ORGINAL ARTICLE

lajss.org Latin American Journal of Solids and Structures

www.lajss.org

Postbuckling analysis of nonlocal functionally graded beams

Kurt Soncco [a], **Karl Nils Betancourt** [a], **Roman Arciniega** [a*], **JN Reddy** [b]

[a] School of Civil Engineering, Universidad Peruana de Ciencias Aplicadas, Surco, Lima, Peru. E-mail: pccikson@upc.edu.pe, pccikbet@upc.edu.pe, roman.arciniega@upc.edu.pe
[b] J. Mike Walker '66 Department of Mechanical Engineering, Texas A&M University, College Station, TX, United States.
E-mail: jnreddy@tamu.edu

* Corresponding author

https://doi.org/10.1590/1679-78256699

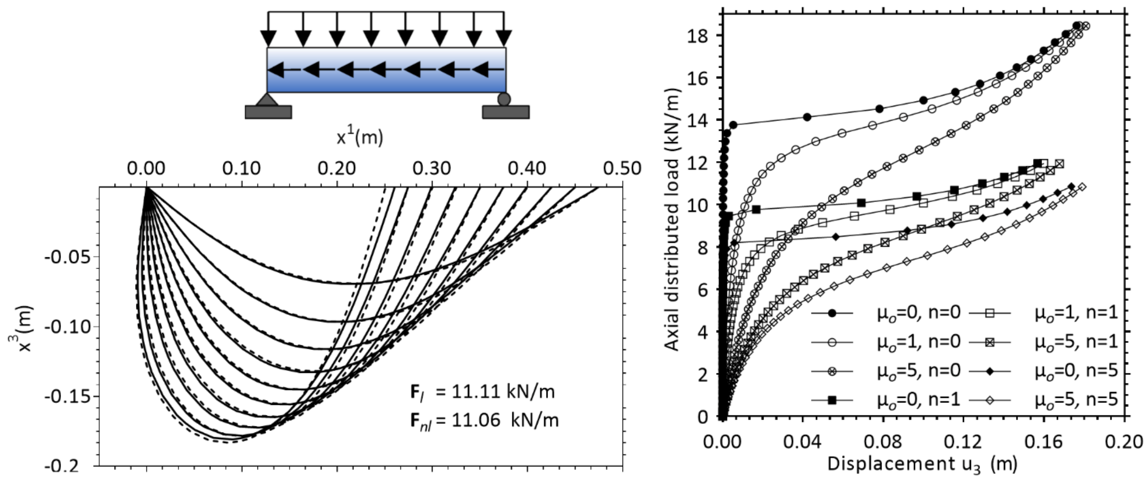
Abstract

The main goal of this research is to study the postbuckling behavior of nonlocal functionally graded beams. Eringen's nonlocal differential model is used to evaluate the influence of the material length scale in the bending response. An improved shear deformation beam theory with five independent parameters is utilized, which is suitable for the use of 3D constitutive equations. A finite element model is derived with spectral high-order interpolation functions to avoid shear locking. The formulation is verified by comparing the present results with the ones found in the literature. Functionally graded beams with different boundary conditions, nonlocal parameters, and power law indices are analyzed. It is shown that the present model can accurately predict the behavior of nonlocal beams due to the use of high-order terms in the displacement field in comparison with classical beam formulations. Finally, new benchmark problems are analyzed to show the capabilities of the present model to evaluate the effect of the nonlocal parameter and the power law index on postbuckling beam behavior.

Keywords

Non-local theories, finite element analysis, functionally graded beams, post-buckling behavior

Graphical Abstract



Received August 12, 2021. In revised form August 12, 2021. Accepted September 12, 2021. Available online September 13, 2021.
https://doi.org/10.1590/1679-78256699

1 INTRODUCTION

Over the last two decades, the interest in generalized continuum mechanical models capable of predicting the response of structural elements with material length scales has increased. Nonlocal theories started in the 20th century with the contributions of Mindlin (1964), Kröner (1967), Toupin (1962), Eringen (1972a), Eringen (1983), Eringen and Edelen (1972), among others. Nowadays, progress has been achieved in the fields of nanotechnology and the evaluation of biosensors, microsensors, and microscopes due to the improvements included in nonlocal theories. A review of this can be found in the work of Chandel et al. (2020), who provided an overview of the different models of analysis of nanostructures under different loadings and boundary conditions, as well as their application in different fields.

According to Srinivasa and Reddy (2017), within non-classical continuum mechanics, Eringen's theory can be categorized as a strain based nonlocal theory. Other examples of non-classical models with displacements as independent variables are high strain gradient models (see Toupin, 1964; Mindlin and Eshel, 1968; Yang et al., 2002). Other group of nonlocal theories is referred to as "peridynamics," and it is based on the original work of Silling (2000). It was developed to address the need of modeling discontinuities and avoid spatial derivatives. For a comprehensive review of this theory, see the work of Silling and Lehoucq (2010).

Eringen´s approach is based on the works of Kröner (1967) and Kunin (1968). Eringen and Edelen (1972) and Eringen (1972b) presented the fundamentals of nonlocal elasticity, stating that a point in the continuum is influenced by its interactions with all parts of the body. Eringen's theory addresses this by using a nonlocal modulus or kernel function in the constitutive equations of the continuum. Different forms of this functions have been obtained for applications in plane waves (Eringen, 1972b; Eringen, 1974), two-dimensional (Eringen and Ari, 1980) problems, and three-dimensional problems (Eringen, 1978). Since the integral constitutive equations that result from the direct use of these functions are hard to solve, Eringen (1983) proposed an approximation by means of a partial differential equation relating the nonlocal and classical stress tensors.

Applications of Eringen's nonlocal theory are numerous on isotropic beam and plate structures. Peddieson et al. (2003) developed an analytical solution of Euler-Bernoulli beams. Their work focuses in simply supported and cantilever beams under distributed loads. Their research led to several advances in the solution of nonlocal beams. The analytical formulation for many different theories including the nonlocal differential constitutive relations of Eringen were developed by Reddy (2007). He obtained the variational statements of this beam models using the Hamilton principle and using Navier's solution method calculated numerical solutions for static bending, buckling, and vibration. Aydogdu (2009a) studied the nonlocal bending, buckling and vibration of different beam models. He obtained the analytical solutions and compared them for different values of the nonlocal modulus and length-to-thickness ratios. Reddy (2010) developed a formulation to study the bending of Euler-Bernoulli and Timoshenko beams as well as of the first-order plate theory using Eringen's differential formulation and including the effect of von Kármán strains. His theoretical development is the start point for the future evaluation of these structures using the finite element method. Reddy and Reddy and El-Borgi (2014) used the finite element method to obtain the bending solution of Euler-Bernoulli and Timoshenko beams under different boundary conditions and distributed loads. Their formulation only included linear terms of the nonlocal parameter, and 1D constitutive equations were considered to account for the excessive stiffening effect of 3D constitutive equations. They identified that the effect of the nonlocal parameter on the deflection of beams depends on the boundary conditions and the applied distributed load, that is, it increased the deflection of simply supported beams, while causing an increase in the stiffness of cantilever beams. This inconsistency was addressed by Khodabakhshi and Reddy (2015) using the two-phase constitutive model elaborated by Eringen (1987) in its integral form. They found that the paradox regarding different types of boundary conditions could be solved using the integral version of Eringen's model and that the nonlocal parameters cause a softening effect in beams. Fernández-Sáez et al. (2016) showed a general numerical method to solve Eringen's integral model and compared their results against the ones obtained with the differential model.

Analytical solutions for the one-dimensional model of Eringen have also been reported by Tuna and Kirca (2016) in the case of bending of Euler-Bernoulli and Timoshenko beams subjected to different boundary conditions. Finite element solutions using non-uniform meshes were obtained by Tuna and Kirca (2017) for bending, buckling, and free vibration using the one-dimensional kernel in the integral form of Eringen's model. Their results had good agreement against analytical solutions of Euler-Bernoulli beams under different boundary conditions. On the other hand, exact solutions for the two-phase model used in Euler-Bernoulli beams were developed by Wang et al. (2016) and Zhu et al. (2017) for the static and buckling problems, respectively, and a satisfactory softening effect was found for different boundary conditions. Further work of this research to obtain analytical solutions for the two-phase model was carried out by Wang et al. (2019) using the Timoshenko beam theory. Their research concluded that the shear effect was evident in nonlocal beams and that extended research into high-order beam theories is necessary. With respect to the bending analysis of first-order plates by means of the standard integral formulation of Eringen, Ansari et al. (2018) proposed a finite element model including the effect of Winkler and Pasternak elastic foundations. Fakher and Hosseini-Hashemi (2020) developed the exact solution for the vibration analysis of Euler-Bernoulli beams considering the von Kármán nonlinearity. They used the two-phase integral Eringen model and evaluated different boundary conditions. They mentioned that the increase in the nonlocal parameter and the reduction of the local phase fraction causes a reduction the frequency of beams. Moreover, they found that the inclusion of the von Kármán nonlinearity causes greater nonlocal effects in the natural frequency of beams.

