown in Figure 2 for aspect ratio 10 which depicts the curvilinear behaviour. Normal stresses obtained for two layer laminated plate by present theory, HSDT, FSDT and CPT are more of less identical for aspect ratio 4 and 10. Variation of normal stresses through the thickness of antisymmetric laminated plate is shown in Figures 5 and 6. For three layer cross-ply laminated plate, distribution of this stress by CPT shows little departure in 90^0layer as shown in Figure 10. Inplane shear stresses $\overline{\tau}_{xy}$ obtained for orthotropic and symmetric laminated plates by present theory are comparable with HSDT, whereas FSDT overestimates the inplane shear stress and CPT underestimates it compared to the results of present theory and HSDT for aspect ratio 4 and 10. Inplane shear stresses obtained for two layer laminated plate by present theory, HSDT, FSDT and CPT are more or less identical for aspect ratio 4 and 10 as shown in Tables 1 and 2.

Transverse shear stresses $(\overline{\tau}_{zx}, \overline{\tau}_{zy})$: Transverse shear stresses for orthotropic, two layer antisymmetric and three layer symmetric laminated plate are presented in Tables 1 and 2 for aspect ratio 4 and 10. Transverse shear stresses $(\overline{\tau}_{zx}, \overline{\tau}_{zy})$obtained by present theory for orthotropic and symmetric laminated plate are comparable with HSDT and FSDT, whereas CPT yields much higher value for aspect ratios 4 and 10. Variation of transverse shear stresses through the thickness of orthotropic and symmetric laminated plate is shown in Figures 3- 4 and 11-12 respectively for aspect ratio 10. The variations of $\overline{\tau}_{zx}$ and $\overline{\tau}_{zy}$ are different from each other with change in sign. The distribution of these stresses by CPT shows the considerable departure in the middle layer as compared to that of other theories (see Figures 11 and 12). Transverse shear stresses for two layer laminated plate obtained by present theory are in good agreement with HSDT, FSDT and CPT for aspect ratio 4 and 10 and its variation through the thickness with change in sign is shown in Figures 7 and 8 for aspect ratio 10.

5 CONCLUSIONS

Thermal response of orthotropic, two layer antisymmetric and three layer symmetric cross-ply laminated plates under non-linear thermal load across the thickness of plate has been studied by using present trigonometric shear deformation theory. The results are compared with classical plate theory, first order shear deformation theory and higher order shear deformation theory. Present theory gives good prediction of the thermal response of laminated plates in respect of displacements and stresses. The effect of non-linear variation of thermal load through the thickness of laminated plate shows the significant effect on inplane normal and transverse shear stresses as observed from this investigation which validates the efficacy of the present theory.

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Appendix

(a) Elements of stiffness matrix $[K]$

$$k_{11} = A_{66}\frac{n^2\pi^2}{b^2} + A_{11}\frac{\pi^2 m^2}{a^2},\ k_{12} = (A_{12}+A_{66})\frac{\pi^2 nm}{ba},\ k_{13} = -B_{11}\frac{m^3\pi^3}{a^3} - (B_{12}+2B_{66})\frac{\pi^3 mn^2}{ab^2},$$

$$k_{14} = E_{11}\frac{m^2\pi^2}{a^2} + E_{66}\frac{\pi^2 n^2}{b^2},\ k_{15} = (E_{12}+E_{66})\frac{\pi^2 nm}{ba}$$

$$k_{21} = k_{12},\ k_{22} = A_{22}\frac{n^2\pi^2}{b^2} + A_{66}\frac{\pi^2 m^2}{a^2},\ k_{23} = -B_{22}\frac{n^3\pi^3}{b^3} - (B_{12}+2B_{66})\frac{\pi^3 m^2 n}{a^2 b},$$

$$k_{24} = (E_{12}+E_{66})\frac{\pi^2 nm}{ba},\ k_{25} = E_{22}\frac{n^2\pi^2}{b^2} + E_{66}\frac{\pi^2 m^2}{a^2},$$

$$k_{31} = k_{13},\ k_{32} = k_{23},\ k_{33} = D_{11}\frac{m^4\pi^4}{a^4} + (2D_{12}+4D_{66})\frac{\pi^4 m^2 n^2}{a^2 b^2} + D_{22}\frac{n^4\pi^4}{b^4},$$

$$k_{34} = -F_{11}\frac{m^3\pi^3}{a^3} - (F_{12}+2F_{66})\frac{\pi^4 mn^2}{ab^2},\ k_{35} = -F_{22}\frac{n^3\pi^3}{b^3} - (F_{12}+2F_{66})\frac{\pi^3 m^2 n}{a^2 b}$$

$$k_{41} = k_{14},\ k_{42} = k_{24},\ k_{43} = k_{34},\ k_{44} = H_{11}\frac{m^2\pi^2}{a^2} + H_{66}\frac{\pi^2 n^2}{b^2} + C_{55},\ k_{45} = (H_{12}+H_{66})\frac{\pi^2 nm}{ba}$$

$$k_{51} = k_{15},\ k_{52} = k_{25},\ k_{53} = k_{35},\ k_{54} = k_{45},\ k_{55} = H_{66}\frac{m^2\pi^2}{a^2} + H_{22}\frac{\pi^2 n^2}{b^2} + C_{44}$$

(b) Elements of load vector [f]

$$f_1 = -\frac{m\pi}{a}\left[(L_{11}+L_{12})T_{1mn} + (P_{11}+P_{12})T_{2mn} + (R_{11}+R_{12})T_{3mn}\right]$$

$$f_2 = -\frac{n\pi}{b}\left[(L_{12}+L_{22})T_{1mn} + (P_{12}+P_{22})T_{2mn} + (R_{12}+R_{22})T_{3mn}\right]$$

$$f_3 = \frac{m^2\pi^2}{a^2}\left[(S_{11}+S_{12})T_{1mn} + (T_{11}+T_{12})T_{2mn} + (U_{11}+U_{12})T_{3mn}\right]$$
$$+\frac{n^2\pi^2}{b^2}\left[(S_{12}+S_{22})T_{2mn} + (P_{12}+P_{22})T_{2mn} + (U_{12}+U_{22})T_{3mn}\right]$$

$$f_4 = -\frac{m\pi}{a}\left[(V_{11}+V_{12})T_{1mn} + (W_{11}+W_{12})T_{2mn} + (X_{11}+X_{12})T_{3mn}\right]$$

$$f_5 = -\frac{n\pi}{b}\left[(V_{12}+V_{22})T_{1mn} + (W_{12}+W_{22})T_{2mn} + (X_{12}+X_{22})T_{3mn}\right]$$