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Magneto-Electro-Viscoelastic Torsional Waves in Aeolotropic Tube under Initial Compression Stress

Abstract
This study examines the effect of electric and magnetic field on torsional waves in hetrogeneous viscoelastic cylindrically aeolotropic tube subjected to initial compression stresses. A new equation of motion and phase velocity of torsional waves propagating in cylindrically aeolotropic tube subjected to initial compression stresses, nonhomogeneity, electric and magnetic field have been derived. The study reveals that the initial stresses, nonhomogeneity, electric and magnetic field present in the aeolotropic tube of viscoelastic solid have a notable effect on the propagation of torsional waves. The results have been discussed graphically. This investigation is very significant for potential application in various fields of science such as detection of mechanical explosions in the interior of the earth.

Keywords
Aeolotropic Material, Viscoelastic Solids, Non-Homogeneous, Bessel Functions.

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1 INTRODUCTION

The mutual interactions between an externally applied magnetic field and the elastic deformation in the solid body, give rise to the coupled field of magneto-elasticity. Since electric currents also give rise to magnetic field and vice-versa, the combined effect is also sometimes known as magneto-electro-elasticity. It is evident that since many component fields are interacting, a large number of unknowns are involved and the solution of even the most elementary problems becomes difficult and cumbersome. We therefore almost always have to take certain assumptions to solve the problems. The interaction of elastic and electromagnetic fields has numerous applications in various field of science such as detection of mechanical explosions in the interior of the earth. In spite of the fact that Maxwell equations governing electro-magnetic field have been known for long time, the interest in the coupled field is helpful in the field such as geophysics, optics, acoustics, damping of acoustic waves in magnetic fields, geomagnetics and oil prospecting etc.

Much literature is available on torsional surface wave propagation in homogeneous elastic and viscoelastic media. Pal (2000) presented a note on torsional body forces in a viscoelastic half space. Dey et al. (1996, 2000, 2002, 2003) investigated the effect of torsional surface waves in non-homogeneous anisotropic medium, torsional surface waves in an elastic layer with void pores, torsional surface waves in an elastic layer with void pores over an elastic half space with void pores and effect of gravity and initial stress on torsional surface waves in dry sandy medium. Kaliski (1959) purposed dynamic equations of motion coupled with the field of temperatures and resolving functions for elastic and inelastic bodies in a magnetic field, Narain (1978) discussed magneto-elastic torsional waves in a bar under initial stress, White (1981) studied cylindrical waves in transversely isotropic media. Das et al. (1978) investigated axisymmetric vibrations of orthotropic shells in a magnetic field. The contribution of various researchers on torsional wave propagation such as Suhubi (1965), Abd-alla (1994), Datta (1985) and Selim (2007) cannot be ignored. Kakar and Kakar (2012) discussed torsional waves in fiber reinforced medium subjected to magnetic field. Kakar and Gupta (2013) presented a note on torsional surface waves in a non-homogeneous isotropic layer over viscoelastic half-space. Tang et al. (2010) discussed transient torsional vibration responses of finite, semi-infinite and infinite hollow cylinders. Kakar and Kumar (2013) investigated surface waves in electro-magneto-thermo two layer heterogeneous viscoelastic medium involving time rate of change of strain and recently, Kakar (2013) presented a note on interfacial waves in non-homogeneous electro-magneto-thermoelastic orthotropic granular half space.

In this study an attempt has been made to investigate the torsional wave propagation in non-homogeneous viscoelastic cylindrically aeolotropic material permeated by an electro-magneto field. The graphs have been plotted showing the effect of variation of elastic constants and the presence of electro-magneto field. It is observed that the torsional elastic waves in a viscoelastic solid body propagating under the influence of a superimposed electro-magneto field can be different significantly from that of those propagating in the absence of an electro-magneto field.

2 BASIC EQUATIONS

The problem is dealing with electro-magnetoelasticity. Therefore the basic equations will be electromagnetism and elasticity. The Maxwell equations of the electromagnetic field in a region with no charges ($\rho = 0$) and no currents ($J = 0$), such as in a vacuum, are (Thidé, 1997)

$$\vec{\nabla}\cdot\vec{E}=0, \tag{1a}$$

$$\vec{\nabla}\cdot\vec{B}=0, \tag{1b}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\overrightarrow{\partial B}}{\partial t}, \tag{1c}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\overrightarrow{\partial E}}{\partial t}. \tag{1d}$$

where, $\vec{E}$, $\vec{B}$, μ_0 and ε_0 are electric field, magnetic field induction, permeability and permittivity of the vacuum. For vacuum, $\mu_0 = 4\pi\times10^{-7}$ and $\varepsilon_0 = 8.85\times10^{-12}$ in SI units. These equations lead directly to $\vec{E}$ and $\vec{B}$ satisfying the wave equation for which the solutions are linear combinations of plane waves traveling at the speed of light, $c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$. In addition, $\vec{E}$ and $\vec{B}$ are mutually perpendicular to each other to the direction of wave propagation.