Functionally graded materials (FGMs), conceived first in Japan (Hirano et al., 1988), are the combination of two different materials by gradually changing their relative distributions. The mechanical and thermal behaviors of FG structures make them suitable for applications in different areas, such as aerospace, tribology, nanotechnology, biology, and high temperature technology. Many studies in FGM micro and nano beams can be found in the literature, which considers Eringen's nonlocal theory. For instance, Eltaher et al. (2013) developed a static and stability analysis of Euler-Bernoulli beams using the finite element method. They found that the increase in the nonlocal parameter increase the deflections in the cases of simply supported and clamped-hinged beams. Furthermore, the critical buckling load decreases with the influence of the nonlocal parameter. Reddy et al. (2014) used the differential model and the finite element method to analyze the nonlocal behavior of Euler-Bernoulli and Timoshenko beams including von Kármán nonlinearity. In this research they only included linear terms of the nonlocal parameter to evaluate simply supported and encastred (i.e., clamped-pinned) beams and found that the increase of both power-law index and nonlocal parameter enhances the deflections of beams. This type of beams was also evaluated by Nazemnezhad and Hosseini-Hashemi (2014) in the case of vibration. They examined the effect of the thickness to length ration, gradient index, boundary conditions, the length of the beam and the nonlocal parameter. By means of analytical methods, they obtained that the nonlocal parameter causes a reduction in the natural frequency of vibration of functionally graded beams. More recently, Srividhya et al. (2018) developed a static analysis of plates accounting for moderate rotations in the third order plate theory of Reddy (1984), and using Eringen's differential nonlocal model and two different homogenization techniques: the rule of mixtures (or Voigt rule) and the Mori-Tanaka scheme. In their research, they evaluated the effect of the length-to-thickness ratio and the nonlocal parameter in a simply supported plate. Additionally, they showed the axial and shear stress distribution of the third-order plate theory and compared it against the first-order shear deformation theory. More recently, Zhang and Qing (2020) analyzed the buckling response of functionally graded curved sandwich microbeams. They assessed the nonlocal response, utilizing a variation of the original Eringen integral model, which was proposed by Romano and Barretta (2017).

A review of the literature shows that few nonlinear studies have been conducted on nonlocal functionally graded beams. Most of them are based on classical formulations with moderate nonlinearity only. The present work presents a finite element formulation to study the postbuckling behavior of nonlocal functionally graded beams. The fundamentals of the model can be found in Arciniega and Reddy's previous works; see Arciniega and Reddy (2007b). Numerical results are presented for typical benchmark problems with applications to functionally graded nonlocal beams under different boundary conditions.

2 THEORETICAL FORMULATION

2.1 Eringen's nonlocal elasticity theory

Eringen's nonlocal theory states that the stress $\bar{\boldsymbol{\sigma}}$ at a point **X** of the continuum depends not only on the strains at that point **X**, but also on the strain field at every point of the body. Hence, according to Eringen, the nonlocal stress tensor $\boldsymbol{\sigma}$ can be expressed as

$$\boldsymbol{\sigma} = \int_V \boldsymbol{\alpha}\left(\left|x' - x\right|, \boldsymbol{\tau}\right) \bar{\boldsymbol{\sigma}}\, dV \tag{1}$$

where $\bar{\boldsymbol{\sigma}}$ is the local stress tensor and the Kernel function $\alpha(|\boldsymbol{x}' - \boldsymbol{x}|, \tau)$ represents the nonlocal modulus, $|\boldsymbol{x}' - \boldsymbol{x}|$ being the Euclidian's norm of the distance, and τ is a material constant that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength or crack length, respectively).

Due to the complexity of equation (1), Eringen proposed an equivalent differential model as a linear transformation using the Laplacian operator ∇^2, as shown in the following equation:

$$\boldsymbol{\sigma} - \boldsymbol{\mu}_o^2 \nabla^2 \boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}, \qquad \boldsymbol{\mu}_o = \boldsymbol{\tau}^2 \boldsymbol{l}^2 = \boldsymbol{e}_o^2 \boldsymbol{a}^2 \tag{2}$$

where $\boldsymbol{\sigma}$ is the nonlocal stress tensor, e_O is a material constant, and a and l are the internal and external characteristics lengths, respectively.

The original equation of Eringen (1983) used the engineering stress tensor. However, equation (2) can be extended to include the second Piola-Kirchoff stress tensor, which is conjugated with Green-Lagrangian strain tensor (see Reddy 2013 and 2015).

In this section, we derive the mathematical formulation of the theory by using the tensor notation, which is independent of any coordinate system. The simplicity of this approach allows us to show many important equations in a simple and compact way.

Let $\{x^i\}$ be a set of Cartesian coordinates with orthonormal basis $\{\boldsymbol{e}_i\}$. The neutral axis of the beam is defined by the coordinate x^1. The displacement field is assumed to be of the following form (Arciniega and Reddy, 2007a)

$$\mathbf{v}(x^1, x^3) = \mathbf{u}(x^1) + x^3 \boldsymbol{\phi}(x^1) + (x^3)^2 \boldsymbol{\psi}(x^1) \tag{3}$$

where $\boldsymbol{u} = u_i \boldsymbol{e}_i$ denotes the displacement vector of the neutral axis, $\boldsymbol{\varphi} = \varphi_i \mathbf{e}_i$ and $\boldsymbol{\psi} = \psi_3 \mathbf{e}_3$ are difference vectors (i=1,3). Equation (3) contains five independent variables. The quadratic term $\boldsymbol{\psi}$ is included to avoid Poisson's locking, therefore, no reduced constitutive equations are needed.

For the given displacement field, we define the Green-Lagrange strain tensor as

$$\begin{aligned} &\mathbf{e} = \mathbf{e}^{(0)} + x^3 \mathbf{e}^{(1)}, \\ &\mathbf{e}^{(i)} = \varepsilon_{11}^{(i)} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \varepsilon_{33}^{(i)} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \varepsilon_{13}^{(i)} \boldsymbol{e}_1 \otimes \boldsymbol{e}_3, \qquad i = 0,1 \end{aligned} \tag{4}$$

where high order terms are neglected. We can express equation (4) in indicial notation as:

$$\begin{aligned} &\varepsilon_{11}^{(0)} = u_{1,1} + \frac{1}{2}\left(u_{1,1}^2 + u_{3,1}^2\right) && \varepsilon_{11}^{(1)} = \varphi_{1,1} + u_{1,1}\varphi_{1,1} + u_{3,1}\varphi_{3,1} \\ &\varepsilon_{13}^{(0)} = \frac{1}{2}\left(\varphi_1 + u_{3,1} + u_{1,1}\varphi_1 + u_{3,1}\varphi_3\right) && \varepsilon_{13}^{(1)} = \frac{1}{2}\left(\varphi_{3,1} + \varphi_{1,1}\varphi_1 + 2u_{3,1}\psi_3 + \varphi_{3,1}\varphi_3\right) \\ &\varepsilon_{33}^{(0)} = \varphi_3 + \frac{1}{2}\left(\varphi_1^2 + \varphi_3^2\right) && \varepsilon_{33}^{(1)} = 2\psi_3 + 2\varphi_3\psi_3 \end{aligned} \tag{5}$$

when it is written in terms of the five components of the displacement field. It's important to highlight that the formulation used does not satisfy the zero traction free boundary conditions, which requires the use of shear correction factor in the definition of stress resultants.

2.3 Principle of virtual work

The weak form can be easily constructed using the principle of virtual displacements (see Reddy, 2015). The virtual work statement is nothing but the weak form of the equilibrium equations, and it is valid for linear and nonlinear constitutive relations. We define the configuration solution of the beam by the triplet $\Phi := (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\psi})$. Thus

$$\mathcal{G}\left(\Phi, \delta\Phi\right) = \mathcal{G}_{\text{int}}\left(\Phi, \delta\Phi\right) - \mathcal{G}_{\text{ext}}\left(\Phi, \delta\Phi\right) = \int_{x_1} \left(\mathbf{N}^{(0)} \cdot \delta\boldsymbol{\varepsilon}^{(0)} + \mathbf{N}^{(1)} \cdot \delta\boldsymbol{\varepsilon}^{(1)}\right) dx^1 - \int_{x_1} \mathbf{p} \cdot \delta\mathbf{u}\, dx^1 \tag{6}$$

where $\delta\Phi := (\delta\mathbf{u}, \delta\boldsymbol{\varphi}, \delta\boldsymbol{\psi})$, $\mathbf{N}^{(i)}$ is the nonlocal stress resultant tensor and $\mathbf{p}$ is the body forces acting on the beam per unit length.

