11 (2014) 459 - 487

Latin American Journal of
Solids and Structures
www.lajss.org

Modified Couple Stress-Based Third-Order Theory for Nonlinear Analysis of Functionally Graded Beams

Abstract

A microstructure-dependent nonlinear third-order beam theory which accounts for through-thickness power-law variation of a two-constituent material is developed using Hamilton's principle. The formulation is based on a modified couple stress theory, power-law variation of the material, and the von Kármán nonlinear strains. The modified couple stress theory contains a material length scale parameter that can capture the size effect in a functionally graded material beam. The influence of the material length scale parameter on linear bending is investigated. The finite element models are also developed to determine the effect of the geometric nonlinearity and microstructure-dependent constitutive relations on linear and nonlinear response.

Keywords

Modified couple stress theory, equations of motion, general third-order beam theory, von Kármán nonlinearity, functionally graded beam, nonlinear finite element model.

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Received in 02 Apr 2013
In revised form 30 Apr 2013

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1 BACKGROUND

1.1 Microstructural Effects

In recent years a number of attempts have been made to bring microstructural length scales into the continuum description of beams and plates. Microstructure-dependent theories were developed for the Bernoulli-Euler beam by Park and Gao(2006 and 2008) and for the Timoshenko and Reddy-Levinson beams and Mindlin plates by Ma, Gao, and Reddy (2008, 2010 and 2011), using a modified couple stress theory proposed by Yang et al. (2002), which contains only one material length scale parameter. Recently, Reddy (2011) developed a complete formulation of functionally graded beams with microstructure-dependent constitutive relations for Bernoulli–Euler and Timoshenko beams. The formulation accounted for temperature-dependent material properties and the von Kármán nonlinearity, which may have significant contribution to the response of beam-like elements used in micro- and nano-scale devices such as biosensors and atomic force microscopes (see, e.g., Li et al. (2003) and Pei et al. (2004)). The present paper is an extension of the work of Reddy (2011) to a general third-order theory that contains the Reddy third-order beam theory as well as the Bernoulli–Euler and Timoshenko beam theories as special cases. The following review of literature provides a background for the present study.

1.2 An Overview of Beam Theories

Mathematical models used to determine the response of beams under external loads are deduced from the three-dimensional elasticity theory through a series assumption concerning the kinematics of deformation and constitutive behavior. The kinematic assumptions exploit the fact that such structures do not experience significant strains or stresses associated with the cross-sectional dimensions (e.g., transverse normal and shear strains and stresses). For example, the solution of the three-dimensional elasticity problem associated with a straight beam is reformulated as a one-dimensional problem in terms of displacements whose form is presumed on the basis of an educated guess concerning the nature of the deformation.

The theories based on assumed form of the displacement field have received the most attention, especially in the context finite element model development. In these theories, the displacement field of the beam is expanded in terms of increasing powers of the coordinate in the thickness (i.e., height) direction. The higher-order terms would have diminishing returns compared to the lower-order terms due to the smallness of thickness compared to the length. The third-order expansion of the displacement field is optimal because it gives quadratic variation of transverse strain and stress, and requires no “shear correction factor” compared to the Timoshenko theory, where the transverse strains and stresses are constant through the plate thickness (see Reddy (2002 and 2007)).

The simplest shear deformation beam theory is the Timoshenko beam theory, which is based on the displacement field

$$\begin{gathered}\mathbf{u} = u_1\,\hat{\mathbf{e}}_1 + u_2\,\hat{\mathbf{e}}_2 + u_3\,\hat{\mathbf{e}}_3 \quad \text{where,} \\ u_1(x,y,z) = u(x,y) + z\theta_x(x,y), \qquad u_2 = 0, \qquad u_3(x,y,z) = w(x,y)\end{gathered} \tag{1}$$

Here θ_x is the rotation of a transverse normal line

$$\theta_x = \frac{\partial u_1}{\partial z} \tag{2}$$

The Bernoulli–Euler hypothesis of straight lines normal to the axis of the beam before deformation remain (a) straight after deformation, (b) inextensible, and (c) rotate as rigid lines to remain perpendicular the bent axis of the beam (see Fig. 1) require

$$\theta_x = -\frac{\partial w}{\partial x} \tag{3}$$

Consequently, the transverse shear strain γ_{xz} is zero.

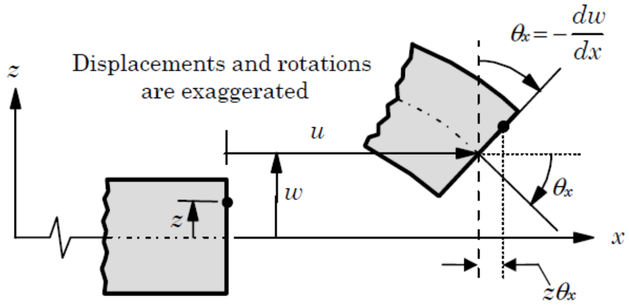


Figure 1 Kinematics of the Euler–Bernoulli beam theory

The Timoshenko beam theory is based on the first two assumptions of the Bernoulli–Euler hypothesis, and the normality of the assumption is not invoked, making the rotation θ_x to be independent of $\partial w/\partial x$. As a result, the transverse shear strain γ_{xz} is nonzero but independent of z. This leads to the introduction of a shear correction factor in the evaluation of the transverse shear force. The finite element models of the theory require only C^0-continuity, that is, the variables of the theory (u, w, θ_x) be continuous between elements; however, they can exhibit spurious transverse shear strains even in pure bending, known as the shear locking, as the beam length-to-height ratio, L/h, becomes very large. The spurious transverse shear stiffness stems from an interpolation inconsistency that prevents the condition of Eq. (2) from being satisfied as the beam becomes thin, that is, $L/h \to \infty$. The shear locking phenomenon can be alleviated by using a reduced integration in the evaluation of transverse shear stiffness terms in the element stiffness matrix or by using higher-order approximations of the variables (u, w, θ_x). Although the reduced integration solution is the most economical alternative, the process allows some elements to exhibit spurious displacement modes, i.e., deformation modes that result in zero strain at the numerical integration points.

Second-order and higher-order beam theories relax the Bernoulli-Euler hypothesis further by allowing the straight lines normal to the beam axis before deformation to become curves after deformation. However, most published theories still assume inextensibility of these lines. Second-order plate theories are not popular because of the fact that they too require shear correction factors and while not improving over the Timoshenko beam theory. The third-order theories provide a slight increase in accuracy relative to the Timoshenko beam theory, at the expense of an increase in computational effort, but do not require a shear correction factor. From the finite element model development standpoint, the third-order beam theory of Reddy (see Reddy (1984, 1987, 1990, 2002 and 2011)), which is based on the displacement field

$$\begin{gathered}\mathbf{u} = u_1\,\hat{\mathbf{e}}_1 + u_2\,\hat{\mathbf{e}}_2 + u_3\,\hat{\mathbf{e}}_3 \quad \text{where,} \\ u_1 = u + z\theta_x - \frac{4z^3}{3h^2}\left(\theta_x + \frac{\partial w}{\partial x}\right), \qquad u_2 = 0, \qquad u_3 = w\end{gathered} \tag{4}$$

requires C^1-continuity of the transverse displacement component but less sensitive to shear locking. In the Reddy beam theory, the transverse shear stresses are zero at the top and bottom of the beam.

In this study, we consider a general third-order theory based on the following displacement field

$$\mathbf{u} = u_1\,\hat{\mathbf{e}}_1 + u_2\,\hat{\mathbf{e}}_2 + u_3\,\hat{\mathbf{e}}_3 \quad \text{where,}$$
$$u_1 = u + z\theta_x + z^2\phi_x + z^3\psi_x, \qquad u_2 = 0, \qquad u_3 = w + z\theta_z + z^2\phi_z \tag{5}$$

Finite element model of this theory requires only C^0-continuity of all field variables $(u, w, \theta_x, \phi_x, \psi_x, \theta_z, \phi_z)$, and no shear correction factors are needed.

1.3 Functionally Graded Materials

Elimination of stress concentrations in structural elements is mitigated by suitable design of materials, and functionally gradient materials (FGMs) are a class of materials that are designed to eliminate stress concentrations (see Hasselman and Youngblood (1978), Yamanouchi et al. (1990) and Koizumi (1993)). These materials were proposed as thermal barrier materials for applications in space planes, space structures, nuclear reactors, turbine rotors, flywheels, and gears, to name only a few. One reason for increased interest in FGMs is that it may be possible to create certain types of FGM structures capable of adapting to operating conditions. A most common FGM is one in which two materials are mixed to achieve a composition that provides certain functionality. For example, for thermal-barrier structures, two-constituent FGMs are made of a mixture of ceramic and metals. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture due to high temperature gradient in a very short period of time. The gradation in properties of the material reduces thermal stresses, residual stresses, and stress concentrations. A number of other investigations dealing with thermal stresses and deformations in beams, plates, and cylinders have been published in the literature (see, for example, Noda and Tsuji (1991), Obta, Noda, and Tsuji (1991), Reddy and Chin (1998), Praveen and Reddy (1998), Praveen, Chin, and Reddy (1999), Reddy (2000), and Aliaga and Reddy (2004), among others). The work of Praveen and Reddy (1998), Reddy (2000) and Aliaga and Reddy (2004) considered von Kármán nonlinearity, and Aliaga and Reddy (2004) used the third-order plate theory of Reddy (1984, 1987, 1990 and 2011).

1.4 Present Study

The objective of the present study is to develop a general third-order beam theory that accounts for through-thickness power-law variation of a two-constituent material with temperature dependent material properties, modified couple stress theory, and the von Kármán nonlinear strains. In particular, we extend the modified couple stress theory of Yang et al. (2002) to the case of functionally graded beams (Reddy (2011)) using the third-order kinematics with the von Kármán nonlinearity and derive the equations of motion using Hamilton's principle (see Reddy (2002 and

2013)). Also, a nonlinear finite element model is developed for the general third-order theory of beams. To date no such study is reported in the literature. Since most nanoscale devices involve beam-like elements that may be functionally graded and undergo moderately large rotations, the newly developed third-order beam theory can be used to capture the size effects in functionally graded microbeams. Moreover, the bending-extensional coupling is captured through the von Kármán nonlinear strains.

2 CONSTITUTIVE MODELS

2.1 Material Variation through the Thickness

Consider a straight beam of rectangular cross section, with width b and total height (or thickness) h. The x coordinates is taken along the length of the beam and it passes through the geometric centroid of the cross section, and the z-axis is taken transverse to the beam length such that the beam cross section is in the yz plane. We assume that the material of the beam is isotropic but varies from one kind of material at the bottom, $z = -h/2$ to another material on the top, $z = h/2$. A typical material property of the FGM through the beam thickness is assumed to be represented by a power-law (see Praveen and Reddy (1998) and Reddy (2000))

$$P(z,T) = [P_c(T) - P_m(T)]f(z) + P_m(T), \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{6}$$

Where, P_c and P_m are the values of a typical material property, such as the modulus, density, and conductivity, of a ceramic material and metal, respectively;n denotes the volume fraction exponent, called power-law index. When $n = 0$, we obtain the single-material plate (with ty P_c). When FGMs are used in high-temperature environment, the material properties are temperature-dependent and they can be expressed as (see Reddy (2011))

$$P_\alpha(T) = c_0(c_{-1}T^{-1} + 1 + c_1T + c_2T^2 + c_3T^3), \qquad \alpha = c \text{ or } m \tag{7}$$

where c_0 is a constant appearing in the cubic fit of the material property with temperature; and c_{-1}, c_1, c_2, and c_3 coefficients of T^{-1}, T, T^2, and T^3, obtained after factoring out c_0 from the cubic curve fit of the property. The modulus of elasticity, conductivity, and the coefficient of thermal expansion are considered to vary according to Eqs. (6) and (7).