Also, the term Ohm's law is used to refer to various generalizations. The simplest example of this is:

$$\vec{J} = \sigma \vec{E}, \tag{2a}$$

where, $\vec{J}$ is the current density at a given location in a resistive material $\vec{\mathrm{E}}$ is the electric field at that location, and σ is a material dependent parameter called the conductivity. If an external magnetic field induction $\vec{\mathrm{B}}$ is present and the conductor is not at rest but moving at velocity $\vec{V}$, then an extra term must be added to account for the current induced by the Lorentz force on the charge carriers (Thidé, 1997).

$$\vec{J} = \sigma(\vec{E} + \vec{V} \times \vec{B}) = \sigma(\vec{E} + \frac{\partial v}{\partial t} \times \vec{B}). \tag{2b}$$

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a vacuum. The homogeneous form of the equation, written in terms of either the electric field $\vec{\mathrm{E}}$ or the magnetic field induction $\vec{\mathrm{B}}$, takes the form: (Thidé, 1997)

$$\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{\mathrm{E}} = 0, \tag{3a}$$

$$\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{\mathrm{B}} = 0. \tag{3b}$$

where,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

The dynamical equations of motion in cylindrical coordinate (r, θ, z) are (Love, 1944)

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{1}{r}(s_{rr} - s_{\theta\theta}) + T_R = \rho \frac{\partial^2 u}{\partial t^2}, \tag{4a}$$

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial s_{\theta z}}{\partial z} + \frac{2 s_{r\theta}}{r} + T_\theta = \rho \frac{\partial^2 v}{\partial t^2}, \tag{4b}$$

$$\frac{\partial s_{rz}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{rz}}{r} + T_Z = \rho \frac{\partial^2 w}{\partial t^2}. \tag{4c}$$

where, $s_{rr}, s_{r\theta}, s_{rz}, s_{rr}.s_{\theta\theta}, s_{\theta z}, s_{zz}$ are the respective stress components, T_R, T_θ, T_Z are the respective body forces and u, v, w are the respective displacement components.

The stress-strain relations are

$$s_{rr} = \delta_{11}^0 e_{rr} + \delta_{12}^0 e_{\theta\theta} + \delta_{13}^0 e_{zz}, \tag{5a}$$

$$s_{\theta\theta} = \delta_{21}^0 e_{rr} + \delta_{22}^0 e_{\theta\theta} + \delta_{23}^0 e_{zz}, \tag{5b}$$

$$s_{zz} = \delta_{31}^0 e_{rr} + \delta_{32}^0 e_{\theta\theta} + \delta_{33}^0 e_{zz}, \tag{5c}$$

$$s_{rz} = \delta_{44}^0 e_{rz}, \tag{5d}$$

$$s_{\theta z} = \delta_{55}^0 e_{\theta z}, \tag{5e}$$

$$s_{r\theta} = \delta_{66}^0 e_{r\theta}. \tag{5f}$$

where, δ_{ij} = elastic constants (ij = 1, 2......6).

The elastic constants of viscoelastic medium are (Christensen, 1971)

$$\delta_{ij}^0 = \delta_{ij} + \delta_{ij}^{/} \frac{\partial}{\partial t} + \delta_{ij}^{//} \frac{\partial^2}{\partial t^2} \ (ij = 1,\ 2......6). \tag{6}$$

where, $\delta_{ij}^{/}$ and $\delta_{ij}^{//}$ are the first and second order derivatives of δ_{ij}.

The strain components are

$$e_{rr} = \frac{1}{2}\frac{\partial u}{\partial r}, \tag{7a}$$

$$e_{\theta\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right), \tag{7b}$$

$$e_{zz} = \frac{1}{2}\frac{\partial w}{\partial z}, \tag{7c}$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial v}{\partial z}\right), \tag{7d}$$

$$e_{rz} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right), \tag{7e}$$

$$e_{zz} = \frac{1}{2}\frac{\partial w}{\partial z}, \tag{7f}$$

The rotational components are

$$\Omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right), \tag{8a}$$

$$\Omega_\theta = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right), \tag{8b}$$

$$\Omega_z = \frac{1}{r}\left(\frac{\partial (rv)}{\partial r} - \frac{\partial u}{\partial \theta}\right). \tag{8c}$$

Equations governing the propagation of small elastic disturbances in a perfectly conducting visco-elastic solid having electromagnetic force $\left(\vec{J}\times\vec{\mathrm{B}}\right)$ (the Lorentz force, $\vec{J}$ is the current density and $\vec{\mathrm{B}}$ being magnetic induction vector) as the only body force are (using Eq. (4))

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{1}{r}(s_{rr} - s_{\theta\theta}) + \left(\vec{J}\times\vec{\mathrm{B}}\right)_R = \rho\frac{\partial^2 u}{\partial t^2}, \tag{9a}$$

