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Algebraic solutions and computational strategy for planar multibody systems subjected to impact with friction

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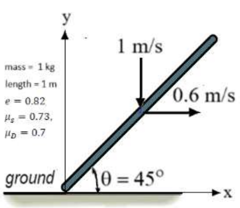
Abstract

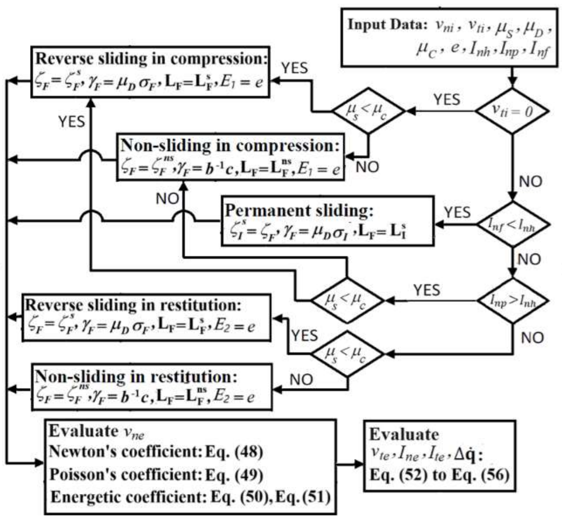
This paper presents a precise and effective formulation for modeling and simulating impact with friction for planar multibody systems. Nonlinear equations of motion are formulated using normal impulse at contact point as a time-like independent variable. The derived equations demonstrate that sliding during impact period could persist, or it could cease leading either to a persistence of sticking or to a reverse sliding. To distinguish between these different modes, Routh's incremental method is employed. A critical coefficient of friction, dependent solely on the system configuration, is identified and used to determine whether the contact mode is sliding or sticking. Three definitions of coefficient of restitution are introduced to model the plasticity of the impact. Analytical algebraic solutions and a computational strategy are offered to identify the mode of impact and to evaluate the impact variables. An example is thoroughly examined to demonstrate the effectiveness of the formulation, validate the algebraic solutions, and evaluate the outcomes of the three definitions of the coefficient of restitution.

Keywords

Impact with friction, rough collision, multibody system, Routh's method.

Graphical Abstract





Impact variables	Units	Newton's	Poisson's	Energetic
v_{ne}	m/s	0.82	0.5248	0.589
v_{te}	m/s	0	0	0
I_{ne}	N.s	0.9125	0.728	0.7681
I_{te}	N.s	0.3075	0.1968	0.2209
$\dot{\mathbf{q}}_e$	$\begin{bmatrix} m/s \\ m/s \\ rad/s \end{bmatrix}$	$\begin{bmatrix} 0.9075 \\ -0.0875 \\ -2.5668 \end{bmatrix}$	$\begin{bmatrix} 0.7968 \\ -0.272 \\ -2.2537 \end{bmatrix}$	$\begin{bmatrix} 0.8209 \\ -0.2319 \\ -2.3218 \end{bmatrix}$
W_{ne}	N.m	−0.0489	−0.0752	−0.0528
ΔT	N.m	0.0101	−0.1139	−0.0916

1 INTRODUCTION

The modeling and simulation of many multibody systems often involve impacts with friction. These contact-impacts phenomena can result from unexpected collisions or be inherent to their functional processes. Such phenomena are prevalent across nearly all fields of engineering. Intended impacts occur in various applications, such as walking machines, spot welding, drilling devices, tool manipulation, forging machines, writing on surfaces, assembling components, cooperative manipulators, and impulsive manipulation.

Impacts generate large forces, energy dissipation, and abrupt changes in velocities and accelerations, among other challenges. Additionally, friction management is complex due to the existence of two distinct modes: sticking and slipping. The nature of the problem, and consequently the solution, differs between these two modes. For example, a rigid body sliding on a surface could encounter the Painlevé paradox if friction is considered, a paradox that was solved recently and still is a focus of considerable research (Charles et al., 2018; Elkaranshawy et al., 2017a). When impact occurs alongside friction, the discontinuity of Coulomb friction introduces nonlinearity, further compounded by the nonlinearity caused by velocity discontinuities.