For straight beams, Eringen´s differential model can be expressed in terms of the linearized nonlocal stress resultant tensors, namely

$$\mathbf{N}^{(i)} - \mu_0^2 \nabla^2 \mathbf{N}^{(i)} = \overline{\mathbf{N}}^{(i)}, \qquad i = 0,1 \tag{7}$$

Let $\overline{\mathbf{N}}^{(i)}$ denote the local stress resultant tensor, which is a symmetric tensor. The tensor $\overline{\mathbf{N}}^{(i)}$ is defined as

$$\begin{aligned} \overline{\mathbf{N}}^{(i)} &= \mathbb{B}^{(i)} \boldsymbol{\varepsilon}^{(0)} + \mathbb{B}^{(i+1)} \boldsymbol{\varepsilon}^{(1)}, \\ \mathbb{B}^{(k)} &= \int_{-h/2}^{h/2} (x^3)^k \mathbb{C} dx^3, \qquad i = 0,...,2 \end{aligned} \tag{8}$$

The components of the tensor $\mathbb{B}^{(i)}$ are the material stiffness coefficients and $\mathbb{C}$ is the fourth-order elasticity tensor. Furthermore, the linearized equilibrium equations are obtained from the virtual work statement through integration by parts. It means

$$\begin{aligned} &\delta u_1 : \frac{dN_{11}^{(0)}}{dx_1} - f\left(x_1\right) = 0 \qquad \delta u_3 : \frac{dN_{13}^{(0)}}{dx_1} - q\left(x_1\right) = 0 \\ &\delta \phi_1 : -N_{13}^{(0)} + \frac{dN_{11}^{(1)}}{dx_1} = 0 \qquad \delta \phi_3 : -N_{33}^{(0)} + \frac{dN_{13}^{(1)}}{dx_1} = 0 \\ &\delta \psi_3 : 2N_{33}^{(1)} + \frac{dN_{13}^{(2)}}{dx_1} = 0 \end{aligned}$$

Next, we obtain an explicit expression for nonlocal stress resultants. From Eq. (9), we use the first three equilibrium equations. Substituting equation (8) in (7), and after some manipulations, we obtain:

$$\mathbf{N}^{(i)} = N_{11}^{(i)} \mathbf{e}_1 \otimes \mathbf{e}_1 + N_{33}^{(i)} \mathbf{e}_3 \otimes \mathbf{e}_3 + N_{13}^{(i)} \mathbf{e}_1 \otimes \mathbf{e}_3, \qquad i = 0,1 \tag{10}$$

where

$$\begin{aligned} N_{11}^{(0)} &= B_{1111}^{(0)} \varepsilon_{11}^{(0)} + B_{1133}^{(0)} \varepsilon_{33}^{(0)} + B_{1111}^{(1)} \varepsilon_{11}^{(1)} + B_{1133}^{(1)} \varepsilon_{33}^{(1)} - \mu_0 \frac{df_1}{dx^1} \\ N_{11}^{(1)} &= B_{1111}^{(1)} \varepsilon_{11}^{(0)} + B_{1133}^{(1)} \varepsilon_{33}^{(0)} + B_{1111}^{(2)} \varepsilon_{11}^{(1)} + B_{1133}^{(2)} \varepsilon_{33}^{(1)} - \mu_0 f_3 \\ N_{33}^{(0)} &= B_{1133}^{(0)} \varepsilon_{11}^{(0)} + B_{3333}^{(0)} \varepsilon_{33}^{(0)} + B_{1133}^{(0)} \varepsilon_{11}^{(0)} + B_{3333}^{(1)} \varepsilon_{33}^{(1)} \\ N_{33}^{(1)} &= B_{1133}^{(0)} \varepsilon_{11}^{(0)} + B_{3333}^{(0)} \varepsilon_{33}^{(0)} + B_{1133}^{(2)} \varepsilon_{11}^{(1)} + B_{3333}^{(2)} \varepsilon_{33}^{(1)} \\ N_{13}^{(0)} &= B_{1313}^{(0)} \varepsilon_{13}^{(0)} + B_{1313}^{(1)} \varepsilon_{13}^{(1)} - \mu_0 \frac{df_3}{dx^1} \\ N_{13}^{(1)} &= B_{1313}^{(1)} \varepsilon_{13}^{(0)} + B_{1313}^{(2)} \varepsilon_{13}^{(1)} \end{aligned} \tag{11}$$

f_1, f_3 are the axial and transverse body forces, $B_{ijkl}^{(r)}$ are the components the material stiffness and $\varepsilon_{kl}^{(r)}$ are the components of the strain tensor of order *r*.

2.4 Functionally Graded materials

Functionally graded materials are composite materials, where the properties of two materials (usually ceramics and metals) varied in a predetermined manner from the bottom to the top surface of the beam (or plate). These materials are considered microscopically inhomogeneous but isotropic and are commonly used to mitigate severe stress variations that occur between the layers of a laminated composite structure. Functionally graded materials are used in applications where both effects, heat transfer and stress resistance, are important. For example, in applications involving a combination of ceramic and metal, the ceramic provides less thermal conductivity whereas the metal provides resistance against stress due to its ductility.

In these two-phase functionally graded materials, the properties are assumed to vary through the thickness of the beam. Therefore, the tensor $\mathbb{C}$ is a function of the thickness coordinate, x^3. The elastic coefficients of an FGM are expressed as:

$$C_{ijkl} = C_{ijkl}{}^{c} f_c + C_{ijkl}{}^{m} f_m \tag{12}$$

In the equation above, super index c and m represent the properties from ceramic and metal, respectively, C_{ijkl} refers to the material stiffness coefficients, and f is the volume fraction of the constituent expressed as (see Reddy, 2022):

$$f_c = \left(\frac{x^3}{h} + \frac{1}{2}\right)^n, \quad f_m = 1 - f_c \tag{13}$$

where n is the power law index exponent that represents the variation of the material along the thickness. The μ_o parameter is assumed constant.

3 FINITE ELEMENT FORMULATION

Let Ω be the domain of the neutral axis of the beam, which is discretized into *N* elements, such that:

$$\Omega = \bigcup_{e=1}^{N} \Omega^e \tag{14}$$

Recall that $\widehat{\Omega}^e = [-1,1]$ is a parent domain in ξ-space and $x^1(\xi): \widehat{\Omega}^e \in R \rightarrow \Omega^e$. The finite element equations are obtained by interpolating the components of the field variables written in terms of the base vectors. Namely,

$$\begin{aligned} &\mathbf{u}^{hp}\left(x^1\right) = \left(\sum_{j=1}^{m} u_k^{(j)} \phi^{(j)}\left(\xi\right)\right)\mathbf{e}_k, \quad \boldsymbol{\varphi}^{hp}\left(x^1\right) = \left(\sum_{j=1}^{m} \varphi_k^{(j)} \phi^{(j)}\left(\xi\right)\right)\mathbf{e}_k, \\ &\boldsymbol{\psi}^{hp}\left(x^1\right) = \left(\sum_{j=1}^{m} \psi_3^{(j)} \phi^{(j)}\left(\xi\right)\right)\mathbf{e}_3, \qquad k = 1,3 \end{aligned} \tag{15}$$

where ϕ^j stands for the one-dimensional interpolation functions and $u_k^{(j)}$, $\varphi_k^{(j)}$ and $\psi_3^{(j)}$ are the values of these variables at the nodes of each element.

With regard to the interpolation polynomials, Chinosi et al. (1998), Hakula et al. (1996), and Pontaza and Reddy (2005) have shown that the use of high order interpolation functions brings many advantages because it allows more accurate approximations and precludes membrane and shear locking in the finite element solutions. Moreover, the use of equally spaced gauss points for the evaluation of the interpolation functions leads to numerical problems due to oscillations at the edges of the elements (see Karniadakis and Sherwin (2005)). This undesired effect is more evident as the order of the interpolation functions increases. To overcome this, non-equally spaced Gauss-Lobatto-Legendre points are used, which are suitable for high-order expansions with no oscillatory effect. Therefore, the selected one-dimensional coordinates for $\widehat{\Omega}^e$ are the roots ξ_i of the expression

$$\left(\xi - 1\right)\left(\xi + 1\right) L_P^{'}\left(\xi\right) = 0 \tag{16}$$

where $L_p'(\xi)$ is the derivative of the one-dimensional interpolation functions $L_p(\xi)$. The selected polynomials of order $p = m - 1$ are expressed using the p-order Legendre polynomial P_{m-1} by means of the equation

$$\psi_j = \prod_{i=1, j\neq i}^{m} \frac{\xi - \xi_j}{\xi_j - \xi_i} \tag{17}$$

After the discretization of the continuum system in (14), we arrive at a set of nonlinear algebraic equations which are solved by the spherical arc-length method with a line search procedure (Zhou and Murray, 1995; Ritto-Correa and Camotim, 2008). The computational program was implemented in MATLAB and full integration was used to evaluate the variational energy terms.

4 NUMERICAL RESULTS

4.1 Preliminary comments

The results presented here are divided in two parts. In the first section, we aim to verify the validity of our model. For this reason, macro and micro beams are evaluated using the Eringen's nonlocal parameter μ_0 and the power-law index n, and compared with those available in the literature. Next, novel results involving post-buckling behavior with distributed loads are shown. The problems are evaluated using a 16-element mesh and spectral interpolation functions of Gauss-Legendre-Lobatto type (with P=4), to avoid the numerical problems explained in the previous section. Finally, the shear correction factor required for these exercises is 5/6, for all rectangular sections.

First we analyze an isotropic simply supported beam to compare the results of our formulation against the linear analytical solutions given by Aydogdu (2009a). For this parametric analysis, we considered a Poisson's ratio of 0.3, an elastic modulus of 1GPa, and 1m as the height of the beam, which was subjected to a uniformly distributed load of 1 N/m. In Table 1, results for the nondimensional center transverse deflection $\bar{u}_3 = 100u_3(EI)/(q(x^1)L^4)$ are shown, where I is the moment of inertia of the rectangular section of the beam, and L its length.

Table 1. Comparison of nondimensional center transverse deflection under a uniformly distributed load.