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial s_{\theta z}}{\partial z} + \frac{2s_{r\theta}}{r} + \left(\vec{J}\times\vec{\mathrm{B}}\right)_\theta = \rho\frac{\partial^2 v}{\partial t^2}, \tag{9b}$$

$$\frac{\partial s_{rz}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{rz}}{r} + \left(\vec{J}\times\vec{\mathrm{B}}\right)_Z = \rho\frac{\partial^2 w}{\partial t^2}. \tag{9c}$$

Let us assume the components of magnetic field intensity $\vec{\mathrm{H}}$ are $\mathrm{H}_r = \mathrm{H}_\theta = 0$ and $\mathrm{H}_z = \mathrm{H}$ constant. Therefore, the value of magnetic field intensity is (Thidé, 1997).

$$\vec{\mathrm{H}}(0,0,\mathrm{H}) = \vec{\mathrm{H}}_0 + \vec{\mathrm{H}}_i \tag{10}$$

where, $\vec{\mathrm{H}}_0$ is the initial magnetic field intensity along z-axis and $\vec{\mathrm{H}}_i$ is the perturbation in the magnetic field intensity.

The relation between magnetic field intensity $\vec{\mathrm{H}}$ and magnetic field induction $\vec{\mathrm{B}}$ is

$$\vec{\mathrm{B}} = \mu_0\vec{\mathrm{H}} \text{ (For vacuum, } \mu_0 = 4\pi\times10^{-7} \text{ SI units.)} \tag{11}$$

From Eq. (1), Eq. (2), Eq. (3) and Eq. (10), we get

$$\nabla^2 \vec{\mathrm{H}} = \mu_0 \sigma \left\{ \frac{\partial \vec{\mathrm{H}}}{\partial t} + \vec{\nabla} \times \left(\frac{\partial v}{\partial t} \times \vec{\mathrm{H}} \right) \right\} \tag{12}$$

The components of Eq. (12) can be written as (Thidé, 1997).

$$\frac{\partial \mathrm{H}_r}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathrm{H}_{r,} \tag{13a}$$

$$\frac{\partial \mathrm{H}_\theta}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathrm{H}_{\theta,} \tag{13b}$$

$$\frac{\partial \mathrm{H}_z}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathrm{H}. \tag{13c}$$

3 FORMULATION OF THE PROBLEM

Let us consider a semi-infinite hollow cylindrical tube of radii α and β. Let the elastic properties of the shell are symmetrical about z-axis, and the tube is placed in an axial magnetic field surrounded by vacuum. Since, we are investigating the torsional waves in an aeolotropic cylindrical tube therefore the displacement vector has only v component. Hence,

$$u = 0, \tag{14a}$$

$$w = 0 \tag{14b}$$

$$v = v(r, z). \tag{14c}$$

Therefore, from Eq. (14) and Eq. (7), we get,

$$e_{rr} = e_{\theta\theta} = e_{zz} = e_{zr} = 0, \tag{15a}$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{\partial v}{\partial z}\right), \tag{15b}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right). \tag{15c}$$

From Eq. (14) and Eq. (8), we get,

$$\Omega_r = \frac{1}{2}\left(\frac{\partial v}{\partial z}\right), \tag{16a}$$

$$\Omega_\theta = 0, \tag{16b}$$

$$\Omega_z = \frac{\partial v}{\partial r} + \frac{v}{r}. \tag{16c}$$

Using Eq. (14), Eq. (15) and Eq. (6), the Eq. (5) becomes

$$s_{rr} = s_{\theta\theta} = s_{zz} = s_{rz} = 0, \tag{17a}$$

$$s_{r\theta} = (\delta_{66} + \delta_{66}^{/}\frac{\partial}{\partial t} + \delta_{66}^{//}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}), \tag{17b}$$

$$s_{\theta z} = (\delta_{55} + \delta_{55}^{/}\frac{\partial}{\partial t} + \delta_{55}^{//}\frac{\partial^2}{\partial t^2})(-\frac{1}{2}\frac{\partial v}{\partial r}). \tag{17c}$$

where, $\delta_{ij}^{/}$ and $\delta_{ij}^{//}$ are the first and second order derivatives of δ_{ij}.