The present study is concerned with the development of the equations of motion associated with a general third-order beam theory which accounts for microstructural length scale, two-constituent material variation through beam height, and the von Kármán nonlinearity. Solutions of bending, vibration, and buckling are presented for the linear case to study various parametric effects on the response. Following this introduction, the governing equations of the general third-order FGM beams with the microstructure-dependent length scale parameter are presented. Existing beam theories are derived as special cases of the theory presented herein. The equations derived are then specialized to the linear case to develop the bending, buckling, and vibration frequencies.

3 THE GENERAL THIRD-ORDER BEAM THEORY

3.1 Modified Couple Stress Theory

According to the modified couple stress theory, the strain energy of an elastic beam can be expressed as

$$U = \frac{1}{2}\int_0^L \int_A \left(\sigma_{ij}\,\varepsilon_{ij} + m_{ij}\,\chi_{ij}\right) dAdx \tag{8}$$

where σ_{ij} are the components of the symmetric part of the Cauchy stress tensor, m_{ij} denote the components of the deviatoric part of the symmetric couple stress tensor, and χ_{ij} are the components of the symmetric curvature tensor

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right), \qquad i,j = 1,2,3 \tag{9}$$

where ω_i are the components of the rotation vector

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \qquad \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \qquad \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{10}$$

In the following sections we replace (x_1, x_2, x_3) with (x, y, z),$\chi_{12} = \chi_{xy}$, $\chi_{23} = \chi_{yz}$, and $\omega_2 = \omega_y$.

3.2 Equations of Equilibrium

We begin with the following displacement field for a straight beam bent by forces in the xz plane (i.e., bending about the y-axis):

$$\begin{gathered} \mathbf{u} = u_1\,\hat{\mathbf{e}}_1 + u_2\,\hat{\mathbf{e}}_2 + u_3\,\hat{\mathbf{e}}_3 \quad \text{where,} \\ u_1 = \sum_{i=0}^{m} z^i\,\phi_x^{(i)}(x,t), \quad u_2 = 0, \quad u_3 = \sum_{i=0}^{p} z^i\,\psi_z^{(i)}(x,t) \end{gathered} \tag{11}$$

where ($\phi_x^{(0)} = u,\ \psi_z^{(0)} = w$) are midplane displacements along the x and z directions, respectively, and $\phi_x^{(i)}$ and $\phi_x^{(i)}$ have the meaning

$$\phi_x^{(i)} = \frac{1}{(i)!}\left(\frac{\partial^i u_1}{\partial z^i}\right)_{z=0}, \qquad \psi_z^{(i)} = \frac{1}{(i)!}\left(\frac{\partial^i u_3}{\partial z^i}\right)_{z=0} \tag{12}$$

For a general third order beam theory, we take $m = 3$ and $p = 2$ in Eq. (11). Then the displacement field would be same as (5) with $\phi_x^{(0)} = u$, $\phi_x^{(1)} = \theta_x$, $\phi_x^{(2)} = \phi_x$, $\phi_x^{(2)} = \psi_x$, $\psi_z^{(0)} = w$,

$\psi_z^{(1)} = \theta_z$ and $\psi_z^{(2)} = \phi_z$. The nonzero von Kármán nonlinear strains (i.e., simplified Green-Lagrange strain tensor components) associated with the displacement field (11) are

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial \psi_z^{(0)}}{\partial x}\right)^2 = \sum_{i=0}^{m} z^i\, \varepsilon_{xx}^{(i)}$$
$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \sum_{i=0}^{\tilde{p}} z^i\, \gamma_{xz}^{(i)}, \quad \varepsilon_{zz} = \frac{\partial u_3}{\partial z} = \sum_{i=0}^{p-1} z^i\, \varepsilon_{zz}^{(i)} \tag{13}$$

and rotation and curvature components are

$$\begin{aligned}
\omega_y &= \frac{1}{2}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) = \sum_{i=0}^{\tilde{p}} z^i\, \omega_y^{(i)} \\
\chi_{xy} &= 2\chi_{12} = \frac{\partial \omega_y}{\partial x} = \sum_{i=0}^{\tilde{p}} z^i\, \chi_{xy}^{(i)} \\
\chi_{yz} &= 2\chi_{23} = \frac{\partial \omega_y}{\partial z} = \sum_{i=0}^{\tilde{p}-1} z^i\, \chi_{yz}^{(i)}
\end{aligned} \tag{14}$$

Where $\tilde{p} = \max\,(m-1, p)$ and

$$\begin{aligned}
\varepsilon_{xx}^{(i)} &= \frac{\partial \phi_x^{(i)}}{\partial x} + \delta(i)\frac{1}{2}\left(\frac{\partial \psi_z^{(0)}}{\partial x}\right)^2, \quad \delta(i) = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases} \\
\gamma_{xz}^{(i)} &= (i+1)\phi_x^{(i+1)} + \frac{\partial \psi_z^{(i)}}{\partial x}, \quad \varepsilon_{zz}^{(i)} = (i+1)\psi_z^{(i+1)} \\
\omega_y^{(i)} &= \frac{1}{2}\left\{(i+1)\phi_x^{(i+1)} - \frac{\partial \psi_z^{(i)}}{\partial x}\right\} \\
\chi_{xy}^{(i)} &= \frac{1}{2}\left\{(i+1)\frac{\partial \phi_x^{(i+1)}}{\partial x} - \frac{\partial^2 \psi_z^{(i)}}{\partial x^2}\right\} \\
\chi_{yz}^{(i)} &= \frac{1}{2}(i+1)\left\{(i+2)\phi_x^{(i+2)} - \frac{\partial \psi_z^{(i+1)}}{\partial x}\right\}
\end{aligned} \tag{15}$$

Next, Hamilton's Principle is used to derive the equations of motion, incorporating modified couple stress terms and through-thickness variation of the material (see Reddy (2002)):

$$0 = \int_{t_2}^{t_1} (-\delta K + \delta U + \delta V)\, dt \tag{16}$$

Where δK is the virtual kinetic energy, δU is the virtual strain energy, and δV is the virtual work done by external forces. The virtual kinetic energy expression is

$$\begin{aligned}\delta K &= \int_0^L \int_A \rho\, \dot{u}_i \delta \dot{u}_i \, dA\, dx \\ &= \int_0^L \int_A \rho \left[\left(\sum_{i=0}^{m} z^i\, \dot{\phi}_x^{(i)} \right)\left(\sum_{j=0}^{m} z^j\, \delta\dot{\phi}_x^{(j)} \right) + \left(\sum_{i=0}^{p} z^i\, \dot{\psi}_z^{(i)} \right)\left(\sum_{j=0}^{p} z^j\, \delta\dot{\psi}_x^{(j)} \right) \right] dx \\ &= \int_0^L \left[\sum_{i=0}^{m}\sum_{j=0}^{m} m_{i+j}\, \dot{\phi}_x^{(i)} \delta\dot{\phi}_x^{(j)} + \sum_{i=0}^{p}\sum_{j=0}^{p} m_{i+j}\, \dot{\psi}_x^{(i)} \delta\dot{\psi}_x^{(j)} \right] dx \end{aligned} \tag{17}$$

where

$$m_i = \int_A \rho\, z^i\, dA \tag{18}$$

The expression for the virtual strain energy is

$$\begin{aligned}\delta U &= \int_0^L \int_A \left(\sigma_{xx}\, \delta\varepsilon_{xx} + \sigma_{zz}\, \delta\varepsilon_{zz} + \sigma_{xz}\delta\gamma_{xz} + m_{xy}\delta\chi_{xy} + m_{yz}\, \delta\chi_{yz} \right) dA dx \\ &= \int_0^L \left[\sum_{i=0}^{m} M_{xx}^{(i)}\, \delta\varepsilon_{xx}^{(i)} + \sum_{i=0}^{p-1} M_{zz}^{(i)}\, \delta\varepsilon_{zz}^{(i)} + \sum_{i=0}^{\tilde{p}} M_{xz}^{le(i)}\, \delta\gamma_{xz}^{(i)} + \sum_{i=0}^{\tilde{p}} M_{xy}^{(i)}\, \delta\chi_{xy}^{(i)} + \sum_{i=0}^{\tilde{p}-1} M_{yz}^{(i)}\, \delta\chi_{yz}^{(i)} \right] dx \end{aligned} \tag{19}$$

Various stress resultants used in Eq. (19) are defined as

$$\begin{aligned} M_{xx}^{(i)} &= \int_A z^i\, \sigma_{xx} dA, \qquad M_{xy}^{(i)} = \int_A z^i\, m_{xy} dA \\ M_{xz}^{(i)} &= \int_A z^i\, \sigma_{xz} dA, \qquad M_{yz}^{(i)} = \int_A z^i\, m_{yz}\, dA \\ M_{zz}^{(i)} &= \int_A z^i\, \sigma_{zz} dA \end{aligned} \tag{20}$$

and virtual work done by external forces is

$$\begin{aligned}\delta V &= -\int_0^L [\, (f_x \delta u_1 + f_z \delta u_3 + c_y \delta\omega_y + q_x^t \delta u_1(x, \frac{h}{2}) + q_x^b \delta u_1(x, -\frac{h}{2}) \\ &\quad + q_z^t \delta u_3(x, \frac{h}{2}) + q_z^b \delta u_3(x, -\frac{h}{2})] dx \\ &= -\int_0^L [\, f_x \sum_{i=0}^{m} z^i\, \delta\phi_x^{(i)} + f_z \sum_{i=0}^{p} z^i\, \delta\psi_z^{(i)} + c_y \sum_{i=0}^{\tilde{p}} z^i\, \delta\omega_y^{(i)} + q_x^t \sum_{i=0}^{m} \left(\frac{h}{2}\right)^i \delta\phi_x^{(i)} \end{aligned} \tag{21}$$

$$+q_x^b \sum_{i=0}^{m} \left(-\frac{h}{2}\right)^i \delta\phi_x^{(i)} + q_z^t \sum_{i=0}^{p} \left(\frac{h}{2}\right)^i \delta\psi_z^{(i)} + q_z^b \sum_{i=0}^{p} \left(-\frac{h}{2}\right)^i \delta\psi_z^{(i)}]\, dx$$
$$= -\int_0^L [\sum_{i=0}^{m} F_x^{(i)}\, \delta\phi_x^{(i)} + \sum_{i=0}^{p} F_z^{(i)}\, \delta\psi_z^{(i)} + \sum_{i=0}^{m} c_y^{(i)}\, \delta\omega_y^{(i)}]\, dx$$

where

$$f_x^{(i)} = \int_A z^i \bar{f}_x\, dA, \qquad f_z^{(i)} = \int_A z^i \bar{f}_x\, dA, \qquad c_y^{(i)} = \int_A z^i \bar{c}_y\, dA,$$
$$F_\xi^{(i)} = f_\xi^{(i)} + \left(\frac{h}{2}\right)^i \left[q_\xi^t + (-1)^i q_\xi^b\right] \quad (\xi = x, z) \tag{22}$$

Here $\bar{f}_x$, $\bar{f}_z$, and $\bar{c}_y$ denote the distributed axial load, transverse load, and body couple about the y axis (all measured per unit volume of the beam), respectively. Substituting the sions δU, δV, and δK into Hamilton's principle (16), performing integration-by-parts with respect to$t$ as well as x to relieve the generalized displacements $\phi_x^{(i)}$and $\psi_z^{(i)}$ of any differentiations, and using the fundamental lemma of calculus variations, we obtain the following equations of motion:

$$0 = \sum_{j=0}^{m} m_{i+j}\, \ddot{\phi}_x^{(j)} - \frac{\partial M_{xx}^{(i)}}{\partial x} + i\left(M_{xz}^{(i-1)} - \frac{1}{2}\frac{\partial M_{xy}^{(i-1)}}{\partial x} - \frac{1}{2} c_y^{(i-1)}\right)$$
$$+\frac{1}{2} i(i-1) M_{yz}^{(i-2)} - F_x^{(i)} \qquad (i = 0 \text{ to } m) \tag{23}$$

$$0 = \sum_{j=0}^{p} m_{i+j}\, \ddot{\psi}_z^{(j)} - \delta(i)\frac{\partial}{\partial x}\left(M_{xx}^{(0)} \frac{\partial \psi_z^{(0)}}{\partial x}\right) + i\left(M_{zz}^{(i-1)} + \frac{1}{2}\frac{\partial M_{yz}^{(i-1)}}{\partial x}\right)$$
$$-\frac{\partial M_{xz}^{(i)}}{\partial x} - \frac{1}{2}\frac{\partial^2 M_{xy}^{(i)}}{\partial x^2} - F_z^{(i)} - \frac{1}{2}\frac{\partial c_y^{(i)}}{\partial x} \qquad (i = 0 \text{ to } p) \tag{24}$$