During planar impact, sliding may cease and continue until the end of the impact period or restart in the opposite direction. As a result, friction force laws cannot be directly applied to friction impulses calculated throughout the impact. However, the conventional approach to modeling rough collisions assumes that the contact point either continuously slides or remains non-sliding throughout the collision. This method has been widely used in classical dynamics texts, such as those by Whittaker (1904), Goldsmith (1960), and Kane and Levinson (1985). Recognizing that the sliding may cease or change direction during a collision, Routh (1897) proposed an incremental method that differentiates between various types of contact. He applied a semi-graphical, semi-algebraic technique to solve planar rough collisions. Kane and Levinson (1985) considered a case in which sliding reversed direction during a planar collision, and they observed that Whittaker's method could lead to an increase in energy. To address this issue, Wang and Mason (1992) advocated returning to Routh's method. They demonstrated that using Newton's coefficient of restitution with Routh's method fails to resolve this energy inconsistency, whereas employing Poisson's coefficient of restitution with Routh's method prevents the possibility of energy increase during a collision. Meanwhile, Stronge (1990) showed that with the Poisson-Routh method, the normal component of the impulse can dissipate energy even in purely elastic collisions. He proposed an energetic coefficient of restitution that works with Routh's method to resolve the energy inconsistency. A three-dimensional impact with friction is more complicated as the contact point continuously changes its sliding direction. Hence, sliding velocity in three-dimensional is the primary concern in many research work (Battle 1996; Elkaranshawy et al. 2017b; Jia and Wang 2017).

Impact with friction can be considered among the most critical challenges modeled in multibody systems. Rigorous mathematical models were developed for the dynamics of multibody systems with impact and friction (Battle 1996; Elkaranshawy et al. 2017b; Aghili 2020, Elkaranshawy 2007; Elkaranshawy 2011; Peng et al. 2020; Passas and Natsiavas 2022). Admirable review papers have been composed (Corral et al. 2021; Flores 2021; Flores et al. 2023). However, analytical solutions are only readily available in three-dimensional cases for certain special scenarios. While it is possible to develop analytical solutions for planar multibody systems, only a few attempts have been made to create a general approach that can be easily applied to various multibody systems. For instance, Aghili (2020) developed a unified frictional impact model that is consistent in both slip and stick states. However, the model needs to impose specific constraints on the coefficient of friction and the coefficient of restitution.

In this article, single-point rough collisions in planar rigid multibody systems are considered. Coulomb's law of friction and the assumption of infinite tangential stiffness are applied. The Newton's, Poisson's, and energetic coefficients of restitution are used to determine the end of the collision, and Routh's incremental method is employed to model the contact modes at the collision point. The equations of motion are derived, with the normal impulse serving as the integrating variable instead of time. It is shown that algebraic solutions for planar multibody collisions exist. The equations necessary to identify these invariant directions are derived. Without losing generality, assuming a sliding start, the conditions for reaching the sticking mode are determined, along with the coefficient of friction required to maintain this non-sliding mode. If the friction is not sufficient to keep the non-sliding, the sliding resumes in the reverse direction. A classification of all possible sliding behaviors is provided, and algebraic solutions and computational strategy are presented. A numerical simulation for an illustrated example is conducted to demonstrate the effectiveness of the formulation and verify the algebraic solutions. Additionally, it is used to assess the output of the three definitions of the coefficient of restitution.

2 EQUATIONS OF MOTION

The n-dimensional change in generalized joint velocities $\dot{\mathbf{q}}$ during single-point rough impact can be written as

$$\Delta\dot{\mathbf{q}} = \boldsymbol{M}^{-1}\boldsymbol{J}^T\mathbf{I} \tag{1}$$

The differential form of this equation is:

$$\frac{d\dot{\mathbf{q}}}{dt} = \boldsymbol{M}^{-1}\boldsymbol{J}^T\frac{d\mathbf{I}}{dt} \tag{2}$$

where $\boldsymbol{M}(\mathbf{q})$ is the inertia matrix of the multi-body system; $\boldsymbol{M}(.) \in \boldsymbol{R}^{n\times n}$, $\mathbf{I} = [I_n \quad I_t]^T$ is the impulse at the impact point, and $\boldsymbol{J}(\mathbf{q}) = [\boldsymbol{\alpha}^{\mathbf{T}} \quad \boldsymbol{\beta}^{\mathbf{T}}]^T$ is the Jacobian matrix that transforms generalized joint velocities to the velocity of the impact point $\mathbf{v} = [v_n \quad v_t]^T$ as

$$\mathbf{v} = \boldsymbol{J}\dot{\mathbf{q}} \tag{3}$$

where v_n and $\boldsymbol{v_t}$ are the normal and tangential components of the velocity of the impact point respectively, and I_n and I_t are the normal and tangential components of that impulse respectively, $v_n \in \boldsymbol{R}^{1\times1}$, $v_t \in \boldsymbol{R}^{1\times1}$, $\boldsymbol{\alpha} \in \boldsymbol{R}^{n\times1}$, $\boldsymbol{\beta} \in \boldsymbol{R}^{n\times1}$, $I_n \in \boldsymbol{R}^{1\times1}$ and $I_t \in \boldsymbol{R}^{1\times1}$.