L/h	μ_0	Aydogdu (2009a)					Present
		EBT	TBT	RBT	LBT	ABT	
10	0	1.3130	1.3483	1.3483	1.3487	1.3480	1.2174
	1	1.4487	1.4949	1.4927	1.4954	1.4921	1.3311
	2	1.5844	1.6414	1.6371	1.6421	1.6362	1.4449
	3	1.7201	1.7880	1.7815	1.7888	1.7802	1.5586
	4	1.8558	1.9345	1.9260	1.9355	1.9243	1.6724
20	0	1.3130	1.3218	1.3218	1.3219	1.3218	1.1930
	1	1.3469	1.3564	1.3566	1.3566	1.3563	1.2215
	2	1.3808	1.3910	1.3909	1.3912	1.3908	1.2499
	3	1.4148	1.4256	1.4254	1.4258	1.4253	1.2783
	4	1.4487	1.4602	1.4599	1.4604	1.4598	1.3068
50	0	1.3130	1.3144	1.3144	1.3144	1.3144	1.1862
	1	1.3184	1.3199	1.3199	1.3199	1.3199	1.1907
	2	1.3239	1.3253	1.3253	1.3253	1.3253	1.1953
	3	1.3313	1.3307	1.3307	1.3308	1.3308	1.1998
	4	1.3347	1.3362	1.3362	1.3362	1.3362	1.2044

In Table 1, EBT stands for the Euler-Bernoulli beam theory, TBT for the Timoshenko beam theory, RBT for the Reddy Beam Theory (Reddy, 1984), LBT for the Levinson Beam Theory (Levinson, 1981), and ABT for the Aydogdu Beam Theory (Aydogdu, 2009b). Good agreement is found against the results provided by Aydogdu (2009a). It is observed that our formulation generates smaller deflections, because of the use of 3D constitutive relations.

In the following analysis, the geometry and material parameters are as in Figure 1.

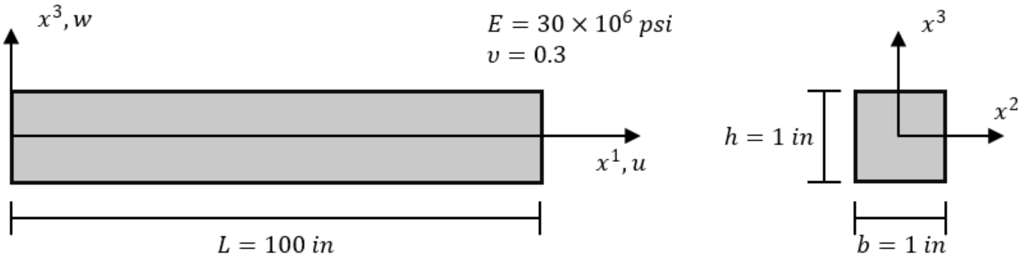


Figure 1. Geometry and material properties of the beam.

We evaluate a simply supported beam under a uniformly distributed load. The results are shown in Figures 2-4. Figure 2 shows the center deflection for different loads. In this figure, the influence of the Poisson ratio is shown. The present model allows us to evaluate the effect of this material property, because of the use of the 3D constitutive relations stated in section 2. Additionally, our formulation does not exhibit any locking. As it can be seen, the presence of Poisson's ratio reduces the deflections.

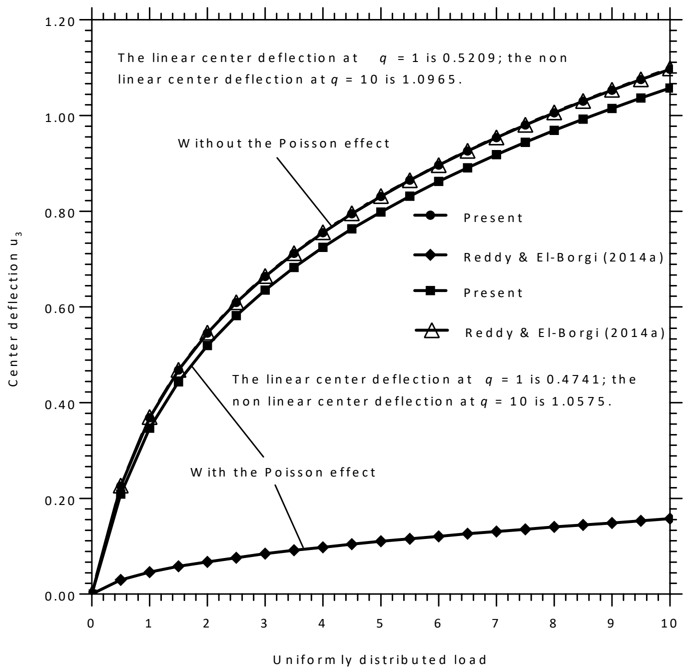


Figure 2. Simply supported beams under uniformly distributed loads with Poisson coefficient (Reddy and El-Borgi, 2014).

In Figure 3, the same boundary conditions are evaluated for different values of the nonlocal parameter μ_0 and zero as the Poisson modulus. As it is shown, the present model matches with the reference results; as the nonlocal parameter increases, the deflections of the beam also increase. To observe the effect of Poisson's ratio using the same nonlocal parameters, the center deflection is also depicted in Figure 4. It is observed that the Poisson effect makes the beam stiffer with lower deflections in comparison with Figure3.

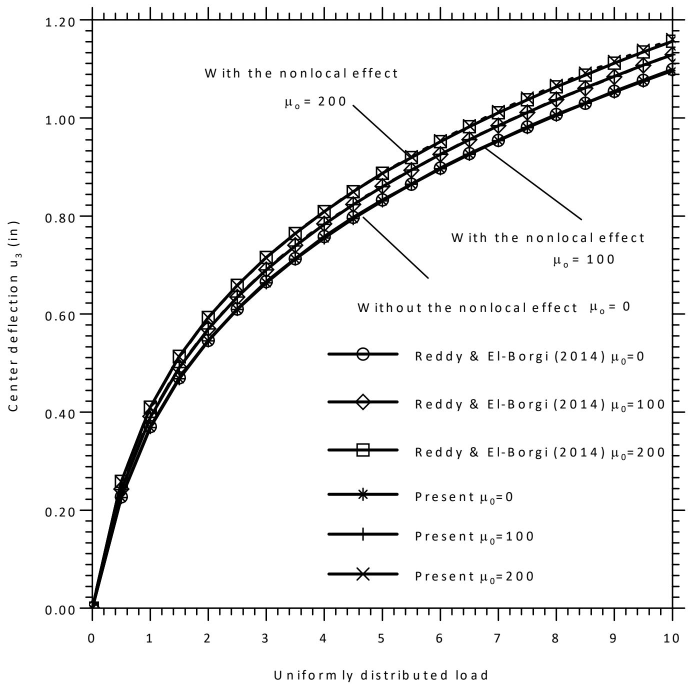


Figure 3. Simply supported beams under uniformly distributed loads without Poisson coefficient.

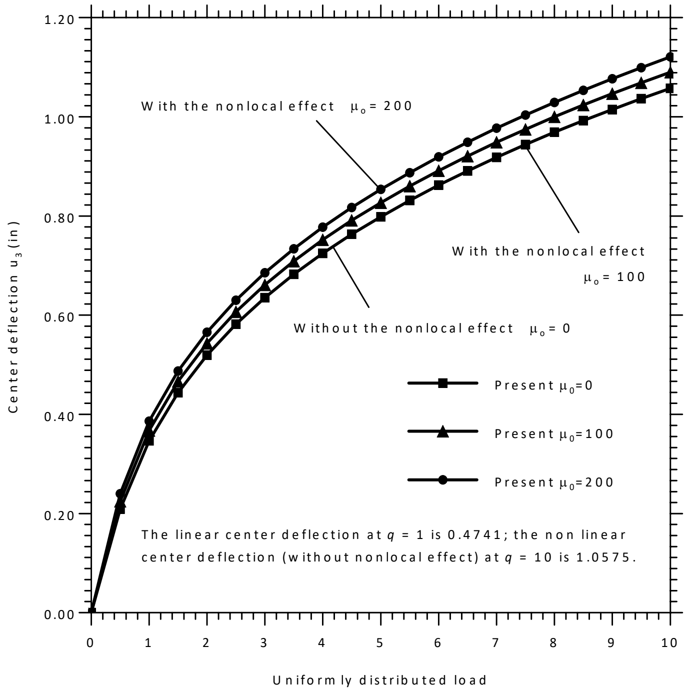


Figure 4. Simply supported beam under uniformly distributed load with the effect of Poisson coefficient.

4.3 Analysis of functionally graded macrobeams

Next, we study the behavior of an encastred (clamped-pinned) beam with the same geometry as shown in the previous section. For this case, the following FGM material properties are considered: $E_1 = 30 \times 10^6$ psi and $E_2 = 3 \times 10^6$ psi. Figure 5 shows the center deflection for different values of the power law index. Notice that, in accordance with several authors mentioned in the introduction, encastred beam are not affected by the nonlocal parameter because of the limitations of the differential model of Eringen when uniformly distributed loads are applied. Because of this, only the influence of the power law index was evaluated. Moreover, the Poisson's ratio was set to zero in order to compare our model against the literature. It can be seen that, as the FGM index increases, the deflections also increase and the effective elastic modulus of the beam gets closer to E_2.

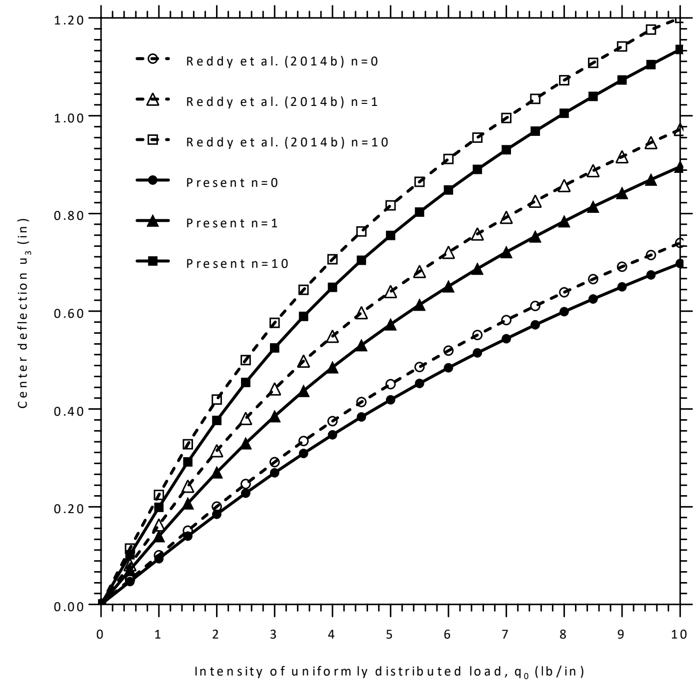


Figure 5. Encastred functionally graded beams under uniformly distributed loads.