In Eqs.(23) and(24), i varies from 0 to m and 0 to p, respectively. There are a total of $(m + p + 2)$ equations of motion.The natural boundary conditions involve specifying the following generalized forces (when the corresponding generalized displacements are not specified):

$$\delta\phi_x^{(i)} : \qquad M_{xx}^{(i)} + \frac{i}{2} M_{xy}^{(i-1)} \qquad (i = 0 \text{ to } m)$$
$$\delta\psi_z^{(i)} : \qquad \delta(i)\left(M_{xx}^{(0)} \frac{d\psi_z^{(0)}}{dx}\right) + M_{xz}^{(i)} - \frac{i}{2} M_{yz}^{(i-1)} + \frac{1}{2}\frac{dM_{yz}^{(i-1)}}{dx} - \frac{1}{2} c_y^{(i)} \tag{25}$$
$$\frac{d\delta\psi_z^{(i)}}{dx} : \qquad -\frac{1}{2} M_{xy}^{(i)} \qquad (i = 0 \text{ to } p)$$

3.3 Generalized Force-Displacement Relations

For the general third-order beam theory, 2-D plane stress state is assumed to write the relations between symmetric part of stress and strain for isotropic and linear elastic material as

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{(1-\nu)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}-\alpha\Delta T\\ \varepsilon_{zz}-\alpha\Delta T\\ \gamma_{xz}\end{Bmatrix} \tag{26}$$

$$\begin{Bmatrix}m_{xy}\\ m_{yz}\end{Bmatrix} = G\ell^2\begin{Bmatrix}2\chi_{12}\\ 2\chi_{23}\end{Bmatrix} \tag{27}$$

where ℓ is a material length scale parameter, which is the square root of the ratio of the modulus of curvature to the modulus of shear, and it is a physical a property measuring the effect of couple stress (see Mindlin (1963)).

The stress resultants can be expressed as

$$\begin{Bmatrix}M_{xx}^{(i)}\\ M_{zz}^{(i)}\\ M_{xz}^{(i)}\end{Bmatrix} = \int_A \begin{Bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\end{Bmatrix} z^i dA = \sum_{k=i}^{\tilde{p}+i}\begin{bmatrix}A_{11}^{(k)} & A_{12}^{(k)} & 0\\ A_{12}^{(k)} & A_{11}^{(k)} & 0\\ 0 & 0 & B_{11}^{(k)}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}^{(k-i)}\\ \varepsilon_{zz}^{(k-i)}\\ \gamma_{xz}^{(k-i)}\end{Bmatrix} - \begin{Bmatrix}X_T^{(i)}\\ Z_T^{(i)}\\ 0\end{Bmatrix} \tag{28}$$

$$\begin{Bmatrix}M_{xy}^{(i)}\\ M_{yz}^{(i)}\end{Bmatrix} = \int_A \begin{Bmatrix}m_{xy}\\ m_{yz}\end{Bmatrix} z^i dA = \sum_{k=i}^{\tilde{p}+i}\frac{1}{2}S_{11}^{(k)}\begin{Bmatrix}\chi_{xy}^{(k-i)}\\ \chi_{yz}^{(k-i)}\end{Bmatrix} \tag{29}$$

where

$$\begin{aligned}
A_{11}^{(k)} &= \frac{1}{1-\nu^2}\int_A z^k\ E(z,T)\,dA, \qquad A_{12}^{(k)} = \frac{\nu}{1-\nu^2}\int_A z^k\ E(z,T)\,dA\\
B_{11}^{(k)} &= \frac{1}{2(1+\nu)}\int_A z^k\ E(z,T)\,dA, \qquad S_{11}^{(k)} = \frac{\ell^2}{(1+\nu)}\int_A z^k\ E(z,T)\,dA\\
X_T^{(k)} &= Z_T^{(k)} = \frac{1}{(1-\nu)}\int_A z^k\ E(z,T)\,\alpha(z,T)\Delta T\,dA
\end{aligned} \tag{30}$$

3.4 Virtual work statements: weak forms

The weak forms of equations of motion (23)–(24) for static case are (see Reddy (2004))

$$\begin{aligned}
0 = \int_{x_a}^{x_b}\Bigg[&\frac{\partial\delta\phi_x^{(i)}}{\partial x}M_{xx}^{(i)} + i\left(\delta\phi_x^{(i)}M_{xz}^{(i-1)} + \frac{1}{2}\frac{\partial\delta\phi_x^{(i)}}{\partial x}M_{xy}^{(i-1)} - \frac{1}{2}\delta\phi_x^{(i)}c_y^{(i-1)}\right)\\
&+\frac{1}{2}i(i-1)\delta\phi_x^{(i)}M_{yz}^{(i-2)} - \delta\phi_x^{(i)}F_x^{(i)}\Bigg]dx - \delta\phi_x^{(i)}(x_a,t)Q_a^{(i)} - \delta\phi_x^{(i)}(x_b,t)Q_b^{(i)}\\
&\qquad (i = 0 \text{ to } m)
\end{aligned} \tag{31}$$

$$0=\int_{x_a}^{x_b}\left[\delta(i)\frac{\partial\delta\psi_z^{(i)}}{\partial x}\left(M_{xx}^{(0)}\frac{\partial\psi_z^{(0)}}{\partial x}\right)+i\left(\delta\psi_z^{(i)}M_{zz}^{(i-1)}-\frac{1}{2}\frac{\partial\delta\psi_z^{(i)}}{\partial x}M_{yz}^{(i-1)}\right)\right.$$
$$\left.+\frac{\partial\delta\psi_z^{(i)}}{\partial x}M_{xz}^{(i)}-\frac{1}{2}\frac{\partial^2\delta\psi_z^{(i)}}{\partial x^2}M_{xy}^{(i)}-\delta\psi_z^{(i)}F_z^{(i)}-\frac{1}{2}\delta\psi_z^{(i)}\frac{\partial c_y^{(i)}}{\partial x}\right]dx \tag{32}$$
$$-\delta\psi_z^{(i)}(x_a,t)\bar{Q}_a^{(i)}-\delta\psi_z^{(i)}(x_b,t)\bar{Q}_b^{(i)}-\frac{\partial\delta\psi_z^{(i)}}{\partial x}(x_a,t)\hat{Q}_a^{(i)}-\frac{\partial\delta\psi_z^{(i)}}{\partial x}(x_b,t)\hat{Q}_b^{(i)}$$
$$(i=0\text{ to }p)$$

where the generalized force are

$$\begin{aligned}
Q_a^{(i)}&=\left[-M_{xx}^{(i)}-\frac{i}{2}M_{xy}^{(i-1)}\right]_{x_a},\qquad Q_b^{(i)}=\left[M_{xx}^{(i)}+\frac{i}{2}M_{xy}^{(i-1)}\right]_{x_b}\qquad(i=0\text{ to }m)\\
\bar{Q}_a^{(i)}&=-\left[\delta(i)\left(M_{xx}^{(0)}\frac{d\psi_z^{(0)}}{dx}\right)+M_{xz}^{(i)}-\frac{i}{2}M_{yz}^{(i-1)}+\frac{1}{2}\frac{dM_{yz}^{(i-1)}}{dx}-\frac{1}{2}c_y^{(i)}\right]_{x_a},\\
\bar{Q}_b^{(i)}&=\left[\delta(i)\left(M_{xx}^{(0)}\frac{d\psi_z^{(0)}}{dx}\right)+M_{xz}^{(i)}-\frac{i}{2}M_{yz}^{(i-1)}+\frac{1}{2}\frac{dM_{yz}^{(i-1)}}{dx}-\frac{1}{2}c_y^{(i)}\right]_{x_b}\\
\hat{Q}_a^{(i)}&=\left[\frac{1}{2}M_{xy}^{(i)}\right]_{x_a},\qquad\hat{Q}_b^{(i)}=\left[-\frac{1}{2}M_{xy}^{(i)}\right]_{x_b}\qquad(i=0\text{ to }p)
\end{aligned}\tag{33}$$

In terms of displacement the weak forms can be given as

$$0=\int_{x_a}^{x_b}\left[\frac{\partial\delta\phi_x^{(i)}}{\partial x}\left\{\sum_{k=i}^{m+i}A_{11}^{(k)}\left(\frac{\partial\phi_x^{(k-i)}}{\partial x}+\delta(k-i)\frac{1}{2}\left(\frac{\partial\psi_z^{(0)}}{\partial x}\right)^2\right)\right.\right.$$
$$\left.+\sum_{k=i}^{p-1+i}A_{12}^{(k)}(k-i+1)\psi_z^{(k-i+1)}-X_T^{(i)}\right\}$$
$$+i\delta\phi_x^{(i)}\left(\sum_{k=i-1}^{m+i-2}B_{11}^{(k)}(k-i+2)+\sum_{k=i-1}^{p+i-1}B_{11}^{(k)}\frac{\partial\psi_z^{(k-i+1)}}{\partial x}\right)$$
$$+\frac{i}{8}\frac{\partial\delta\phi_x^{(i)}}{\partial x}\left(\sum_{k=i-1}^{m+i-2}S_{11}^{(k)}(k-i+2)\frac{\partial\phi_x^{(k-i+2)}}{\partial x}-\sum_{k=i-1}^{p+i-1}S_{11}^{(k)}\frac{\partial^2\psi_z^{(k-i+1)}}{\partial x^2}\right) \tag{34}$$
$$+\frac{1}{8}i(i-1)\delta\phi_x^{(i)}\left(\sum_{k=i-2}^{m+i-4}S_{11}^{(k)}(k-i+3)(k-i+4)\phi_x^{(k-i+4)}\right.$$
$$\left.\left.-\sum_{k=i-2}^{p+i-3}S_{11}^{(k)}(k-i+3)\frac{\partial\psi_z^{(k-i+3)}}{\partial x}\right)-\delta\phi_x^{(i)}F_x^{(i)}-\frac{i}{2}\delta\phi_x^{(i)}c_y^{(i-1)}\right]dx$$
$$-\delta\phi_x^{(i)}(x_a,t)Q_a^{(i)}-\delta\phi_x^{(i)}(x_b,t)Q_b^{(i)}\qquad(i=0\text{ to }m)$$