For perfectly conducting medium, (i.e. $\sigma \to \infty$), it can be seen that Eq. (2) becomes

$$\vec{\mathrm{E}} = \left[-\frac{\mu_0 \mathrm{H}}{c}\frac{\partial v}{\partial t}, 0, 0 \right] \tag{18}$$

Eq. (1) and Eq. (18), the Eq. (13) becomes,

$$\vec{\mathrm{H}}_i = \left[0, \mathrm{H}\frac{\partial v}{\partial z}, 0 \right] \tag{19}$$

From the above discussion, the electric and magnetic components in the problem are related as

$$\left[-\frac{\mu_0 \mathrm{H}}{c}\frac{\partial v}{\partial t}, 0, 0 \right] = \left[0, \mathrm{H}\frac{\partial v}{\partial z}, 0 \right] \tag{20}$$

Using Eq. (19) and Eq. (1) to get the components of body force in terms of SI system of units as:

$$\mathrm{T} = \left[0, \mu_e H^2 \frac{\partial^2 v}{\partial z^2}, 0 \right] \tag{21}$$

Eq. (17) and Eq. (20) satisfy the Eq. (4a) and Eq. (4c), therefore, the remaining Eq. (4b) becomes

$$\left\{\begin{array}{l}\frac{\partial}{\partial r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})+\frac{\partial}{\partial z}(\delta_{55}+\delta_{55}^{/}\frac{\partial}{\partial t}+\delta_{55}^{//}\frac{\partial^2}{\partial t^2})(-\frac{1}{2}\frac{\partial v}{\partial r}) \\ +\frac{2}{r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})-\left(\mu_e H^2+\varepsilon_e E^2+\frac{p}{2}\right)\frac{\partial^2 v}{\partial z^2}\end{array}\right\}=\rho\frac{\partial^2 v}{\partial t^2} \tag{22}$$

where, p is initial compression stress, μ_e and ε_e are the permeability and permittivity of the material.

Let

$$C_{ij}=\delta_{ij}r^l, C_{ij}^{/}=\delta_{ij}^{/}r^l, C_{ij}^{//}=\delta_{ij}^{//}r^l \text{ and } \rho=\rho_0 r^m \tag{23}$$

where, δ_{ij}, $\delta_{ij}^{/}$, $\delta_{ij}^{//}$ and ρ_0 are constants, r is the radius vector and l, m are non-homogeneities.

From Eq. (23), we get Eq. (17) as

$$s_{r\theta}=(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r}), \tag{24a}$$

$$s_{r\theta}=(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r}), \tag{24b}$$

Using Eq. (23), the Eq. (22) becomes

$$\left\{\begin{array}{l}\frac{\partial}{\partial r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})+\frac{\partial}{\partial z}(\delta_{55}+\delta_{55}^{/}\frac{\partial}{\partial t}+\delta_{55}^{//}\frac{\partial^2}{\partial t^2})r^l(-\frac{1}{2}\frac{\partial v}{\partial r}) \\ +\frac{2}{r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})-\left(\mu_e H^2+\varepsilon_e E^2+\frac{p}{2}\right)\frac{\partial^2 v}{\partial z^2}\end{array}\right\}=\rho_0 r^m\frac{\partial^2 v}{\partial t^2} \tag{25}$$

where, p is initial compression stress, μ_e and ε_e are the permeability and permittivity of the material.

4 SOLUTION OF THE PROBLEM

Let $v=\xi(r)e^{i(\varsigma z+\zeta t)}$ (Watson, 1944) be the solution of Eq. (25). Hence, Eq. (25) reduces to

$$\frac{\partial^2\xi}{\partial r^2}+\frac{(l+1)}{r}\frac{\partial\xi}{\partial r}-\frac{(l+1)}{r^2}\xi+\Theta_1^2\xi+\Theta_2^2\frac{\xi}{r^l}=0 \tag{26}$$

where,

$$\Theta_1^2 = \frac{2\rho_0\zeta^2 - (\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2)\varsigma^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2}, \tag{27a}$$

$$\Theta_2^2 = \frac{\left(\frac{\mu_e H^2}{2} + \frac{\varepsilon_e E^2}{2} + p\right)\varsigma^2}{(\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2)}. \tag{27b}$$

Eq. (26) is in complex form, therefore we generalize its solution for $l = 0$ and $l = 2$

4.1 Solution for $l = 0$

For, $l = 0$ the Eq. (26) becomes,

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} + (\Xi^2 - \frac{1}{r^2})\xi = 0 \tag{28}$$

where,

$$\Xi^2 = \Theta_1^2 + \Theta_2^2 \tag{29}$$

The solution of Eq. (28) is

$$v = \{PJ_1(Gr) + QX_1(Gr)\}e^{i(\varsigma z + \zeta t)} \tag{30}$$

From Eq. (24) and Eq. (30)

$$s_{r\theta} = \{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2\}\left[\frac{P}{2}\{GJ_0(Gr) - \frac{2}{r}J_1(Gr) + \frac{Q}{2}\{GX_0(Gr) - \frac{2}{r}X_1(Gr)\right]e^{i(\varsigma z + \zeta t)} \tag{31}$$