Equations (2) and (3) gives:

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{D}\frac{d\mathbf{I}}{dt} \tag{4}$$

where $\boldsymbol{D}$ is the Jacobian inertia $\boldsymbol{D} \in \boldsymbol{R}^{3\times3}$ which depends only upon system configuration and is a symmetric positive definite matrix given by:

$$\boldsymbol{D} = \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \tag{5}$$

where

$$a = \boldsymbol{\alpha}^T\boldsymbol{M}^{-1}\boldsymbol{\alpha}, \quad b = \boldsymbol{\beta}^T\boldsymbol{M}^{-1}\boldsymbol{\beta}, \quad c = \boldsymbol{\beta}^T\boldsymbol{M}^{-1}\boldsymbol{\alpha}, \tag{6}$$

and $a \in \boldsymbol{R}^{1\times1}$, $b \in \boldsymbol{R}^{1\times1}$, $c \in \boldsymbol{R}^{1\times1}$.

3 CONTACT MODES

Coulomb's friction law with stiction and infinite tangential stiffness at the impact point are assumed. Accordingly, one can recognize three contact modes as follows.

3.1 Sliding Mode

The following form is used in sliding mode:

$$\frac{dI_t}{dt} = -\mu_D\sigma_I\frac{dI_n}{dt} \tag{7}$$

where μ_D is the kinetic coefficient of friction and $\sigma_I = \frac{v_t}{v_t}$ is the unit vector defining the sliding direction which could be either 1 or -1.

Therefore

$$\frac{dI_t}{dI_n} = -\mu_D\sigma_I \tag{8}$$

The scalar I_n has been selected as an impact parameter, since it is a time-like variable that starts with zero at the starting of impact and continuously increases through the impact period until the end of impact. Substituting Eq. (8) in Eq. (2) yields

$$\frac{d\dot{\mathbf{q}}^s}{dI_n} = \boldsymbol{M}^{-1}\boldsymbol{J}^T[1 \quad -\mu_D\sigma_I]^T = \mathbf{L}_{\boldsymbol{I}}^s \tag{9}$$

In this paper $()^s$ is preserved for the sliding while $()^{ns}$ is preserved for the non-sliding (sticking). Substituting Eq. (8) in Eq. (4) yields

$$\frac{dv_n^s}{dI_n} = a - \mu_D c\sigma_I = \frac{1}{\zeta_I^s} \tag{10}$$

$$\frac{dv_t}{dI_n} = c - \mu_D b \sigma_I = \frac{1}{\epsilon_I} \quad (11)$$

Equation (10) shows that v_n depends upon the sliding direction σ_I, i.e., upon v_t . On the other hand, v_t does not depend upon v_n as can be seen in Eq. (11).

3.2 Non-sliding Mode

For the non-sliding (sticking mode), the following are valid:

$$v_t = 0, \ \frac{dv_t}{dt} = 0 \quad (12)$$

Hence, substituting $\mathbf{v} = \begin{bmatrix} v_n & v_t \end{bmatrix}^T$ and $\mathbf{I} = \begin{bmatrix} I_n & I_t \end{bmatrix}^T$ in Eq. (4) and utilizing (12) gives

$$\frac{dI_t}{dI_n} = -b^{-1}c \quad or \quad \frac{dI_t}{dt} = -b^{-1}c\frac{dI_n}{dt} \quad (13)$$

a critical coefficient of friction can be introduced as

$$\mu_c = -b^{-1}c \quad (14)$$

However, according to Coulomb friction law during non-sliding

$$\frac{dI_t}{dt} \le \mu_s \frac{dI_n}{dt} \quad (15)$$

where μ_s is the static coefficient of friction. For sticking mode to persist once it has been reached, the following must be satisfied:

$$\mu_s \ge \mu_c \quad (16)$$

Otherwise, the friction would not be enough to keep the non-sliding mode and the sliding restarts in an altered direction. Substituting Eq. (13) in Eq. (2) yields

$$\frac{d\dot{\mathbf{q}}^{ns}}{dI_n} = \boldsymbol{M}^{-1}\boldsymbol{J}^T \begin{bmatrix} 1 & -b^{-1}c \end{bmatrix}^T = \mathbf{L}^{ns} \quad (17)$$