$$
\begin{aligned}
0 = \int_{x_a}^{x_b} \Bigg[& \delta(i) \frac{\partial \delta\psi_z^{(i)}}{\partial x} \frac{\partial \psi_z^{(0)}}{\partial x} \left\{ \sum_{k=0}^{m} A_{11}^{(k)} \left(\frac{\partial \phi_x^{(k)}}{\partial x} + \delta(k) \frac{1}{2} \left(\frac{\partial \psi_z^{(0)}}{\partial x} \right)^2 \right) \right. \\
& \left. + \sum_{k=0}^{p-1} A_{12}^{(k)} (k+1) \psi_z^{(k+1)} - X_T^{(0)} \right\} + i \delta\psi_z^{(i)} \left\{ \sum_{k=i-1}^{p+i-2} A_{11}^{(k)} (k-i+2) \psi_z^{(k-i+2)} \right. \\
& \left. + \sum_{k=i-1}^{m+i-1} A_{12}^{(k)} \left(\frac{\partial \phi_x^{(k-i-1)}}{\partial x} + \delta(k-i-1) \frac{1}{2} \left(\frac{\partial \psi_z^{(0)}}{\partial x} \right)^2 \right) - Z_T^{(i-1)} \right\} \\
& - \frac{i}{8} \frac{\partial \delta\psi_z^{(i)}}{\partial x} \left(\sum_{k=i-1}^{m+i-3} S_{11}^{(k)} (k-i+2)(k-i+3) \phi_x^{(k-i+3)} - \sum_{k=i-1}^{p+i-2} S_{11}^{(k)} (k-i+2) \frac{\partial \psi_z^{(k-i+2)}}{\partial x} \right) \\
& + \frac{\partial \delta\psi_z^{(i)}}{\partial x} \left(\sum_{k=i}^{m+i-1} B_{11}^{(k)} (k-i+1) \phi_x^{(k-i+1)} + \sum_{k=i}^{p+i} B_{11}^{(k)} \frac{\partial \psi_z^{(k-i)}}{\partial x} \right) \\
& - \frac{1}{8} \frac{\partial^2 \delta\psi_z^{(i)}}{\partial x^2} \left(\sum_{k=i}^{m+i-1} S_{11}^{(k)} (k-i+1) \frac{\partial \phi_x^{(k-i+1)}}{\partial x} - \sum_{k=i}^{p+i} S_{11}^{(k)} \frac{\partial^2 \psi_z^{(k-i)}}{\partial x^2} \right) \\
& - \delta\psi_z^{(i)} F_z^{(i)} - \frac{1}{2} \delta\psi_z^{(i)} \frac{\partial c_y^{(i)}}{\partial x} \Bigg] dx - \delta\psi_z^{(i)}(x_a,t) \bar{Q}_a^{(i)} - \delta\psi_z^{(i)}(x_b,t) \bar{Q}_b^{(i)} \\
& - \frac{\partial \delta\psi_z^{(i)}}{\partial x}(x_a,t) \hat{Q}_a^{(i)} - \frac{\partial \delta\psi_z^{(i)}}{\partial x}(x_b,t) \hat{Q}_b^{(i)} \qquad (i = 0 \text{ to } p)
\end{aligned} \tag{35}
$$

3.5 Finite element model

Let's approximate the degrees of freedom for static case as following

$$
\phi_x^{(i)} \approx \sum_{j=1}^{m_i} \Delta_{x_j}^{(i)} \varphi_{xj}^{(i)}(x), \qquad \psi_z^{(i)} \approx \sum_{j=1}^{p_i} \Delta_{z_j}^{(i)} \varphi_{zj}^{(i)}(x) \tag{36}
$$

where $\varphi_{xj}^{(i)}(x)$ and $\varphi_{zj}^{(i)}(x)$ are the interpolation functions for $\phi_x^{(i)}$ and $\psi_z^{(i)}$ respectively. In case of conventional beam (i.e. without microstructure dependance) the interpolation function can be taken as Lagrange interpolation polynomials, whereas in case of microstructure dependent beam $\varphi_{xj}^{(i)}(x)$ are Lagrange polynomials and $\varphi_{zj}^{(i)}(x)$ are Hermite interpolation polynomials as the first derivative of $\psi_z^{(i)}$ are also primary variable in the virtual work statement (35). $\Delta_{x_j}^{(i)}$ and $\Delta_{z_j}^{(i)}$ are the nodal values associated with $\phi_x^{(i)}$ and $\psi_z^{(i)}$ respectively (see Reddy (2004)). Substituting $\phi_x^{(i)}$ and $\psi_z^{(i)}$ in the virtual work statement (34) and (35), we get the finite element equation as

$$
\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \dots & \mathbf{K}^{1\mathrm{N}} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \dots & \mathbf{K}^{2\mathrm{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}^{\mathrm{N}1} & \mathbf{K}^{\mathrm{N}2} & \dots & \mathbf{K}^{\mathrm{NN}} \end{bmatrix} \begin{Bmatrix} \mathbf{\Delta}^1 \\ \mathbf{\Delta}^2 \\ \vdots \\ \mathbf{\Delta}^{\mathrm{N}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \vdots \\ \mathbf{F}^{\mathrm{N}} \end{Bmatrix} \tag{37}
$$

where $N = (m + p + 2)$ is the total no. of equations of motion and

$$\Delta^i = \begin{cases} \Delta_x^{(i-1)}, & \text{for } i = 1 \text{ to } m+1 \\ \Delta_z^{(i-m-2)}, & \text{for } i = m+2 \text{ to } m+p+2 \end{cases} \tag{38}$$

The components of the elemental stiffness matrix $\mathbf{K}$ in (37) are

$$\begin{aligned}
K_{ij}^{\alpha\beta} = &\int_{x_a}^{x_b} \{ A_{11}^{(\alpha+\beta-2)} + \frac{1}{8}(\alpha-1)(\beta-1)S_{11}^{(\alpha+\beta-4)}\} \frac{d\varphi_{xi}^{(\alpha-1)}}{dx}\frac{d\varphi_{xj}^{(\beta-1)}}{dx} \\
&+(\alpha-1)(\beta-1)\{B_{11}^{(\alpha+\beta-4)} - \frac{1}{8}(\alpha-2)(\beta-2)S_{11}^{(\alpha+\beta-6)}\}\varphi_{xi}^{(\alpha-1)}\varphi_{xj}^{(\beta-1)}dx \\
&\qquad \text{for } \alpha,\beta = 1 \text{ to } (m+1) \\
K_{ij}^{\alpha\beta} = &\int_{x_a}^{x_b} [\,\delta(\beta-m-2)\frac{1}{2}A_{11}^{(\alpha-1)}\frac{d\psi_z^{(0)}}{dx}\frac{d\varphi_{xi}^{(\alpha-1)}}{dx}\frac{d\varphi_{zj}^{(0)}}{dx} \\
&+(\beta-m-2)A_{12}^{(\alpha+\beta-m-4)}\frac{d\varphi_{xi}^{(\alpha-1)}}{dx}\varphi_{zj}^{(\beta-m-2)} \\
&+(\alpha-1)\{B_{11}^{(\alpha+\beta-m-4)} + \frac{1}{8}(\alpha-2)(\beta-m-2)S_{11}^{(\alpha+\beta-m-6)}\}\varphi_{xi}^{(\alpha-1)}\frac{d\varphi_{zj}^{(\beta-m-2)}}{dx} \\
&-\frac{1}{8}(\alpha-1)S_{11}^{(\alpha+\beta-m-4)}\frac{d\varphi_{xi}^{(\alpha-1)}}{dx}\frac{d^2\varphi_{zj}^{(\beta-m-2)}}{dx^2}]dx, \\
&\qquad \text{for } \alpha = 1 \text{ to } m+1, \text{and } \beta = m+2 \text{ to } m+p+2 \\
K_{ij}^{\alpha\beta} = &\int_{x_a}^{x_b} [\,\delta(\alpha-m-2)A_{11}^{(\beta-1)}\frac{d\psi_z^{(0)}}{dx}\frac{d\varphi_{zi}^{(0)}}{dx}\frac{d\varphi_{xj}^{(\beta-1)}}{dx} \\
&+(\alpha-m-2)A_{12}^{(\alpha+\beta-m-4)}\varphi_{zi}^{(\alpha-m-2)}\frac{d\varphi_{xj}^{(\beta-1)}}{dx} \\
&+(\beta-1)\{B_{11}^{(\alpha+\beta-m-4)} - \frac{1}{8}(\alpha-m-2)(\beta-2)S_{11}^{(\alpha+\beta-m-6)}\}\frac{d\varphi_{zi}^{(\alpha-m-2)}}{dx}\varphi_{xj}^{(\beta-1)} \\
&-\frac{1}{8}(\beta-1)S_{11}^{(\alpha+\beta-m-4)}\frac{d^2\varphi_{zi}^{(\alpha-m-2)}}{dx^2}\frac{d\varphi_{xj}^{(\beta-1)}}{dx}]dx, \\
&\qquad \text{for } \alpha = m+2 \text{ to } m+p+2, \text{and } \beta = 1 \text{ to } m+1 \\
K_{ij}^{\alpha\beta} = &\int_{x_a}^{x_b} [\frac{1}{2}\delta(\alpha-m-2)\delta(\beta-m-2)A_{11}^{(0)}\left(\frac{d\psi_z^{(0)}}{dx}\right)^2\frac{d\varphi_{zi}^{(0)}}{dx}\frac{d\varphi_{zj}^{(0)}}{dx} \\
&+\delta(\alpha-m-2)(\beta-m-2)A_{12}^{(\beta-m-3)}\frac{d\psi_z^{(0)}}{dx}\frac{d\varphi_{zi}^{(0)}}{dx}\varphi_{zj}^{(\beta-m-2)} \\
&+\frac{1}{2}\delta(\beta-m-2)(\alpha-m-2)A_{12}^{(\alpha-m-3)}\frac{d\psi_z^{(0)}}{dx}\varphi_{zi}^{(\alpha-m-2)}\frac{d\varphi_{zj}^{(0)}}{dx} \\
&+(\alpha-m-2)(\beta-m-2)A_{11}^{(\alpha+\beta-2m-6)}\varphi_{zi}^{(\alpha-m-2)}\varphi_{zj}^{(\beta-m-2)} \\
&+\{B_{11}^{(\alpha+\beta-2m-4)} + \frac{1}{8}(\alpha-m-2)(\beta-m-2)S_{11}^{(\alpha+\beta-2m-6)}\}\frac{d\varphi_{zi}^{(\alpha-m-2)}}{dx}\frac{d\varphi_{zj}^{(\beta-m-2)}}{dx} \\
&+\frac{1}{8}S_{11}^{(\alpha+\beta-2m-4)}\frac{d^2\varphi_{zi}^{(\alpha-m-2)}}{dx^2}\frac{d^2\varphi_{zj}^{(\beta-m-2)}}{dx^2}]dx, \\
&\qquad \text{for } \alpha,\beta = m+2 \text{ to } m+p+2
\end{aligned} \tag{39}$$

The components of force vector are

$$F_i^{(\alpha)} = \int_{x_a}^{x_b} [X_T^{(\alpha-1)} \frac{d\varphi_{xi}^{(\alpha-1)}}{dx} + \left(F_x^{(\alpha-1)} + \frac{1}{2}(\alpha-1)c_y^{(\alpha-2)}\right)\varphi_{xi}^{(\alpha-1)}]dx \quad \text{for } \alpha = 1 \text{ to } m+1$$

$$F_i^{(\alpha)} = \int_{x_a}^{x_b} [\delta(\alpha-m-2)X_T^{(0)} \frac{d\psi_z^{(0)}}{dx}\frac{d\varphi_{zi}^{(0)}}{dx} + (\alpha-m-2)Z_T^{(\alpha-m-3)}\varphi_{zi}^{(\alpha-m-2)} + \left(F_z^{(\alpha-m-2)} + \frac{1}{2}\frac{dc_y^{(\alpha-m-2)}}{dx}\right)\varphi_{zi}^{(\alpha-m-2)}]dx \quad \text{for } \alpha = m+2 \text{ to } m+p+2 \tag{40}$$

The nonlinear equations (37) are solved using Newton's iterative method (see Reddy (2004)), which involves the computation of the coefficients of the element tangent stiffness matrix $\mathbf{T}^e$, which has similar structure as the stiffness matrix in Eq. (37). By applying Newton's iterative method, the algebraic equation for $(s+1)th$ iteration can be given as

$$\begin{bmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} & \dots & \mathbf{T}^{1\mathrm{N}} \\ \mathbf{T}^{21} & \mathbf{T}^{22} & \dots & \mathbf{T}^{2\mathrm{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}^{\mathrm{N}1} & \mathbf{T}^{\mathrm{N}2} & \dots & \mathbf{T}^{\mathrm{NN}} \end{bmatrix} \begin{Bmatrix} \delta\mathbf{\Delta}^1 \\ \delta\mathbf{\Delta}^2 \\ \vdots \\ \delta\mathbf{\Delta}^{\mathrm{N}} \end{Bmatrix}_{(\mathrm{s}+1)}^{(\mathrm{r}+1)} = \begin{Bmatrix} \mathbf{R}^1 \\ \mathbf{R}^2 \\ \vdots \\ \mathbf{R}^{\mathrm{N}} \end{Bmatrix}_{(\mathrm{s}+1)}^{(\mathrm{r})} \tag{41}$$

where $\mathbf{T}^e$ is the tangent matrix and its components are

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i{}^{\alpha}}{\partial \Delta_j{}^{\beta}} \tag{42}$$