5 BOUNDARY CONDITIONS AND FREQUENCY EQUATION

The boundary conditions that must be satisfied are

B1. For $r = \alpha$, (α is the internal radius of the tube)

$$s_{r\theta} + \tau_{r\theta} = \tau_{(r\theta)_0}$$

B2. For $r = \beta$, (β is the external radius of the tube)

$$s_{r\theta} + \tau_{r\theta} = \tau_{(r\theta)_0}$$

where $\tau_{r\theta}$ and $\tau_{(r\theta)_0}$ are the Maxwell stresses in the body and in the vacuum, respectively. There will be no impact of these Maxwell stresses. Hence,

$$\tau_{r\theta} = \tau_{(r\theta)_0} = 0 \tag{32}$$

On simplification, Eq. (18) and Eq. (30) gives

$$\mathrm{E} = -\frac{\mu_0 \mathrm{H}}{c} i\zeta \{PJ_1(Gr) + QX_1(Gr)\} e^{i(\varsigma z + \zeta t)} \tag{33}$$

Let,

$\mathrm{E}_0 = \Psi e^{i(\varsigma z + \zeta t)}$

Hence, Eq. (3) becomes

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \gamma^2 \Psi = 0 \tag{34}$$

$$\text{where, } \gamma^2 = \frac{\zeta^2}{c^2} - \varsigma^2 \tag{35}$$

The solution of the Eq. (34) becomes

$$\Psi = RJ_0(\gamma r) + SX_0(\gamma r) \tag{36}$$

where J_0 and X_0 are Bessel functions of order zero. R and S are constants.

From Eq. (37) and Eq. (40)

$$\Psi = \{RJ_0(\gamma r) + SX_0(\gamma r)\} e^{i(\varsigma z + \zeta t)} \tag{37}$$

The boundary conditions B1 and B2 with the help of the Eq. (31) and (32) turn into:

$$P\{G\alpha J_0(G\alpha) - 2J_1(G\alpha)\} + Q\{G\alpha X_0(G\alpha) - 2X_1(G\alpha)\} = 0 \tag{38}$$

$$P\{G\beta J_0(G\beta) - 2J_1(G\beta)\} + Q\{G\beta X_0(Ga) - 2X_1(G\beta)\} = 0 \tag{39}$$

Eliminating P and Q from Eq. (38) and Eq. (39)

$$\begin{vmatrix} G\alpha J_0(G\alpha) - 2J_1(G\alpha) & G\alpha X_0(G\alpha) - 2X_1(G\alpha) \\ G\beta J_0(G\beta) - 2J_1(G\beta) & G\beta X_0(Ga) - 2X_1(G\beta) \end{vmatrix} = 0 \tag{40}$$

On solving Eq. (40), we get the obtained frequency equation

$$\frac{G\alpha J_0(G\alpha)-2J_1(G\alpha)}{G\beta J_0(G\beta)-2J_1(G\beta)}=\frac{G\alpha X_0(G\alpha)-2X_1(G\alpha)}{G\beta X_0(G\beta)-2X_1(G\beta)}=0 \tag{41}$$

On the theory of Bessel functions, if tube under consideration is very thin i.e. $\beta=\alpha+\Delta\alpha$ and neglecting $\Delta\alpha^2, \Delta\alpha^3$........, the frequency equation can be written as (Watson [18])

$$\Xi^3\alpha^2+\Xi-1=0 \tag{42}$$

where,

$$\Xi^2=\frac{2\rho_0\zeta^2-(\delta_{55}+\delta_{55}^{/}i\zeta-\delta_{55}^{//}\zeta^2)\varsigma^2+\left(\frac{\mu_e H^2}{2}+\frac{\varepsilon_e E^2}{2}+p\right)\varsigma^2}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2} \tag{43}$$

Putting the value of Ξ in Eq. (42), the frequency ζ of the wave can be found. Clearly, frequency ζ is dependent on magnetic field, electric field and initial pressure.

Put ,

$$\Xi\alpha=\Phi \tag{44}$$

The phase velocity $c_1=\zeta/\varsigma$ can be written as

$$\frac{c_1^2}{c_0^2}=\Phi^2\left(\frac{\lambda}{2\pi\alpha}\right)^2+\mathrm{K}-\frac{\left(\frac{\mu_e H^2}{2}+\frac{\varepsilon_e E^2}{2}+p\right)}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2} \tag{45}$$

where,

$$\lambda=\frac{2\pi}{k},\ \mathrm{K}=\frac{\delta_{55}+\delta_{55}^{/}i\zeta-\delta_{55}^{//}\zeta^2}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2}, \tag{46}$$

$$c_0^2=\frac{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2}{2\rho_0}$$

The terms H, E and p are negative in Eq. (45) which means that the combine effect of magnetic field, electric field and initial pressure reduces the phase velocity of torsional wave.