On the other hand, when an encastred beam is subjected to a sinusoidal distributed load, the nonlocal parameter does influence the deflection, as it can be seen in Figure 6. The reason for this is because of the continuous variation of the load, which can be derived in equation 7. Conversely, when a uniform load is applied to a nonlocal encastred beam the effect of the nonlocal parameter is null. The figure shows that as the nonlocal parameter increases, the center deflection increases too. Moreover, the present IFSDT model shows reduced deflections because of the consideration of 3D constitutive relation, which has been shown previously in Figure 3.

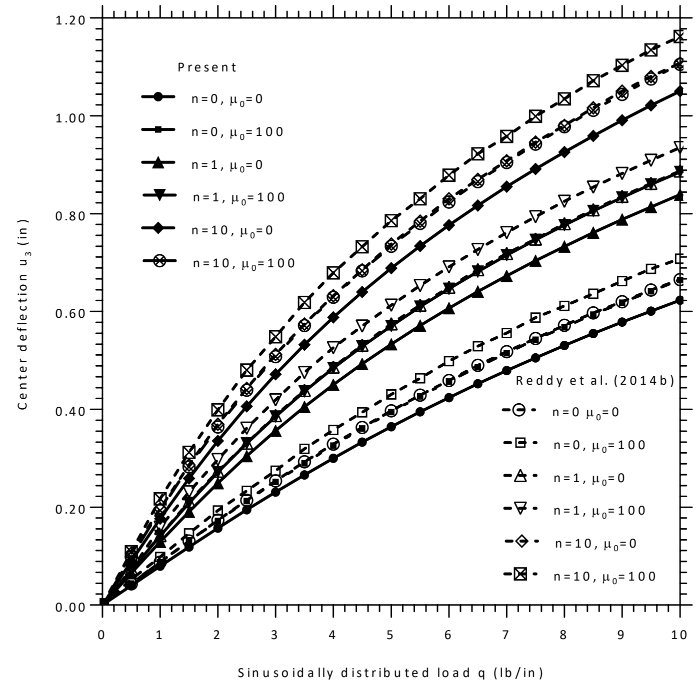


Figure 6. Encastred functionally graded beams under sinusoidally distributed loads.

4.4 Analysis of functionally graded microbeams

In the following examples, the geometry and material parameters analyzed are illustrated in Figure 7. The mentioned parameters have been extracted from the literature.

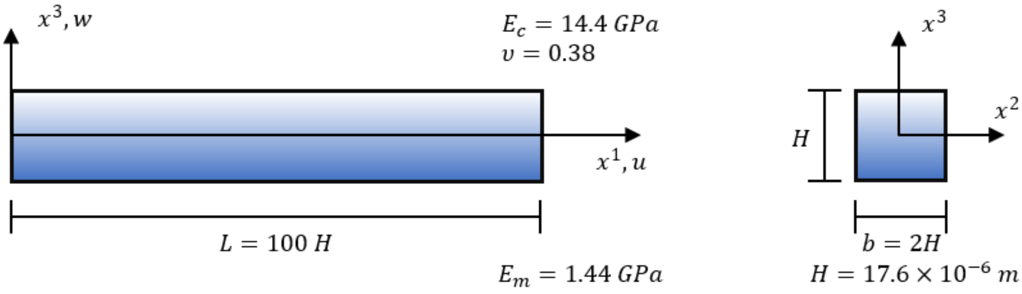


Figure 7. Geometry and material properties of FGM microbeam.

In Figure 8, a simply supported microbeam under sinusoidally distributed load is shown. As before, FGM nonlocal parameters increase the center-deflection of the beam. Yet, the effect on microbeams is amplified, especially by the nonlocal parameter's effect. This observation relates with Eringen's theory on the effect of the neighborhood is greater as the element size is reduced. Additionally, as shown previously, the present model exhibits lower deflection, as the Poisson's effect is considered and relates to the shear effect. As a comparison, the FGM local microbeam results ($\mu_0 = 1, n = 0$) indicates a variation of 25.8% between classical and IFSDT models. Also, the FGM nonlocal microbeam results ($\mu_0 = H/2, n = 1$), the variation of 7.69% between classical Timoshenko and IFSDT models.

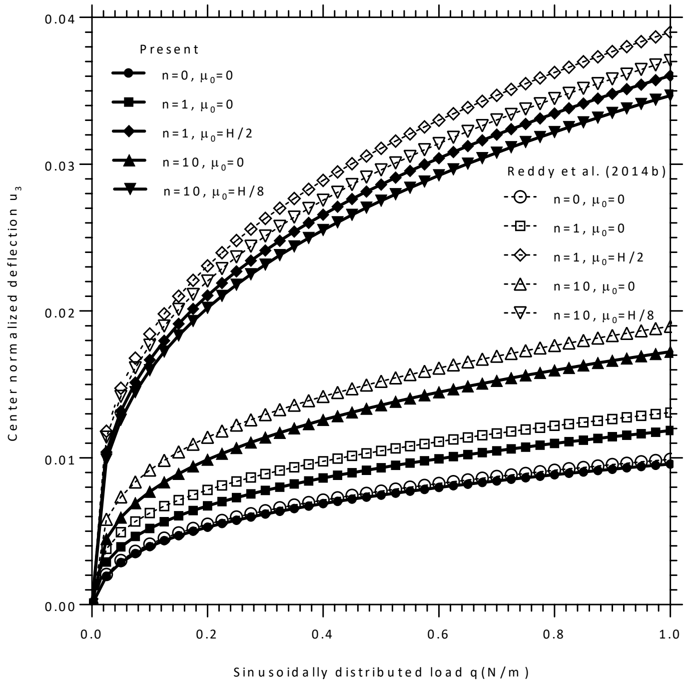


Figure 8. Simply supported functionally graded beam under sinusoidally distributed loads.

Figure 9 shows the results of an encastred microbeam subjected to sinusoidal distributed loads. As the results shown, the increase in the nonlocal parameter does increase the displacements as in Figure 8. Again, the results seen in Figure 9 show that our formulation cause smaller displacements than the Timoshenko beam.

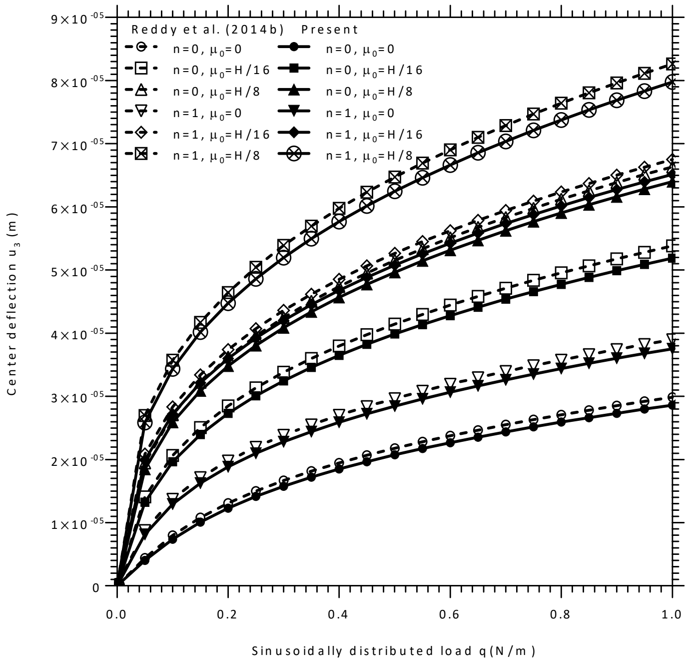


Figure 9. Encastred functionally graded beams under sinusoidally distributed load.

4.5 Postbuckling analysis of nonlocal FGM beams

In this section, we show novel results combining postbuckling behavior and FGM nonlocal models. The authors' motivation is related to the small number of studies available in the literature on this topic. These benchmark problems are analyzed with both boundary conditions (simply supported and encastred), with various power-law index and nonlocal parameters. Some examples have been adapted based on the work of Sze et al. (2004) by considering distributed instead of pinching point loads. This is because concentrated forces have no effect on Eringen's differential model. One possible way of overcoming this problem could be to transform the concentrated forces to distributed loads by means of the Dirac-Delta function (Wang and Liew, 2007; Civalek and Demir, 2011). However, the application of the Dirac-delta function on finite elements models is not efficient, since it requires a large number of integration points to accurately describe it.

The first proposed benchmark is a cantilever beam subjected to distributed axial and transversal loads. This configuration will trigger the postbuckling behavior. A proportion of one to a thousand is set between the transversal and axial loads, respectively. The geometry and material properties are taken from Soncco et al. (2019), and are shown in Figure 10.