Here R^α is the residual vector

$$\begin{aligned} R_i^\alpha &= \sum_{\gamma=1}^{(m+p+2)} \sum_{k=1}^{n_\gamma} K_{ik}^{\alpha\gamma}\,\Delta_k^\gamma - F_i^\alpha \\ &= \sum_{\gamma=1}^{(m+1)} \sum_{k=1}^{m_{\gamma-1}} K_{ik}^{\alpha\gamma}\,\Delta_{xk}^{(\gamma-1)} + \sum_{\gamma=m+2}^{(m+p+2)} \sum_{k=1}^{p_{\gamma-1}} K_{ik}^{\alpha\gamma}\,\Delta_{zk}^{(\gamma-m-2)} - F_i^\alpha \end{aligned} \tag{43}$$

Hence, the components of tangent matrix are

$$
\begin{aligned}
T_{ij}^{\alpha\beta} &= K_{ij}^{\alpha\beta}, \qquad \text{for } \alpha,\beta = 1 \text{ to } m+p+2, \text{ and } \beta \neq m+2 \\
T_{ij}^{\alpha\beta} &= K_{ij}^{\alpha\beta} + \int_{x_a}^{x_b} \frac{1}{2} A_{11}^{(\alpha-1)} \frac{d\psi_z^{(0)}}{dx}\frac{d\varphi_{xi}^{(\alpha-1)}}{dx}\frac{d\varphi_{zj}^{(0)}}{dx} dx, \\
&\qquad \text{for } \beta = m+2, \text{ and } \alpha = 1 \text{ to } m+1 \\
T_{ij}^{\alpha\beta} &= K_{ij}^{\alpha\beta} + \int_{x_a}^{x_b} \frac{1}{2}(\alpha-m-2) A_{12}^{(\alpha-m-3)} \frac{d\psi_z^{(0)}}{dx} \varphi_{zi}^{(\alpha-m-2)} \frac{d\varphi_{zj}^{(0)}}{dx}]dx \\
&\qquad \text{for } \beta = m+2, \text{ and } \alpha = m+2 \\
T_{ij}^{\alpha\beta} &= K_{ij}^{\alpha\beta} + \int_{x_a}^{x_b} \{ \sum_{\gamma=1}^{(m+1)} A_{11}^{(\gamma-1)} \frac{d\phi_x^{(\gamma-1)}}{dx} + \sum_{\gamma=m+2}^{(m+n+2)} (\delta(\gamma-m-2) A_{11}^{(0)} \left(\frac{d\psi_z^{(0)}}{dx}\right)^2 \\
&\quad + (\gamma-m-2) A_{12}^{(\gamma-m-3)} \psi_z^{(\gamma-m-2)}) - X_T^{(0)} \} \frac{d\varphi_{zi}^{(0)}}{dx}\frac{d\varphi_{zj}^{(0)}}{dx} dx, \\
&\qquad \text{for } \beta = m+2, \text{ and } \alpha = m+3 \text{ to } m+p+2
\end{aligned}
\tag{44}
$$

4 SPECIALIZATIONS TO OTHER BEAM THEORIES

4.1 Reddy Third-Order Beam Theory

The Reddy third-order beam theory is based on the displacement field

$$
\begin{aligned}
&\mathbf{u} = u_1\,\hat{\mathbf{e}}_1 + u_2\,\hat{\mathbf{e}}_2 + u_3\,\hat{\mathbf{e}}_3 \quad \text{where} \\
&u_1 = u + z\theta_x - c_1 z^3\left(\theta_x + \frac{\partial w}{\partial x}\right), \qquad u_2 = 0, \qquad u_3 = w
\end{aligned}
\tag{45}
$$

which is a special case of the general beam theory with $m = 3$,$p = 0$, and

$$
\phi_x^{(0)} = u, \qquad \phi_x^{(1)} = \theta_x, \qquad \phi_x^{(2)} = 0, \qquad \phi_x^{(3)} = -c_1\left(\theta_x + \frac{\partial w}{\partial x}\right), \qquad \psi_z^{(0)} = w \tag{46}
$$

The nonzero strains are

$$
\begin{aligned}
\varepsilon_{xx} &= \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + z\frac{\partial \theta_x}{\partial x} + z^3\frac{\partial \psi_x}{\partial x} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)} \\
\gamma_{xz} &= \theta_x + \frac{\partial w}{\partial x} + 3z^2\psi_x = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \\
\omega_y &= \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x} + 3z^2\psi_x\right) = \omega_y^{(0)} + z^2\omega_y^{(2)} \\
\chi_{xy} &= \frac{1}{4}\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2} + 3z^2\frac{\partial \psi_x}{\partial x}\right) = \chi_{xy}^{(0)} + z^2\chi_{xy}^{(2)} \\
\chi_{yz} &= \frac{3}{2} z\psi_x = z\chi_{yz}^{(1)}
\end{aligned}
\tag{47}
$$

where

$$
\begin{aligned}
\varepsilon_{xx}^{(0)} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_{xx}^{(1)} = \frac{\partial \theta_x}{\partial x}, \qquad \varepsilon_{xx}^{(3)} = \frac{\partial \psi_x}{\partial x} = -c_1\left(\frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\
\gamma_{xz}^{(0)} &= \theta_x + \frac{\partial w}{\partial x}, \qquad \gamma_{xz}^{(2)} = 3\psi_x = -c_2\left(\theta_x + \frac{\partial w}{\partial x}\right), \qquad c_2 = 3c_1
\end{aligned} \tag{48}
$$

and

$$
\begin{aligned}
\omega_y^{(0)} &= \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x}\right), \qquad \omega_y^{(2)} = \frac{3}{2}\psi_x = -\frac{c_2}{2}\left(\theta_x + \frac{\partial w}{\partial x}\right) \\
\chi_{xy}^{(0)} &= \frac{1}{2}\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right), \qquad \chi_{xy}^{(2)} = \frac{3}{2}\frac{\partial \psi_x}{\partial x} = -\frac{c_2}{2}\left(\frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\
\chi_{yz}^{(1)} &= 3\psi_x = -c_2\left(\theta_x + \frac{\partial w}{\partial x}\right)
\end{aligned} \tag{49}
$$

The equations of motion of the Reddy third-order beam theory take the form

$$
m_0\frac{\partial^2 u}{\partial t^2} + \overline{m}_1\frac{\partial^2 \theta_x}{\partial t^2} - c_1 m_3\frac{\partial^3 w}{\partial t^2 \partial x} - \frac{\partial M_{xx}^{(0)}}{\partial x} - f_x^{(0)} = 0 \tag{50}
$$

$$
\begin{aligned}
&\overline{m}_1\frac{\partial^2 u}{\partial t^2} + \widehat{m}_2\frac{\partial^2 \theta_x}{\partial t^2} - c_1\overline{m}_4\frac{\partial^3 w}{\partial x \partial t^2} - \frac{\partial \overline{M}_{xx}^{(1)}}{\partial x} + \widehat{M}_{xz}^{(0)} - F_x^{(1)} \\
&-c_2 M_{yz}^{(1)} - \frac{1}{2}\frac{\partial \widehat{M}_{xy}^{(0)}}{\partial x} + \frac{1}{2}\hat{c}_y^{(0)} = 0
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
&m_0\frac{\partial^2 w}{\partial t^2} + c_1\left(m_3\frac{\partial^3 u}{\partial x \partial t^2} + \overline{m}_4\frac{\partial^3 \theta_x}{\partial x \partial t^2} - c_1 m_6\frac{\partial^4 w}{\partial x^2 \partial t^2}\right) \\
&-\frac{\partial}{\partial x}\left(M_{xx}^{(0)}\frac{\partial w}{\partial x}\right) - \frac{\partial \widehat{M}_{xz}^{(0)}}{\partial x} - c_1\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} - F_z^{(0)} - \frac{1}{2}\frac{\partial^2 \widetilde{M}_{xy}^{(0)}}{\partial x^2} + c_2\frac{\partial M_{yz}^{(1)}}{\partial x} + \frac{1}{2}\frac{\partial \tilde{c}_y^{(0)}}{\partial x} = 0
\end{aligned} \tag{52}
$$

where

$$
\begin{aligned}
&\overline{m}_i = m_i - c_1 m_{i+2}, \quad \widehat{m}_i = \overline{m}_i - c_1\overline{m}_{i+2}, \qquad \hat{c}_y^{(0)} = c_y^{(0)} - c_2 c_y^{(2)}, \qquad \tilde{c}_y^{(0)} = c_y^{(0)} + c_2 c_y^{(2)} \\
&\overline{M}_{xx}^{(i)} = M_{xx}^{(i)} - c_1 M_{xx}^{(i+2)}, \qquad \widehat{M}_{xz}^{(i)} = M_{xz}^{(i)} - c_2 M_{xz}^{(i+2)} \\
&\widehat{M}_{xy}^{(0)} = M_{xy}^{(0)} - c_2 M_{xy}^{(2)}, \qquad \widetilde{M}_{xy}^{(i)} = M_{xy}^{(i)} + c_2 M_{xy}^{(i+2)}
\end{aligned} \tag{53}
$$

The natural boundary conditions involve specifying the following generalized forces:

$$\begin{aligned}
&\delta u: && M_{xx}^{(0)} \\
&\delta \theta_x: && \overline{M}_{xx}^{(1)} + \frac{1}{2}\widehat{M}_{xy}^{(0)} \\
&\delta w: && \widehat{M}_{xz}^{(0)} + M_{xx}^{(0)}\frac{\partial w}{\partial x} + c_1\frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{1}{2}\frac{\partial \widehat{M}_{xy}^{(0)}}{\partial x} - c_2 M_{yz}^{(1)} + \frac{1}{2}\tilde{c}_y^{(0)} \\
& && + c_1 f_x^{(3)} + c_1\left(m_3\frac{\partial^2 u}{\partial t^2} + \overline{m}_4\frac{\partial^2 \theta_x}{\partial t^2} - c_1 m_4\frac{\partial^3 w}{\partial x \partial t^2}\right) \\
&\frac{\partial \delta w}{\partial x}: && M_{xx}^{(3)} + \frac{1}{2}\widehat{M}_{xy}^{(0)}
\end{aligned} \tag{54}$$

4.2 The Timoshenko (First-Order) Beam Theory

The Timoshenko first-order beam theory is based on the displacement field (see Reddy (2011))

$$\begin{gathered}
\mathbf{u} = u_1\,\hat{\mathbf{e}}_1 + u_2\,\hat{\mathbf{e}}_2 + u_3\,\hat{\mathbf{e}}_3 \quad \text{where} \\
u_1(x,z,t) \quad = u(x,t) + z\theta_x(x,t), \qquad u_2 = 0, \qquad u_3(x,t) = w(x,t)
\end{gathered} \tag{55}$$

which is again a special case of the general third-order beam theory with $m = 1, p = 0$, and

$$\phi_x^{(0)} = u, \qquad \phi_x^{(1)} = \theta_x, \qquad \psi_z^{(0)} = w \tag{56}$$

The nonzero strains are

$$\begin{aligned}
\varepsilon_{xx} &= \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + z\frac{\partial \theta_x}{\partial x} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \qquad \gamma_{xz} = \gamma_{xz}^{(0)} \\
\omega_y &= \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x}\right) = \omega_y^{(0)} \\
\chi_{xy} &= \frac{1}{4}\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) = \chi_{xy}^{(0)}
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
\varepsilon_{xx}^{(0)} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_{xx}^{(1)} = \frac{\partial \theta_x}{\partial x}, \qquad \gamma_{xz}^{(0)} = \theta_x + \frac{\partial w}{\partial x} \\
\omega_y^{(0)} &= \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x}\right), \qquad \chi_{xy}^{(0)} = \frac{1}{4}\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right)
\end{aligned} \tag{58}$$