Case 1

Since the pipe under consideration is made of an aeolotropic material, then

$$\delta_{ij}^{/}=\delta_{ij}^{//}=0 \tag{47}$$

Hence, from Eq. (42), Eq. (44) and Eq. (47) the frequency equation becomes

$$\Phi_0^3 + \Phi_0 - \alpha = 0 \tag{48}$$

Using Eq. (45) and Eq. (46), the phase velocity is

$$c_2^2 = \frac{\delta_{66}}{2\rho_0}\left\{ \Phi_0^2 \left(\frac{\lambda}{2\pi\alpha}\right)^2 + \frac{\delta_{55}}{\delta_{66}} - \frac{\left(\frac{\mu_e H^2}{2} + \frac{\varepsilon_e E^2}{2} + p\right)}{\delta_{66}} \right\} \tag{49}$$

$$or \qquad \frac{c_2}{c_0} = \left\{ \frac{[\frac{\Phi_0}{2\pi}]^2}{[\frac{\alpha}{\lambda}]^2} + \frac{\delta_{55}}{\delta_{66}} - \frac{\left(\frac{\mu_e H^2}{2} + \frac{\varepsilon_e E^2}{2} + p\right)}{\delta_{66}} \right\}^{\frac{1}{2}} \tag{50}$$

where, $c_0^2 = \delta_{66}/2\rho_0$

The terms H, E and p are negative in Eq. (49) which reduces the phase velocity of torsional wave. This is in complete agreement with the corresponding classical results given by Chandrasekharaiahi (1972).

Case 2

If the pipe under consideration is made of an isotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0, \delta_{55} = \delta_{66} = \chi \tag{51}$$

Using Eq. (49) and Eq. (50), the phase velocity is

$$c_2^2 = \frac{\chi}{2\rho_0}\left\{ \Phi_0^2 \left(\frac{\lambda}{2\pi\alpha}\right)^2 + 1 - \frac{\left(\frac{\mu_e H^2}{2} + \frac{\varepsilon_e E^2}{2} + p\right)}{\chi} \right\} \tag{52}$$

This is in complete agreement with the corresponding classical results given by Narain (1978).

5.1 Solution for l=2

For, $l=2$ the Eq. (26) becomes,

$$\frac{\partial^2\xi}{\partial r^2}+\frac{3}{r}\frac{\partial\xi}{\partial r}+(\Theta_1^2-\frac{(3-\Theta_2^2)}{r^2}\xi=0 \tag{53}$$

Putting $\xi=\frac{1}{r}\mathrm{N}(r)$ in Eq. (53), one get

$$\frac{\partial^2\mathrm{N}}{\partial r^2}+\frac{1}{r}\frac{\partial\mathrm{N}}{\partial r}+\left[\Theta_1^2-\frac{\mathrm{P}^2}{r^2}\right]\mathrm{N}=0 \tag{54}$$

where,

$$\mathrm{P}^2=3-\Theta_2^2 \tag{55}$$

Solution of Eq. (54) will be (Watson, 1944)

$$\mathrm{N}=RJ_{\mathrm{P}}(\Theta_1 r)+SX_{\mathrm{P}}(\Theta_2 r) \tag{56}$$

Putting the value of ξ and N in Eq. (55), we get

$$\mathrm{P}=\frac{1}{r}\{RJ_{\mathrm{P}}(\Theta_1 r)+SX_{\mathrm{P}}(\Theta_1 r)\}e^{i(\varsigma z+\zeta t)} \tag{56}$$

From the Eq. (24) and Eq. (56)

$$s_{r\theta}=(\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2)\left[\begin{array}{l}\frac{R}{2}\{\Theta_1 rJ_{\mathrm{P}-1}(\Theta_1 r)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1 r)\}\\+\frac{S}{2}\{\Theta_1 rX_{\mathrm{P}-1}(\Theta_1 r)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1 r)\}\end{array}\right]e^{i(\varsigma z+\zeta t)}=0 \tag{57}$$

With the help of Eq. (32), Eq. (56) and boundary conditions B1 and B2, we get

$$\frac{R}{2}\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\}+\frac{S}{2}\{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\alpha)\}=0 \tag{58}$$

$$\frac{R}{2}\{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\}+\frac{S}{2}\{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\beta)\}=0 \tag{59}$$

Eliminating R and S from Eq. (58) and Eq. (59)

$$\begin{vmatrix}\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\} & \{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\gamma\alpha)\}\\ \{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\} & \{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\gamma\beta)\}\end{vmatrix}=0 \tag{60}$$

On solving Eq. (60), we get

$$\frac{\{\Theta_1\alpha J_{P-1}(\Theta_1\alpha)-(P+2)J_P(\Theta_1\alpha)\}}{\{\Theta_1\alpha X_{P-1}(\Theta_1\alpha)-(P+2)X_P(\Theta_1\alpha)\}}=\frac{\{\Theta_1\beta J_{P-1}(\Theta_1\beta)-(P+2)J_P(\Theta_1\beta)\}}{\{\Theta_1\beta X_{P-1}(\Theta_1\beta)-(P+2)X_P(\Theta_1\beta)\}} \tag{61}$$