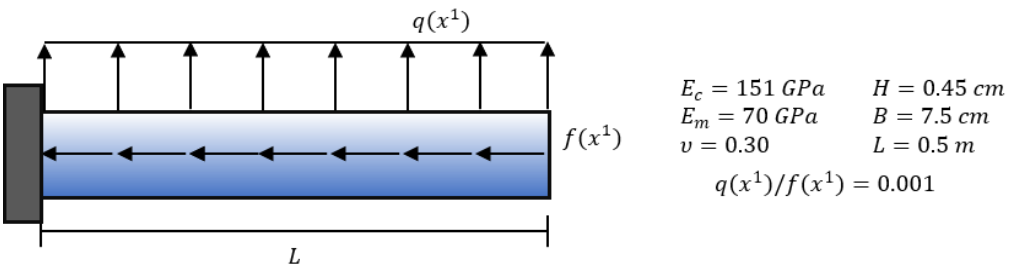


Figure 10. Geometry and material properties for clamped FGM beam - postbuckling analysis.

In Figure 11, the post-buckling behavior of the nonlocal beam is depicted. The axial displacement at the free end vs the intensity of the distributed axial load is shown. The nonlocal parameter produces lower critical loads in comparison with local beams. Moreover, as it can be seen, the stepped buckling behavior changes to a smoother curve as the nonlocal parameter increases. Figure 12 shows an increase in the vertical displacement at the free end in the early stages of the loading process as the nonlocal parameter increases. This is evident when beams with the same values of the power law index are evaluated by only changing the nonlocal parameter. However, after the displacement of the tip gets closer to 0.4 m, the displacement converges for all the different values of the nonlocal parameter. This could be an indicator that under greater loads the influence of the nonlocal parameter is reduced. It is also observed that the power-law index reduces the buckling load of each beam.

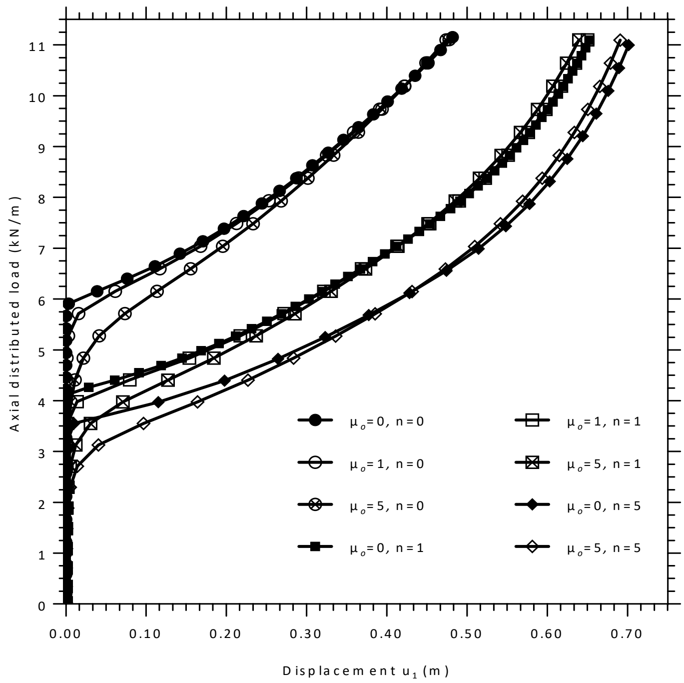


Figure 11. Tip-displacement u_1 for clamped FGM beam under uniformly buckling load.

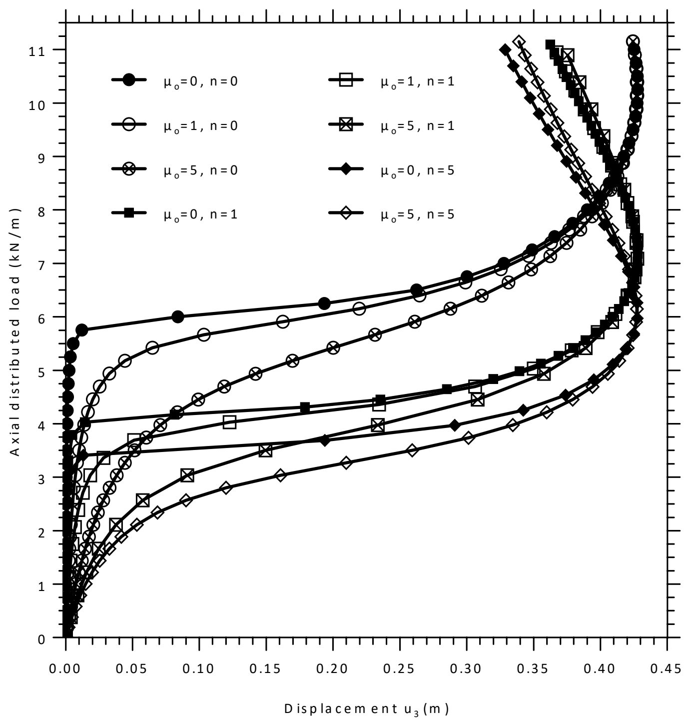


Figure 12. Tip-deflection u_3 for clamped FGM beam under uniformly buckling load.

The second example is a simply supported beam under uniform axial and transversely distributed loads. These loads, as previously mentioned, have a proportional relation to induce the postbuckling behavior on the element. The geometrical and material properties are described in Figure 13.

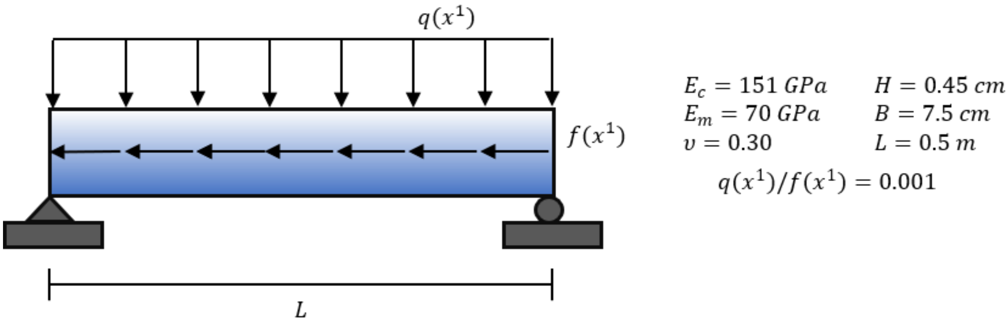


Figure 13. Geometry and material properties for simply supported FGM beam.

The results of the present model can be seen in Figures 14 and 15. This model predicts the postbuckling behavior of a simply supported beam, as proposed by Pagani and Carrera (2016). However, some issues arise with this approximation, especially when the end node's axial displacement approaches near L/2, because the finite element solution becomes unstable. Even with this limitation, the model can describe the influence of the power-law index and nonlocal parameter in the early stages of the load. In Figure 14, the intensity of the axial distributed load against mid-plane displacement u_1 is shown. As the nonlocal parameter increases, the transition between the linear and post-buckled instances becomes smoother, in contrast with the local model. Regarding the power law index, it is seen that it reduces the critical distributed load because the material properties are closer to the metallic constituent, which makes the model more flexible. In Figure 15, the intensity of axial distributed load vs. the mid-plane deflection u_3 is shown and the same effect of the power law index and nonlocal modulus is observed.

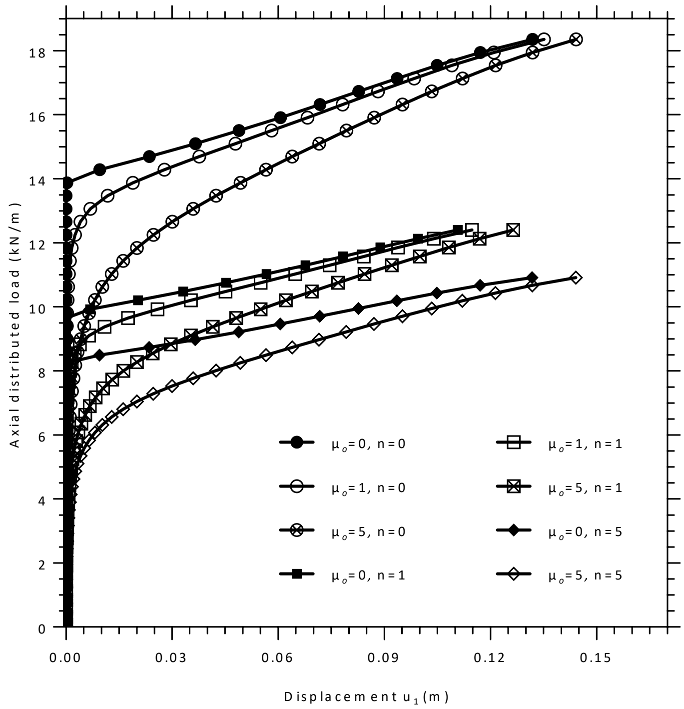


Figure 14. Mid-plane displacement u_1 for simply supported FGM beam under uniform load.