The shear stress resultant $M_{xz}^{(0)}$ in the Timoshenko beam theory is defined as

$$M_{xz}^{(0)} = K \int_A \sigma_{xz} \, dA \tag{59}$$

and K is the shear correction factor. The equations of motion of the Timoshenko beam theory are

$$m_0 \frac{\partial^2 u}{\partial t^2} + m_1 \frac{\partial^2 \theta_x}{\partial t^2} - \frac{\partial M_{xx}^{(0)}}{\partial x} - f_x^{(0)} = 0 \tag{60}$$

$$m_1 \frac{\partial^2 u}{\partial t^2} + m_2 \frac{\partial^2 \theta_x}{\partial t^2} - \frac{\partial M_{xx}^{(1)}}{\partial x} + M_{xz}^{(0)} - f_x^{(1)} - \frac{1}{2}\frac{\partial M_{xy}^{(0)}}{\partial x} - \frac{1}{2} c_y^{(0)} = 0 \tag{61}$$

$$m_0 \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(M_{xx}^{(0)} \frac{\partial w}{\partial x}\right) - \frac{\partial M_{xz}^{(0)}}{\partial x} - F_z^{(0)} - \frac{1}{2}\frac{\partial^2 M_{xy}^{(0)}}{\partial x^2} - \frac{1}{2}\frac{\partial c_y^{(0)}}{\partial x} = 0 \tag{62}$$

The natural boundary conditions involve specifying the following generalized forces:

$$\begin{aligned} &\delta u: \quad M_{xx}^{(0)} \\ &\delta \theta_x: \quad M_{xx}^{(1)} + \frac{1}{2} M_{xy}^{(0)} \\ &\delta w: \quad M_{xz}^{(0)} + M_{xx}^{(0)} \frac{\partial w}{\partial x} + \frac{1}{2}\frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{1}{2} c_y^{(0)} \end{aligned} \tag{63}$$

4.3 The Bernoulli–Euler Beam Theory

The Bernoulli–Euler beam theory is based on the displacement field

$$\begin{aligned} \mathbf{u} &= u_1 \,\hat{\mathbf{e}}_1 + u_2 \,\hat{\mathbf{e}}_2 + u_3 \,\hat{\mathbf{e}}_3 \quad \text{where,} \\ u_1(x,z,t) &= u(x,t) + z\theta_x(x,t), \qquad u_2 = 0, \qquad u_3(x,t) = w(x,t) \end{aligned} \tag{64}$$

This displacement field is a special case of the general third order beam theory with $m = 1, p = 0$, and

$$\phi_x^{(0)} = u, \qquad \phi_x^{(1)} = \frac{dw}{dx}, \qquad \psi_z^{(0)} = w \tag{65}$$

The equations of motion can be obtained from the Reddy third-order beam theory by some changes. We have

$$m_0 \frac{\partial^2 u}{\partial t^2} - m_1 \frac{\partial^3 w}{\partial t^2 \partial x} - \frac{\partial M_{xx}^{(0)}}{\partial x} - f_x^{(0)} = 0 \tag{66}$$

$$m_0 \frac{\partial^2 w}{\partial t^2} + m_1 \frac{\partial^3 u}{\partial x \partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\partial}{\partial x}\left(M_{xx}^{(0)} \frac{\partial w}{\partial x}\right) - \frac{\partial^2 M_{xx}^{(1)}}{\partial x^2} - F_z^{(0)} - \frac{\partial f_x^{(1)}}{\partial x} - \frac{\partial^2 \widetilde{M}_{xy}^{(0)}}{\partial x^2} - \frac{\partial c_y^{(0)}}{\partial x} = 0 \tag{67}$$

The natural boundary conditions are

$$\begin{aligned} &\delta u: \quad M_{xx}^{(0)} \\ &\delta w: \quad M_{xx}^{(0)} \frac{\partial w}{\partial x} + \frac{\partial M_{xx}^{(1)}}{\partial x} + f_x^{(1)} + \frac{\partial M_{xy}^{(0)}}{\partial x} + m_1 \frac{\partial^2 u}{\partial t^2} + m_2 \frac{\partial^3 w}{\partial x \partial t^2} + c_y^{(0)} \\ &\frac{\partial \delta w}{\partial x}: \quad M_{xx}^{(0)} + M_{xy}^{(0)} \end{aligned} \tag{68}$$

In all of the special cases considered in this section, the transverse normal strain is zero, $\varepsilon_{zz} = 0$; this requires us to use the one-dimensional stress-strain relations $\sigma_{xx} = E\varepsilon_{xx}$ and $\sigma_{xz} = G\gamma_{xz}$. That is, we set $\nu = 0$ in the definition of $A_{11}^{(k)}$, $A_{12}^{(k)}$, and $X_T^{(k)}$ of Eq. (29).

5 ANALYTICAL SOLUTION

Analytical solution for a simply supported beam is obtained for a general third-order beam theory neglecting geometric nonlinearity. The equations of equilibrium for linear static case are

$$0 = -\frac{\partial M_{xx}^{(i)}}{\partial x} + i\left(M_{xz}^{(i-1)} - \frac{1}{2}\frac{\partial M_{xy}^{(i-1)}}{\partial x} - \frac{1}{2}c_y^{(i-1)}\right) + \frac{1}{2}i(i-1)M_{yz}^{(i-2)} - F_x^{(i)} \quad (i = 0 \text{ to } m) \tag{69}$$

$$0 = i\left(M_{zz}^{(i-1)} + \frac{1}{2}\frac{\partial M_{yz}^{(i-1)}}{\partial x}\right) - \frac{\partial M_{xz}^{(i)}}{\partial x} - \frac{1}{2}\frac{\partial^2 M_{xy}^{(i)}}{\partial x^2} - F_z^{(i)} - \frac{1}{2}\frac{\partial c_y^{(i)}}{\partial x} \quad (i = 0 \text{ to } p) \tag{70}$$

Static linear equations of motion in terms of displacement

$$0 = -\left(\sum_{k=i}^{m+i} A_{11}^{(k)} \frac{d^2 \phi_x^{(k-i)}}{dx^2} + \sum_{k=i}^{n-1+i} (k-i+1) A_{12}^{(k)} \frac{d\psi_z^{(k-i+1)}}{dx}\right) \tag{71}$$

$$
\begin{aligned}
&+i\left(\sum_{k=i-1}^{m+i-2} B_{11}^{(k)}\,(k-i+2)\phi_x^{(k-i+2)}+\sum_{k=i-1}^{n+i-1} B_{11}^{(k)}\,\frac{d\psi_z^{(k-i+1)}}{dx}\right)\\
&-\frac{i}{8}\left(\sum_{k=i-1}^{m+i-2} S_{11}^{(k)}\,(k-i+2)\frac{d^2\phi_x^{(k-i+2)}}{dx^2}-\sum_{k=i-1}^{n+i-1} S_{11}^{(k)}\,\frac{d^3\psi_z^{(k-i+1)}}{dx^3}\right)-\frac{i}{2}c_y^{(i-1)}\\
&+\frac{1}{8}i(i-1)\left(\sum_{k=i-2}^{m+i-4} S_{11}^{(k)}\,(k-i+3)(k-i+4)\phi_x^{(k-i+4)}\right.\\
&\left.-\sum_{k=i-2}^{n+i-3} S_{11}^{(k)}\,(k-i+3)\frac{d\psi_z^{(k-i+3)}}{dx}\right)-F_x^{(i)} \qquad (i=0\text{ to }m)
\end{aligned}
$$

$$
\begin{aligned}
0=\;&i\left(\sum_{k=i-1}^{m+i-1} A_{12}^{(k)}\,\frac{d\phi_x^{(k-i-1)}}{dx}+\sum_{k=i-1}^{n+i-2} A_{11}^{(k)}\,(k-i+2)\psi_z^{(k-i+2)}\right)\\
&+\frac{i}{8}\left(\sum_{k=i-1}^{m+i-3} S_{11}^{(k)}\,(k-i+2)(k-i+3)\frac{d\phi_x^{(k-i+3)}}{dx}-\sum_{k=i-1}^{n+i-2} S_{11}^{(k)}\,(k-i+2)\frac{d^2\psi_z^{(k-i+2)}}{dx^2}\right)\\
&-\left(\sum_{k=i}^{m+i-1} B_{11}^{(k)}\,(k-i+1)\frac{d\phi_x^{(k-i+1)}}{dx}+\sum_{k=i}^{n+i} B_{11}^{(k)}\,\frac{d^2\psi_z^{(k-i)}}{dx^2}\right)\\
&-\frac{1}{8}\left(\sum_{k=i}^{m+i-1} S_{11}^{(k)}\,(k-i+1)\frac{d^3\phi_x^{(k-i+1)}}{dx^3}-\sum_{k=i}^{n+i} S_{11}^{(k)}\,\frac{d^4\psi_z^{(k-i)}}{dx^4}\right)-F_z^{(i)}-\frac{1}{2}\frac{dc_y^{(i-1)}}{dx}\\
&\qquad (i=0\text{ to }p)
\end{aligned}
\tag{72}
$$

Let the solution for the degrees of freedom be of the form

$$
\phi_x^{(i)}(x)=\sum_{r=1}^{\infty} U_r^{(i)}\cos\frac{r\pi x}{L},\quad \psi_z^{(i)}(x)=\sum_{r=1}^{\infty} W_r^{(i)}\sin\frac{r\pi x}{L}
\tag{73}
$$

and the applied transverse force is given as

$$
f(x)=\sum_{r=1}^{\infty} F_r\sin\frac{r\pi x}{L}
\tag{74}
$$

Substituting $\phi_x^{(i)}$ and $\psi_z^{(i)}$ in the above equation, we get

$$0=\sum_{r=1}^{\infty}[\left(\sum_{k=i}^{m+i} A_{11}^{(k)}\left(\frac{r\pi}{L}\right)^2 U_r^{(k-i)}-\sum_{k=i}^{p-1+i}(k-i+1)A_{12}^{(k)}\left(\frac{r\pi}{L}\right)W_r^{(k-i+1)}\right)$$
$$+i\left(\sum_{k=i-1}^{m+i-2}(k-i+2)B_{11}^{(k)}U_r^{(k-i+2)}+\sum_{k=i-1}^{p+i-1}B_{11}^{(k)}\left(\frac{r\pi}{L}\right)W_r^{(k-i+1)}\right)$$
$$+\frac{i}{8}\left(\sum_{k=i-1}^{m+i-2}S_{11}^{(k)}(k-i+2)\left(\frac{r\pi}{L}\right)^2 U_r^{(k-i+2)}-\sum_{k=i-1}^{p+i-1}S_{11}^{(k)}\left(\frac{r\pi}{L}\right)^3 W_r^{(k-i+1)}\right) \tag{75}$$
$$+\frac{1}{8}i(i-1)\left(\sum_{k=i-2}^{m+i-4}S_{11}^{(k)}(k-i+3)(k-i+4)U_r^{(\boxtimes-i+4)}\right.$$
$$\left.-\sum_{k=i-2}^{p+i-3}S_{11}^{(k)}(k-i+3)\left(\frac{r\pi}{L}\right)W_r^{(k-i+3)}\right)]\cos\frac{r\pi x}{L} \qquad (i=0\text{ to }m)$$

$$0=\sum_{r=1}^{\infty}[i\left(\sum_{k=i-1}^{m+i-1}-A_{12}^{(k)}\left(\frac{r\pi}{L}\right)U_r^{(k-i-1)}+\sum_{k=i-1}^{p+i-2}A_{11}^{(k)}(k-i+2)W_r^{(k-i+2)}\right)$$
$$-\frac{i}{8}(\sum_{k=i-1}^{m+i-3}S_{11}^{(k)}(k-i+2)(k-i+3)\left(\frac{r\pi}{L}\right)U_r^{(k-i+3)}$$
$$-\sum_{k=i-1}^{p+i-2}S_{11}^{(k)}(k-i+2)\left(\frac{r\pi}{L}\right)^2 W_r^{(k-i+2)})$$
$$+\left(\sum_{k=i}^{m+i-1}B_{11}^{(k)}(k-i+1)\left(\frac{r\pi}{L}\right)U_r^{(k-i+1)}+\sum_{k=i}^{p+i}B_{11}^{(k)}\left(\frac{r\pi}{L}\right)^2 W_r^{(k-i)}\right) \tag{76}$$
$$-\frac{1}{8}\left(\sum_{k=i}^{m+i-1}S_{11}^{(k)}(k-i+1)\left(\frac{r\pi}{L}\right)^3 U_r^{(k-i+1)}-\sum_{k=i}^{p+i}S_{11}^{(k)}\left(\frac{r\pi}{L}\right)^4 W_r^{(k-i)}\right)-F_r^{(i)}]\sin\frac{r\pi x}{L}$$
$$(i=0\text{ to }p)$$