If η_1 is the root of the above equation, then

$$\frac{\{\eta_1 J_{P-1}(\eta_1)-(P+2)J_P(\eta_1)\}}{\{\eta_1 X_{P-1}(\eta_1)-(P+2)X_P(\eta_1)\}}=\frac{\{\eta_1 F_1 J_{P-1}(\eta_1 F_1)-(P+2)J_P(\eta_1 F_1)\}}{\{\eta_1 F_1 X_{P-1}(\eta_1 F_1)-(P+2)X_P(\eta_1 F_1)\}} \tag{62}$$

where, $F_1=\beta/\alpha$

On the theory of Bessel functions, if tube under consideration is very thin i.e. $\beta=\alpha+\Delta\alpha$ and neglecting $\Delta\alpha^2, \Delta\alpha^3$........, the frequency equation can be written as (Watson, 1944)

$$(P+2)^2-\left(2P-1+\frac{1}{\Theta_1}\right)(P+2)+\Theta_1^2\alpha^2=0 \tag{63}$$

where,

$$P^2=3-\Theta_2^2 \Rightarrow P^2=3-\frac{\left(\frac{\mu_e H^2}{2}+\frac{\varepsilon_e E^2}{2}+p\right)\varsigma^2}{(\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2)}, \tag{64a}$$

$$\Theta_1^2=\frac{2\rho_0\zeta^2-(\delta_{55}+\delta_{55}^{/}i\zeta-\delta_{55}^{//}\zeta^2)\varsigma^2}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2}. \tag{64b}$$

From the Eq. (62), Eq. (63) and Eq. (64), the phase velocity can be written as (same as above Eq. (45) and Eq. (46))

$$\frac{c^2}{c_0^2}=\eta^2\left(\frac{\lambda}{2\pi\alpha}\right)^2+\frac{\delta_{55}+\delta_{55}^{/}i\zeta-\delta_{55}^{//}\zeta^2}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2} \tag{65}$$

Case 1

Since the pipe under consideration is made of an aeolotropic material, then

$$\delta_{ij}^{/}=\delta_{ij}^{//}=0 \tag{66}$$

The frequency equation is given by

$$\frac{\{\Theta_3\alpha J_{P_1-1}(\Theta_3\alpha)-(P+2)J_{P_1}(\Theta_3\alpha)\}}{\{\Theta_3\alpha X_{P_1-1}(\Theta_3\alpha)-(P+2)X_{P_1}(\Theta_3\alpha)\}}=\frac{\{\Theta_3\beta J_{P_1-1}(\Theta_3\beta)-(P+2)J_{P_1}(\Theta_3\beta)\}}{\{\Theta_3\beta X_{P_1-1}(\Theta_3\beta)-(P+2)X_{P_1}(\Theta_3\beta)\}} \tag{67}$$

$$\eta_2^3+6\eta_2-3\alpha=0 \tag{68}$$

$$P_1^2=3-\frac{\left(\frac{\mu_e H^2}{2}+\frac{\varepsilon_e E^2}{2}+p\right)\varsigma^2}{\delta_{66}},\quad \Theta_3^2=\frac{2\rho_0\zeta^2-\delta_{55}\varsigma^2}{\delta_{66}},\quad \eta_2=\Theta_3\varsigma \ at \ \ P_1=1 \tag{69}$$

Using Eq. (65), Eq. (66), Eq. (67) and Eq. (69), we get (calculations are done in the similar manner as for the Eq. (48) to Eq. (50) for $l=0$ case)

$$\frac{c_3}{c_{01}}=\left[\frac{\left(\frac{\eta_2}{2\pi}\right)^2}{\left(\frac{\alpha}{\lambda}\right)^2}+\frac{\delta_{55}}{\delta_{66}}\right]^{\frac{1}{2}} \tag{70}$$

where, ${c_{01}}^2=\delta_{66}/2\rho_0$

Case 2

If the pipe under consideration is made of an isotropic material, then

$$\delta_{ij}^{/}=\delta_{ij}^{//}=0,\delta_{55}=\delta_{66}=\chi \tag{71}$$

The frequency equation (calculations are done as for the $l=0$ case) is

$$\frac{\{\Theta_4\alpha J_{P_2-1}(\Theta_4\alpha)-(P+2)J_{P_2}(\Theta_4\alpha)\}}{\{\Theta_4\alpha X_{P_2-1}(\Theta_4\alpha)-(P+2)X_{P_2}(\Theta_4\alpha)\}}=\frac{\{\Theta_4\beta J_{P_2-1}(\Theta_4\beta)-(P+2)J_{P_2}(\Theta_4\beta)\}}{\{\Theta_4\beta X_{P_2-1}(\Theta_4\beta)-(P+2)X_{P_2}(\Theta_4\beta)\}} \tag{72}$$

where

$$P_2^2=3-\frac{\left(\frac{\mu_e H^2}{2}+\frac{\varepsilon_e E^2}{2}+p\right)\varsigma^2}{\chi},\qquad \Theta_4^2=\frac{2\rho_0\zeta^2-\chi\varsigma^2}{\chi}.$$