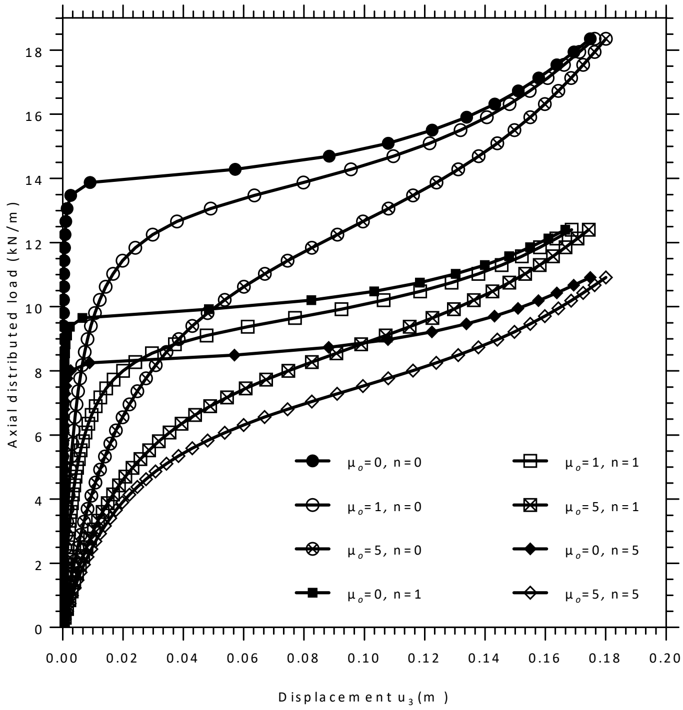


Figure 15. Mid-plane deflection u_3 for simply supported FGM beam under uniformly load.

Finally, the deformed configurations of three different beams are depicted in Figures 16-18. These figures allow us to observe the differences between the local and nonlocal model, when the end node of the simply supported beam is in the same coordinate. Finally, an aspect of the nonlocal model is the lower distributed load required to perform the same deformation configuration as the local model, which implies that the contribution of the nonlocal properties reduces its stiffness, which generates more deformation compared to classical models.

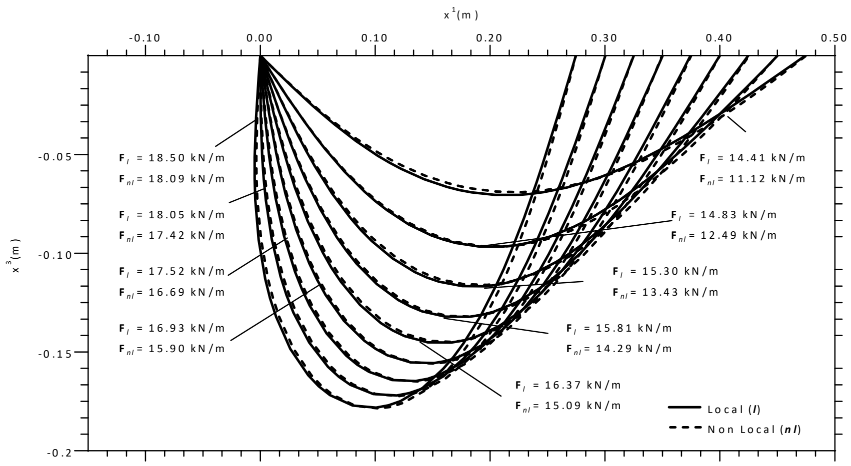


Figure 16. Postbuckling configurations of a simply supported beam under uniform axial load $\mu_o = 5$ $(n = 0)$

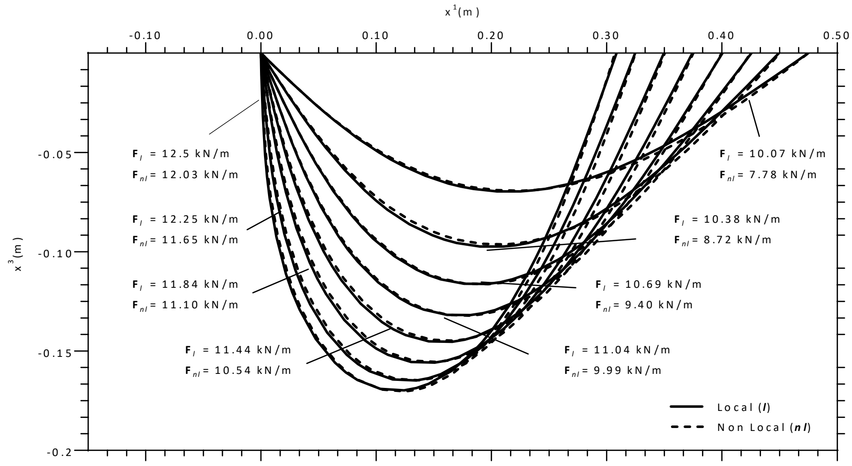


Figure 17. Postbuckling configurations of a simply supported beam under uniform axial load $\mu_o = 5$ $(n = 1)$

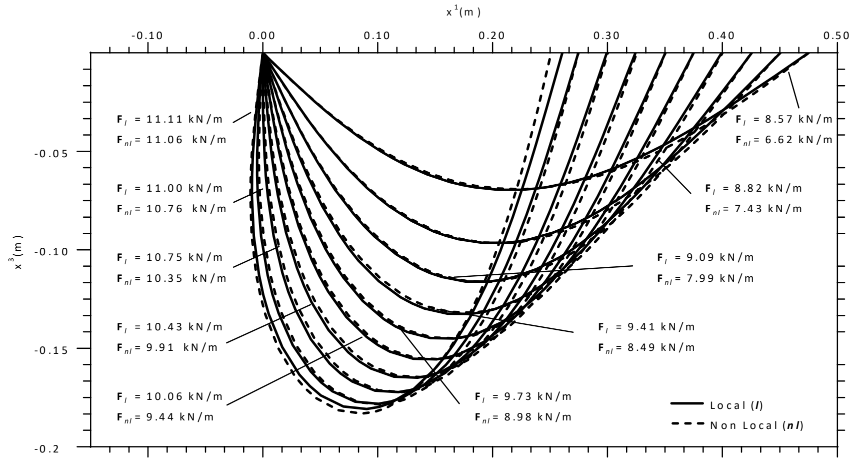


Figure 18. Postbuckling configurations of a simply supported beam under uniform axial load $\mu_o = 5$ $(n = 5)$

5 CONCLUSIONS

In this paper, we have presented a post-buckling study of functionally graded nonlocal beams. The formulation was based on an improved beam theory (IFSDT) with five independent parameters that account for thickness stretch and 3D constitutive relations. A tensor-based finite element model was developed for geometric nonlinear analysis of the nonlocal beam. The Eringen's nonlocal differential formulation was introduced into a finite deformation model. An element with high-order Gauss-Lagrange-Lobatto interpolations was used to avoid membrane and shear locking, and the nonlinear solution scheme was based on a spherical arc-length method using line searches.

We evaluated the effect of the power law index in nonlocal functionally graded macrobeams and microbeams and validated our results against those found in the literature. We observed excellent agreement, where the present formulation showed lower deflections in comparison with classical beam models.

Finally, we have evaluated the post-buckling behavior of functionally graded nonlocal beams subjected to distributed loads. Both a cantilever and simply supported beams are analyzed. In both cases, we found that the nonlocal parameter causes a smoother transition between the linear and postbuckling stages of the loading process and that it increases the deflections in the early stages of the loading. Additionally, we observed that as the tip displacement of the beam got closer to certain value, the effect of the nonlocal parameter is reduced and displacements of different beams with the same power law index, but different nonlocal parameter, nearly converged. Ultimately, we showed the deformed configurations of a simply supported nonlocal beam and local beam with three different combinations of nonlocal modulus and power law index. It was noted that the nonlocal beam requires smaller loads to reach the same deformed configurations.

It was found that the present model causes a softening effect for simply supported functionally graded beams in the postbuckling regime. We believe that research on the integral form of Eringen's model for the evaluation of this non-linear phenomenon on functionally graded and isotropic micro-and nano-beams should be pursued, especially as experiments and empirical data are being observed in this realm Motz et al. (2008); Yayli (2016). Moreover, the authors encourage the study of beams and plates with other nonclassical theories, such as micropolar or micromorphic formulations (see Karttunen et al. 2018; Karttunen et al. 2019; Chowdury and Reddy, 2019; and Nampally and Reddy, 2020), in finite element models.

Acknowledgements

The authors gratefully acknowledge the Universidad Peruana de Ciencias Aplicadas for providing financial support for this project under the VII annual research incentive grant.

Author's Contributions: Conceptualization, K Soncco, N Betancourt, R Arciniega; Methodology, K Soncco, N Betancourt, R Arciniega; Investigation, K Soncco, N Betancourt; Writing – original draft, K Soncco, N Betancourt; Writing - review & editing, K Soncco, N Betancourt, R Arciniega, JN Reddy; Funding acquisition, R Arciniega.

Editor: Marcílio Alves.

References

Ansari, R., Torabi, J. and Norouzzadeh, A. (2018). Bending analysis of embedded nanoplates based on the integral formulation of Eringen's nonlocal theory using the finite element method. Physica B: Condensed Matter 534: 90–7

Arciniega, R A and Reddy, J.N. (2007a). Large deformation analysis of functionally graded shells. 44: 2036–52

Arciniega, R. A. and Reddy, J.N. (2007b). Tensor-based finite element formulation for geometrically nonlinear analysis of shell structures. Computer Methods in Applied Mechanics and Engineering 196: 1048–73

Aydogdu, M. (2009a). A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration. Physica E: Low-Dimensional Systems and Nanostructures 41: 1651–5

Aydogdu, M. (2009b). A new shear deformation theory for laminated composite plates. Composite Structures 89: 94–101

Chandel, V.S., Wang, G. and Talha, M. (2020). Advances in modelling and analysis of nano structures: A review. Nanotechnology Reviews 9: 230–58

Chinosi, C., Della Croce, L. and Scapolla, T. (1998). Hierarchic finite elements for thin Naghdi shell model. International Journal of Solids and Structures 35: 1863–80

Civalek, Ö. and Demir, Ç. (2011). Bending analysis of microtubules using nonlocal Euler-Bernoulli beam theory. Applied Mathematical Modelling 35: 2053–67

Chowdhury, S.R. and Reddy, J.N. (2019). Geometrically exact micropolar Timoshenko beam and its application in modelling sandwich beams made of architected lattice core. *Composite Structures*, 226: 111228.