For the above equations to be true, each coefficient of sine and cosine term should be equal to zero. Hence, for the r^{th} coefficient we have

$$\begin{bmatrix} K_r^{11} & K_r^{12} & \dots & K_r^{1N} \\ K_r^{21} & K_r^{22} & \dots & K_r^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_r^{N1} & K_r^{N2} & \dots & K_r^{NN} \end{bmatrix}\begin{Bmatrix} \Delta_r^1 \\ \Delta_r^2 \\ \vdots \\ \Delta_r^N \end{Bmatrix}=\begin{Bmatrix} F_r^1 \\ F_r^2 \\ \vdots \\ F_r^N \end{Bmatrix} \tag{77}$$

where

$$
\begin{aligned}
K_r^{\alpha\beta} &= \{A_{11}^{(\alpha+\beta-2)} + \frac{1}{8}(\alpha-1)(\beta-1)S_{11}^{(\alpha+\beta-4)}\}\left(\frac{r\pi}{L}\right)^2 \\
&\quad +(\alpha-1)(\beta-1)\{B_{11}^{(\alpha+\beta-4)} + \frac{1}{8}(\alpha-2)(\beta-2)S_{11}^{(\alpha+\beta-6)}\} \\
&\qquad \text{for } \alpha,\beta = 1 \text{ to } (m+1) \\
K_r^{\alpha\beta} &= -(\beta-m-2)A_{12}^{(\alpha+\beta-m-4)}\left(\frac{r\pi}{L}\right) - \frac{1}{8}(\alpha-1)S_{11}^{(\alpha+\beta-m-4)}\left(\frac{r\pi}{L}\right)^3 \\
&\quad +(\alpha-1)\{B_{11}^{(\alpha+\beta-m-4)} - \frac{1}{8}(\alpha-2)(\beta-m-2)S_{11}^{(\alpha+\beta-m-6)}\}\left(\frac{r\pi}{L}\right) \\
&\qquad \text{for } \alpha = 1 \text{ to } m+1, \text{and } \beta = m+2 \text{ to } N \\
K_r^{\alpha\beta} &= -(\alpha-m-2)A_{12}^{(\alpha+\beta-m-4)}\left(\frac{r\pi}{L}\right) - \frac{1}{8}(\beta-1)S_{11}^{(\alpha+\beta-m-4)}\left(\frac{r\pi}{L}\right)^3 \\
&\quad +(\beta-1)\{B_{11}^{(\alpha+\beta-m-4)} - \frac{1}{8}(\alpha-m-2)(\beta-2)S_{11}^{(\alpha+\beta-m-6)}\}\left(\frac{r\pi}{L}\right) \\
&\qquad \text{for } \alpha = m+2 \text{ to } N, \text{and } \beta = 1 \text{ to } m+1 \\
K_{ij}^{\alpha\beta} &= (\alpha-m-2)(\beta-m-2)A_{11}^{(\alpha+\beta-2m-6)} + \frac{1}{8}S_{11}^{(\alpha+\beta-2m-4)}\left(\frac{r\pi}{L}\right)^4 \\
&\quad +\{B_{11}^{(\alpha+\beta-2m-4)} + \frac{1}{8}(\alpha-m-2)(\beta-m-2)S_{11}^{(\alpha+\beta-2m-6)}\}\left(\frac{r\pi}{L}\right)^2, \\
&\qquad \text{for } \alpha,\beta = m+2 \text{ to } N \\
F_r^{\alpha} &= \int_A F_r^{(\alpha-m-2)}\, dA + \left(\frac{h}{2}\right)^{(\alpha-m-2)}\left[(q_t)_r + (-1)^{(\alpha-m-2)}(q_b)_r\right] \\
&\qquad \text{for } \alpha = m+2 \text{ to } N
\end{aligned} \tag{78}
$$

and

$$
\Delta_r^i = \begin{cases} U_r^{(i)}, & \text{for } i = 1 \text{ to } m+1 \\ W_r^{(i)}, & \text{for } i = m+2 \text{ to } N \end{cases} \tag{79}
$$

For the analytical solution, homogeneous and functionally graded simply supported beams with $(n = 1)$ of following geometric and material parameter are considered:

$$
\begin{gathered}
E_1 = 14.4 \text{ GPa}, \qquad E_2 = 1.44 \text{ GPa}, \qquad \nu = 0.38, \\
h = 17.6\times10^{-6} \text{ m}, \qquad b = 2h, \qquad L = 20h,
\end{gathered} \tag{80}
$$

Results for nondimensional central transverse deflection ($\bar{w} = wEI/q_0L^4$) are tabulated for uniform $(q = q_0)$ and sinusoidal load $(q = q_0 \sin(x/L))$ for $q_0 = 1$ N/m for both homogeneous and FGM beam and also, the general third-order beam theory (TOBT)(considering $m = 3$ in horizontal displacement, u_1 and $p = 2$ in transverse displacement, u_3) results are compared with Euler–Bernoulli beam (EBT) and Timoshenko beam (TBT)(see Reddy (2011)) in Table 1 for the load acting as body force and also as the normal traction force on the top of the beam. In Table 2, the non dimensional central transverse deflection for uniform load (acting as body force) for FGM beam are tabulated considering $m = 3$ and $p = 1{,}2{,}3$ for the expression of horizontal displacement,u_1 and transverse displacement, u_3 [see Eq. (11)] respectively for different microstructural length scale factor. By comparing the nondimensional central transverse deflection for $(p = 1{,}2{,}3)$, we can see that the term corresponding to $p = 3$ in the displacement field is not contributing significantly. So, further in the present study, for all the numerical examples $m = 3$ and $p = 2$ are considered for the horizontal and transverse displacement field.

Table 1 Analytical solution for center deflection $\bar{w}\times\mathbf{10^2}$ for simply supported homogeneous and FGM beam for general third-order beam theory.

		Uniform load				Sinusoidal Load			
n	l/h	EBT	TBT	TOBT load as body force	TOBTTractionon top	EBT	TBT	TOBT load as body force	TOBTTractionon top
	0	1.3021	1.3103	1.3098	1.3108	1.0266	1.0333	1.0329	1.0337
	0.2	1.1092	1.1162	1.1155	1.1163	0.8745	0.8802	0.8796	0.8803
	0.4	0.7679	0.7731	0.7723	0.7729	0.6054	0.6096	0.6090	0.6095
	0.6	0.5076	0.5116	0.5109	0.5113	0.4002	0.4034	0.4029	0.4032
	0.8	0.3442	0.3475	0.3470	0.3473	0.2714	0.2741	0.2736	0.2739
	1	0.2435	0.2464	0.2459	0.2461	0.1920	0.1943	0.1939	0.1941
	0	3.0474	3.0624	3.0624	3.0628	2.4027	2.4148	2.4148	2.4152
	0.2	2.4900	2.5023	2.5017	2.5020	1.9632	1.9732	1.9726	1.9729
	0.4	1.6077	1.6165	1.6155	1.6157	1.2676	1.2747	1.2739	1.2741
	0.6	1.0108	1.0175	1.0166	1.0167	0.7970	0.8023	0.8016	0.8018
	0.8	0.6651	0.6707	0.6699	0.6700	0.5244	0.5289	0.5283	0.5284
	1	0.4620	0.4669	0.4663	0.4663	0.3642	0.3682	0.3677	0.3678

Table 2 Analytical solution for center deflection $\bar{w}\times\mathbf{10^2}$ for simply supported homogeneous and FGM beam under uniform loading considering m=3 and p=1,2,3 in displacement field.

p	Microstructure length-scale factor, l	Power-law index, $n = 0$	Power-law index, $n = 1$	Power-law index, $n = 10$
1	$l = 0$	1.12268684	2.62309658	5.65035580
2		1.31080654	3.06284414	6.59229566
3		1.31080654	3.06284398	6.59229547
1	$l = 0.6h$	0.47995749	0.96341184	2.53647121
2		0.51131386	1.01673651	2.71080148
3		0.51131386	1.01673671	2.71080703
1	$l = h$	0.23867030	0.45493901	1.28693138
2		0.24611186	0.46632362	1.32976237
3		0.24611186	0.46632389	1.32976950

6 FINITE ELEMENT SOLUTION

6.1 Linear solution

For numerical example, again the same beam specified as Eq. (80) is considered for the simply supported boundary condition. The linear finite element and analytical solutions are compared in Table 3 for microstructure length scale factor ($\ell = 0, 0.6h, h$) where, h is the height if the beam. In case of conventional beam (i.e., $\ell = 0$), thirty quadratic Lagrange elements are used in half beam (exploiting the symmetry). For microstructure dependent beam, $\phi_x^{(i)}$ are approximated as linear Lagrange interpolation function whereas, Hermite cubic interpolation functions are used for $\psi_z^{(i)}$. Forty-five such elements are used for the half domain to get the solution. The boundary condition can be given as:

$$\begin{array}{llll} w(0)=0, & \theta_z(0)=0, & \phi_z(0)=0 & \\ u(L/2)=0, & \theta_x(L/2)=0, & \phi_x(L/2)=0, & \psi_x(L/2)=0 \\ \dfrac{\partial w}{\partial x}(L/2)=0, & \dfrac{\partial \theta_z}{\partial x}(L/2)=0, & \dfrac{\partial \phi_z}{\partial x}(L/2)=0 & \end{array} \tag{81}$$

Table 3 Comparison of analytical and FEM (linear) solution of center deflection $\bar{\boldsymbol{w}}\times\mathbf{10^2}$ for simply supported homogeneous and FGM beams for general third order beam theory.