Substituting Eq. (13) in Eq. (4) yields

$$\frac{dv_n^{ns}}{dI_n} = a - c^T b^{-1} c = \frac{1}{\zeta_F^{ns}} \quad (18)$$

3.3 Reverse Sliding Mode

For the reverse sliding mode, the equations given in section 3.1 are valid up to the vanishing of the sliding velocity. In this case:

$$\mu_s \le \mu_c \quad (19)$$

Hence, the friction would not be enough to keep the non-sliding mode and the sliding restarts in the reverse direction. In this case Eq. (9) to Eq. (11) can be rewritten as:

$$\frac{d\dot{\mathbf{q}}^s}{dI_n} = \boldsymbol{M}^{-1}\boldsymbol{J}^T \begin{bmatrix} 1 & -\mu_D \sigma_F \end{bmatrix}^T = \mathbf{L}_{\boldsymbol{F}}^{\boldsymbol{s}} \quad (20)$$

$$\frac{dv_n^s}{dI_n} = a - \mu_D c \sigma_F = \frac{1}{\zeta_F^s} \quad (21)$$

$$\frac{dv_t}{dI_n} = c - \mu_D b \sigma_F = \frac{1}{\epsilon_F} \quad (22)$$

where

$$\sigma_F = -\sigma_I \quad (23)$$

4 RESTITUTION LAW

The deformation of an object involves two phases: compression and restitution. At the end of the compression phase $v_n = 0$, and the restitution phase started after that and continues to the end of impact. The coefficient of restitution specifies the end of impact. There are three definitions for the coefficient of restitution: namely the Newton's coefficient, the Poisson's coefficient, and the energetic coefficient. These coefficients are given as

Newton's coefficient

$$e_N = -\frac{v_{ne}}{v_{ni}} \tag{24}$$

Poisson's coefficient

$$e_P = \frac{I_{ne} - I_{nc}}{I_{nc}} \tag{25}$$

energetic coefficient

$$e_E{}^2 = -\frac{W_{nR}}{W_{nc}} = -\frac{W_{ne} - W_{nc}}{W_{nc}} \tag{26}$$

where v_{ni} and v_{ne} are the normal components of the velocity of the impact point at the beginning and at the end of impact respectively, I_{nc} and I_{ne} are the normal impulses at the end of compression period and at the end of impact respectively; W_{ne} ($W_n \le 0$), W_{nc} ($W_{nc} \le 0$) and W_{nR} ($W_{nR} \ge 0$) are the work done by the normal component of reaction force during impact period, compression period, and restitution period, respectively. Utilizing Eq. (10), Eq. (18), and Eq. (21) to obtain the following:

$$I_{nc} = \int_{v_{ni}}^{0} \zeta \, \mathrm{d}v_n, \quad I_{ne} = \int_{v_{ni}}^{v_{ne}} \zeta \, \mathrm{d}v_n \tag{27}$$

$$W_n = \int_0^{I_n} v_n \mathrm{d}I_n, \quad W_{ne} = \int_0^{I_{ne}} v_n \mathrm{d}I_n \tag{28}$$

$$W_{nc} = \int_0^{I_{nc}} v_n \mathrm{d}I_n = \int_{v_{ni}}^{0} v_n \zeta \, \mathrm{d}v_n, \quad W_{nR} = \int_{I_{nc}}^{I_{ne}} v_n \mathrm{d}I_n = \int_0^{v_{ne}} v_n \zeta \, \mathrm{d}v_n \tag{29}$$

where $W_n \left(W_n \le 0\right)$ is the work done by the the normal component of reaction force during impact at any instant, and ζ could be ζ_I^s, ζ_F^s, or ζ_F^{ns}. Also, it can be noticed that friction does not dissipate work during sticking, while it dissipates work during sliding given by

$$W_t = \int_0^{I_t} v_t . dI_t, \quad W_{te} = \int_0^{I_{te}} v_t . dI_t, \tag{30}$$

where I_t, I_{te} are the tangential impulse at any instant and at the end of impact respectively; $W_t \left(W_t \le 0\right)$ and $W_{te} \left(W_{te} \le 0\right)$ are the work done by the friction force during impact at any instant and during impact period respectively.