Eltaher, M.A., Emam, S.A. and Mahmoud, F.F. (2013). Static and stability analysis of nonlocal functionally graded nanobeams. Composite Structures 96: 82–8

Eringen, A.C. (1972a). Nonlocal polar elastic continua. International Journal of Engineering Science 10: 1–16

Eringen, A. C. (1972b). Linear theory of nonlocal elasticity and dispersion of plane waves. International Journal of Engineering Science 10: 425–35

Eringen, A.C. (1974). Nonlocal Elasticity and Waves. In *Thoft-Christensen P. (eds) Continuum Mechanics Aspects of Geodynamics and Rock Fracture Mechanics. NATO Advanced Study Institutes Series (Series C — Mathematical and Physical Sciences), vol 12.* Dordrecht: Springer

Eringen, A.C. (1978). *Nonlocal continuum mechanics and some applications.* Netherlands: D Reidel

Eringen, A.C. (1983). On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. Journal of Applied Physics 54: 4703–10

Eringen, A.C. (1987). Theory of Nonlocal Elasticity and Some Applications. Res mechanica 21: 313–42

Eringen, A.C. and Ari, N. (1980). Nonlocal Stress Field at Griffith Crack.

Eringen, A.C. and Edelen, D.G.B. (1972). On nonlocal elasticity. International Journal of Engineering Science 10: 233–48

Fakher, M. and Hosseini-Hashemi, S. (2020). Nonlinear vibration analysis of two-phase local/nonlocal nanobeams with size-dependent nonlinearity by using Galerkin method. JVC/Journal of Vibration and Control

Fernández-Sáez, J., Zaera, R., Loya, J.A. et al. (2016). Bending of Euler-Bernoulli beams using Eringen's integral formulation: A paradox resolved. International Journal of Engineering Science 99: 107–16

Hakula, H., Leino, Y. and Pitkäranta, J. (1996). Scale resolution, locking, and high-order finite element modelling of shells. Computer Methods in Applied Mechanics and Engineering 133: 157–82

Hirano, T., Yamada, T., Teraki, J. et al. (1988). A study on a functionally gradient material design system for a thrust chamber. Proceedings of the 16th International Symposium on Space Technology and Science

Karniadakis, G. E.; Sherwin, S. (2005). *Spectral/hp Element Methods for Computational Fluids Dynamics*.

Karttunen, A.T., Reddy, J.N. and Romanoff, J. (2018). Micropolar modeling approach for periodic sandwich beams. *Composite Structures* 185: 656–64

Karttunen, A., Reddy, J.N., and Romanoff, J. (2019). Two-scale micropolar plate model for web-core sandwich panels. *International Journal of Solids and Structures,*170: 82-94

Khodabakhshi, P. and Reddy, J.N. (2015). A unified integro-differential nonlocal model. International Journal of Engineering Science 95: 60–75

Khodabakhshi, P. and Reddy, J.N. (2015). A unified integro-differential nonlocal model. International Journal of Engineering Science 95: 60–75

Kröner, E. (1967). Elasticity theory of materials with long range cohesive forces. International Journal of Solids and Structures 3: 731–42

Kunin, I.A. (1968). The Theory of Elastic Media with Microstructure and the Theory of Dislocations. In *Mechanics of Generalized Continua*

Levinson, M. (1981). A new rectangular beam theory. Journal of Sound and Vibration 74: 81–7

Mindlin, R.D. (1964). Micro-structure in linear elasticity. Archive for Rational Mechanics and Analysis 16: 51–78

Mindlin, R.D. and Eshel, N.N. (1968). On first strain-gradient theories in linear elasticity. International Journal of Solids and Structures 4: 109–24

Motz, C., Weygand, D., Senger, J. et al. (2008). Micro-bending tests: A comparison between three-dimensional discrete dislocation dynamics simulations and experiments. Acta Materialia 56: 1942–55

Nampally, P. and Reddy, J.N. (2020). Geometrically nonlinear Euler–Bernoulli and Timoshenko micropolar beam theories. Acta Mechanica 231: 4217–42

Nazemnezhad, R. and Hosseini-Hashemi, S. (2014). Nonlocal nonlinear free vibration of functionally graded nanobeams. Composite Structures 110: 192–9

Pagani, A. and Carrera, E. (2016). Unified formulation of geometrically nonlinear refined beam theories. Mechanics of Advanced Materials and Structures 25: 15–31

Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003). Application of nonlocal continuum models to nanotechnology. In *International Journal of Engineering Science*

Pontaza, J.P. and Reddy, J.N. (2005). Least-squares finite element formulation for shear-deformable shells. Computer Methods in Applied Mechanics and Engineering 194: 2464–93

Reddy, J.N. (1984). A simple higher-order theory for laminated composite plates. Journal of Applied Mechanics, Transactions ASME 51: 745–52

Reddy, J.N. (2007). Nonlocal theories for bending, buckling and vibration of beams. International Journal of Engineering Science 45: 288–307

Reddy, J.N. (2010). Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates. International Journal of Engineering Science 48: 1507–18

Reddy, J.N. (2013). *An Introduction to Continuum Mechanics*, 2nd edn. New York: Cambridge University Press

Reddy, J.N. (2015). *An Introduction to Nonlinear Finite Element Analysis*, 2nd edn. Oxford, UK: Oxford University Press

Reddy, J.N. (2022). *Theories and Analyses of Beams and Axisymmetric Circular Plates*. Boca Raton, FL: CRC Press

Reddy, J.N. and El-Borgi, S. (2014). Eringen's nonlocal theories of beams accounting for moderate rotations. International Journal of Engineering Science 82: 159–77

Reddy, J.N., El-Borgi, S. and Romanoff, J. (2014). Non-linear analysis of functionally graded microbeams using Eringens non-local differential model. International Journal of Non-Linear Mechanics 67: 308–18

Ritto-Corrêa, M. and Camotim, D. (2008). On the arc-length and other quadratic control methods: Established, less known and new implementation procedures. Computers and Structures 86: 1353–68

Romano, G. and Barretta, R. (2017). Nonlocal elasticity in nanobeams: the stress-driven integral model. International Journal of Engineering Science 115: 14–27

Silling, S.A. (2000). Reformulation of elasticity theory for discontinuities and long-range forces. Journal of the Mechanics and Physics of Solids 48: 175–209

Silling, S.A. and Lehoucq, R.B. (2010). *Peridynamic Theory of Solid Mechanics*.

Soncco, K., Jorge, X. and Arciniega, R. (2019). Postbuckling Analysis of Functionally Graded Beams. IOP Conference Series: Materials Science and Engineering 473: 012028

Srinivasa, A.R. and Reddy, J.N. (2017). An overview of theories of continuum mechanics with nonlocal elastic response and a general framework for conservative and dissipative systems. Applied Mechanics Reviews

Srividhya, S., Raghu, P., Rajagopal, A. et al. (2018). Nonlocal nonlinear analysis of functionally graded plates using third-order shear deformation theory. International Journal of Engineering Science 125: 1–22

Sze, K.Y.Y., Liu, X.H.H. and Lo, S.H.H. (2004). Popular benchmark problems for geometric nonlinear analysis of shells. Finite Elements in Analysis and Design 40: 1551–69

Toupin, R.A. (1962). Elastic materials with couple-stresses. Archive for Rational Mechanics and Analysis

Toupin, R.A. (1964). Theories of elasticity with couple-stress. Archive for Rational Mechanics and Analysis

Tuna, M. and Kirca, M. (2016). Exact solution of Eringen's nonlocal integral model for bending of Euler-Bernoulli and Timoshenko beams. International Journal of Engineering Science 105: 80–92

Tuna, M. and Kirca, M. (2017). Bending, buckling and free vibration analysis of Euler-Bernoulli nanobeams using Eringen's nonlocal integral model via finite element method. Composite Structures 179: 269–84

Wang, Q. and Liew, K.M. (2007). Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures. Physics Letters, Section A: General, Atomic and Solid State Physics 363: 236–42

Wang, Y., Huang, K., Zhu, X. et al. (2019). Exact solutions for the bending of Timoshenko beams using Eringen's two-phase nonlocal model. Mathematics and Mechanics of Solids 24: 559–72

Wang, Y.B., Zhu, X.W. and Dai, H.H. (2016). Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. AIP Advances 6

Yang, F., Chong, A.C.M., Lam, D.C.C. et al. (2002). Couple stress based strain gradient theory for elasticity. International Journal of Solids and Structures 39: 2731–43

Yayli, M.O. (2016). Buckling analysis of a microbeam embedded in an elastic medium with deformable boundary conditions. Micro and Nano Letters 11: 741–5

Zhang, P. and Qing, H. (2020). Buckling analysis of curved sandwich microbeams made of functionally graded materials via the stress-driven nonlocal integral model. Mechanics of Advanced Materials and Structures 0: 1–18

Zhou, Z. and Murray, D.W. (1995). An incremental solution technique for unstable equilibrium paths of shell structures. Computers and Structures 55: 749–59

Zhu, X., Wang, Y. and Dai, H.H. (2017). Buckling analysis of Euler–Bernoulli beams using Eringen's two-phase nonlocal model. International Journal of Engineering Science 116: 130–40