Using Eq. (71) and Eq. (72), the phase velocity for this case is (same as above Eq. (45) and Eq. (46))

$$\frac{c_4^2}{c_{02}^2} = \left[\frac{\left(\frac{\eta_2}{2\pi}\right)^2}{\left(\frac{\alpha}{\lambda}\right)^2} + 1 \right] \tag{73}$$

where, $c_{02}^2 = \chi/2\rho_0$

7 NUMERICAL RESULTS

The effect of non-homogeneity, electric field and magnetic field on torsional waves in an aeolotropic material made of viscoelastic solids has been studied. The numerical computation of phase velocity has been made for homogeneous and non-homogeneous pipe. The graphs are plotted for the two cases (l=0 and l=2). Different values of α/λ (diameter/wavelength) for homogeneous in the presence of electro-magneto field and non-homogeneous case in the absence of electro-magneto field are calculated from Eq. (49) and Eq. (65) with the help of MATLAB. The variations elastic constants and presence of electro-magneto field in two curves have been obtained by choosing the following parameters for homogeneous and non-homogeneous aeolotropic pipe (table 1). The curves obtained in fig. 1 clearly show that the phase velocity for homogeneous as well as non-homogeneous case decreases inside the aeolotropic tube. The presence of electro-magneto field also reduces the speed of torsional waves in viscoelastic solids. These curves justify the results obtained in Eq. (50) and Eq. (52) mathematically given by Narain (1978) and Chandrasekharaiahi (1972).

Table 1: Material parameters

	l	Φ_0	α	E (Volt/m)	H (Tesla)	P(Pascal)	$\delta_{55} / \delta_{66}$
Homogeneous Pipe	0	2.33	15	0	0	0	0.9
Inhomogeneous Pipe	2	2.33	15	50	$0.32\text{x}10^4$	0.1	0.9

Table 2: Shows values of $\frac{c_2}{c_0}$ (l =0) and $\frac{c}{c_0}$(l = 2) for different values of α/λ (diameter / wavelength)

α/λ	$\frac{c_2}{c_0}$	$\frac{c}{c_0}$
0.2	1.9849	2.5680
0.4	1.1662	1.5243
0.6	0.9393	1.2380
0.8	0.8455	1.1206
1.0	0.7985	1.0619
1.2	0.7717	1.0286
1.4	0.7557	1.0080
1.6	0.7441	0.9944
1.8	0.7365	0.9850
2.0	0.7310	0.9782

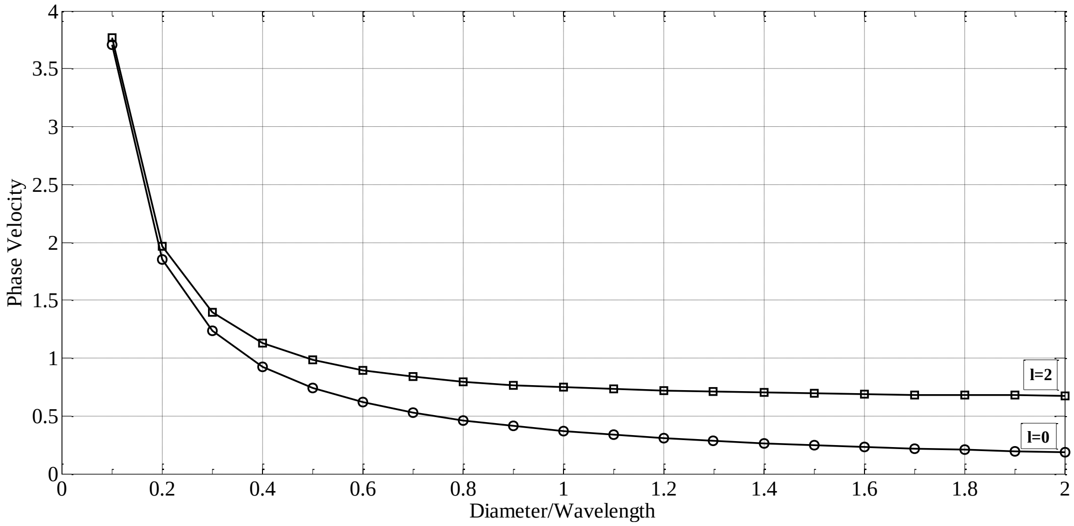

Figure 1: Torsional wave dispersion curves

We see that for homogeneous case when electro-magneto field is present and for non-homogeneous case when electro-magneto field is not present the variation i.e. shape of the curves is same. For non-homogeneous case, the elastic constants and the density of the tube are varying as the square of the radius vector.

6 CONCLUSIONS

The above problem deals with the interaction of elastic and electromagnetic fields in a viscoelastic media. This study is useful for detections of mechanical explosions inside the earth. In this study an attempt has been made to investigate the torsional wave propagation in non-homogeneous viscoelastic cylindrically aeolotropic material permeated by a electric and magnetic field. It has been observed that the phase velocity decreases as the magnetic field and electric field increases.

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