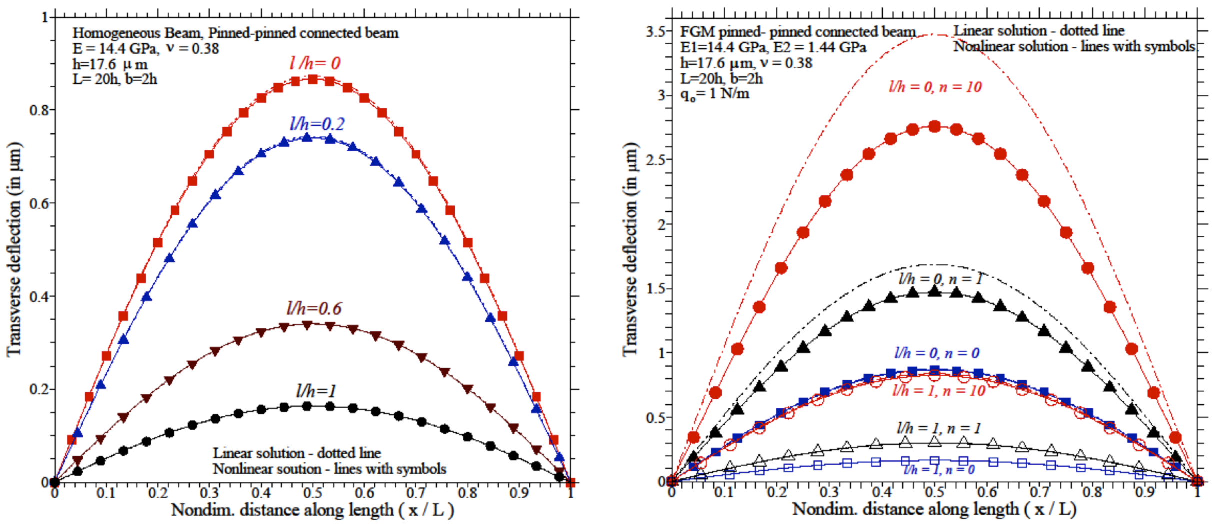
n	l/h	method	Load as body force	Load as traction on top
0	0	Analytical FEM (linear)	1.3098 1.3099	1.3108 1.3109
	0.6	Analytical FEM (linear)	0.5109 0.5109	0.5113 0.5113
	1	Analytical FEM (linear)	0.2459 0.2459	0.2461 0.2461
1	0	Analytical FEM (linear)	3.0624 3.0625	3.0628 3.0630
	0.6	Analytical FEM (linear)	1.0166 1.0166	1.0167 1.0167
	1	Analytical FEM (linear)	0.4663 0.4662	0.4663 0.4663

6.2 Nonlinear solution

Nonlinear solutions are presented in this section for pinned–pinned and clamped–clamped boundary conditions for the aforementioned beam. The deflection under uniformly distributed transverse load acting on the top surface of the beam as traction, $q_0 = 1$ N/m for homogeneous and functionally graded beams with power-law index $n = 1$ and $n = 10$ for various microstructure length scale factors are presented in Figures 2 and 3 for pinned–pinned and clamped–clamped boundary conditions, respectively. Again, for nonlinear solution also, thirty quadratic Lagrange elements are considered for conventional beam (i.e., $\ell = 0$) and, forty-five elements are considered for microstructure dependent beam. The error tolerance for Newton's method (see Reddy (2004)) is taken as 10^{-4}. The linear solutions are also shown in the same figures with dotted lines for comparison. Further, to see the effect of nonlinearity the plots for maximum deflection verses transverse load applied are presented in Figures 4, 5, and 6 for both homogeneous and functionally graded beam considering pinned–pinned and clamped–clamped boundary conditions. Also, in the figures, nonlinear solution for Euler–Bernoulli and Timoshenko beam (see Reddy (2011)) are plotted, for comparison, using the same loads.

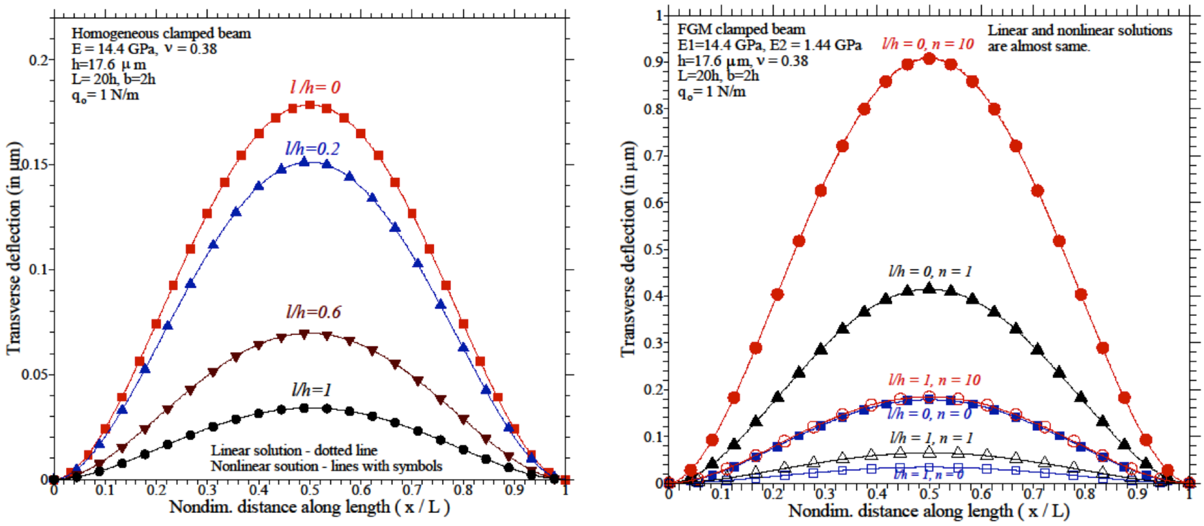
6.3 Summary

In the present study, the governing equations for a general third order theory for functionally graded beam considering the modified couple stress theory and von Kármán nonlinearity are developed using Hamilton's principle. The nonlinear finite element model is also developed using weak formulation of the equations. Analytical solution for the linear case is presented for the pinned-pinned boundary condition, and linear finite element solution is compared with the analytical solution obtained. Nonlinear finite element solutions for different boundary conditions for FGM beams with different power-law index n and different microstructure length-scale factor ℓ are presented. These results are the first of its kind, as there are no solutions available in the literature.



(a)Homogeneous beam (b)FGM beam

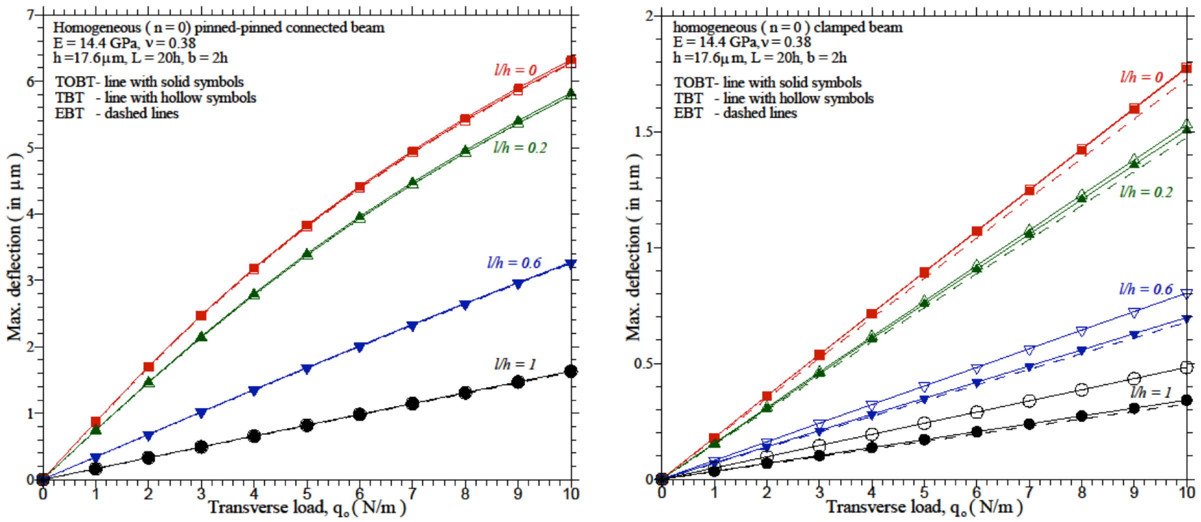
Figure 2 Transverse deflection versus distance along the length of pinned–pinned connected beams



(a)Homogeneous beam

(b)FGM beam

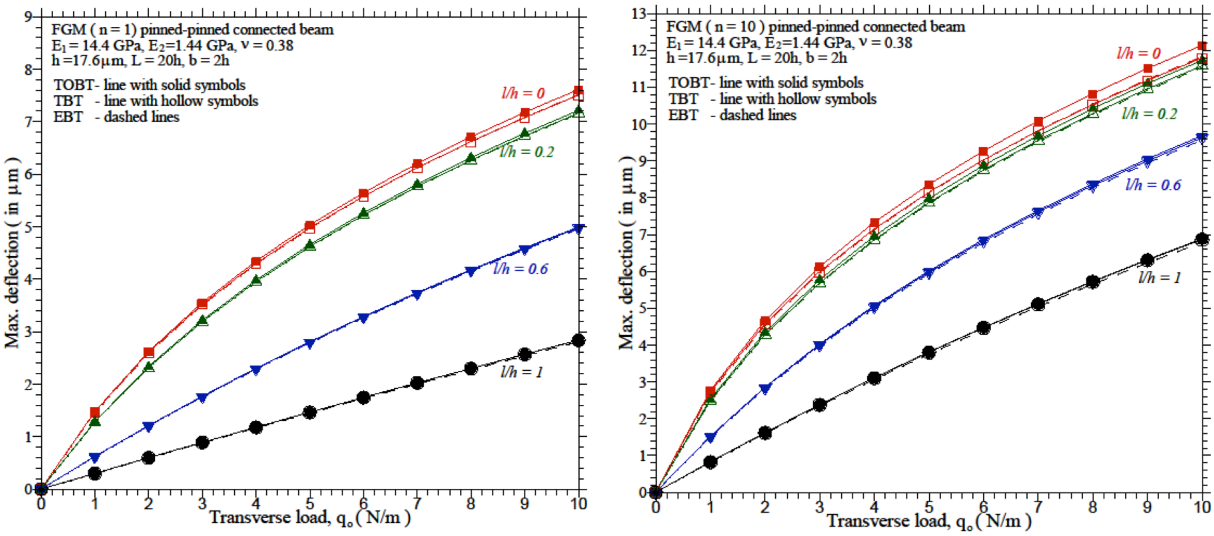
Figure 3 Transverse deflection versus distance along the length of clamped–clamped beams



(a) Pinned-pinned connected beam

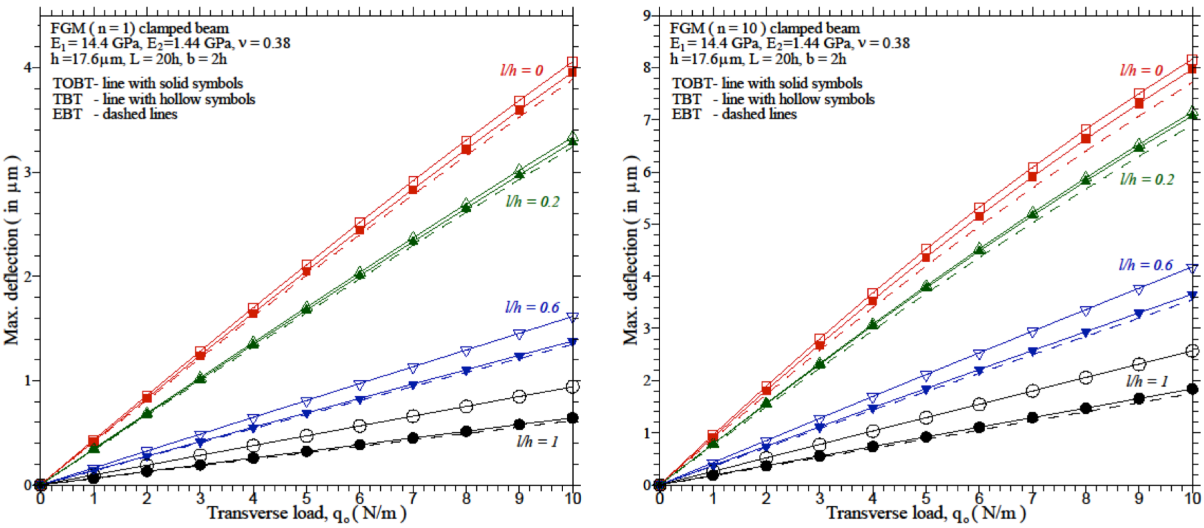
(b) clamped – clamped beam

Figure 4 Maximum transverse deflection versus load for homogeneous beams



(a) FGM beamwith $n = 1$ (b) FGM beamwith $n = 10$

Figure 5 Maximum transverse deflection versus load for pinned–pinned FGM beams



(a) FGM beamwith $n = 1$ (b) FGM beamwith $n = 10$

Figure 6 Maximum transverse deflection versus load for clamped–clamped FGM beams

Acknowledgements The research reported herein was carried out under National Science Foundation research grants CMMI-1000790. The second author also acknowledges support NSF grant CMMI-1030836.

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