5 ALGEBRAIC SOLUTIONS AND COMPUTATIONAL STRATEGY

Let us assume that the impact starts with sliding in a specific direction σ_I, if the tangential velocity vanishes during impact, the normal impulse I_{nh} and the normal velocity v_{nh} at this instant can be obtained by integrating Eq. (10) and Eq. (11). It is important to notice that during that period the tangential velocity persists in its original direction σ_I until it vanishes, consequently $\zeta^s = \zeta_I^s$ is constant and

$$I_{nh} = -\frac{v_{ti}}{\sigma_I^T \left(\mathrm{c} - \mu_D b \sigma_I \right)} \tag{31}$$

$$v_{nh} = v_{ni} + \frac{I_{nh}}{\zeta_I^s} \tag{32}$$

If it is assumed that the tangential velocity does not vanish up to the end of compression phase, the normal impulse at this instant I_{np} can be obtained by integrating Eq. (10) as

$$I_{np} = -\zeta_I^s v_{ni} \tag{33}$$

Also, if it is assumed that the tangential velocity does not vanish up to the end of impact, the normal impulse at this instant I_{nf} can be obtained by integrating Eq. (10) as

$$I_{nf} = \zeta_I^s \left(v_{nf} - v_{ni} \right) \tag{34}$$

It can be noticed that if $I_{nh} > I_{nf}$ the sliding will continue up to the end if impact, if $I_{nh} < I_{np}$ the sliding halts in the compression period, and if $I_{np} < I_{nh} < I_{nf}$ the sliding halts in the restitution period. Hence, $I_{np}, I_{nh}, \ and\ I_{nf}$, which depend upon multibody system inertia and orientation, initial velocity, and coefficient of restitution, identify whether sliding or sticking is encountered. At the same time, if the sliding does not halt up to the end of impact, it is easy to prove that the three definitions for the coefficient of restitution are identical since ζ_I^s is constant. In this case, the normal velocity at the end of impact v_{nf} can be obtained from Eq. (24), Eq. (25), or Eq. (26) as

$$v_{nf} = -e v_{ni} \tag{35}$$

where e could be either e_N , e_P , or e_E . Accordingly, if the initial sliding in σ_I direction halts during the contact period, the sticking mode continues and persists as long as $\mu_s \geq \mu_c$, otherwise the friction would not be enough to keep the sticking mode, and a reverse sliding exists. Therefore, for this planar impact, one can recognize five types of motion given as:

- Permanent sliding if $I_{nh} > I_{nf}$.
- Non-sliding in restitution if $I_{np} < I_{nh} < I_{nf}$ and $\mu_s > \mu_c$.
- Reverse in restitution if $I_{np} < I_{nh} < I_{nf}$ and $\mu_s < \mu_c$.
- Non-sliding in compression if $I_{nh} < I_{np}$ and $\mu_s > \mu_c$.
- Reverse in compression if $I_{nh} < I_{np}$ and $\mu_s < \mu_c$.

If the tangential velocity vanishes in compression phase, the normal impulses at the end of compression period and at the end of impact, and the work done by the normal component of reaction force during compression and restitution can be obtained from Eq. (27) - Eq. (29) as

$$I_{nc} = \zeta_I^s \left(v_{nh} - v_{ni} \right) - \zeta_F v_{nh} \tag{36}$$

$$I_{ne} = \zeta_I^s \left(v_{nh} - v_{ni} \right) + \zeta_F \left(v_{ne} - v_{nh} \right) \tag{37}$$

$$W_{nc} = \frac{1}{2} \zeta_I^s \left(v_{nh}^2 - v_{ni}^2 \right) - \frac{1}{2} \zeta_F v_{nh}^2 \tag{38}$$

$$W_{nR} = \frac{1}{2} \zeta_F v_{ne}^2 \tag{39}$$

While, if the tangential velocity vanishes in restitution phase, these quantities can be obtained from Eq. (27) - Eq. (29) as

$$I_{nc} = -\zeta_I^s v_{ni} \tag{40}$$

$$I_{ne} = \zeta_I^s \left(v_{nh} - v_{ni} \right) + \zeta_F \left(v_{ne} - v_{nh} \right) \tag{41}$$

$$W_{nc} = -\frac{1}{2} \zeta_I^s v_{ni}^2 \tag{42}$$

$$W_{nR} = \frac{1}{2} \zeta_I^s v_{nh}^2 + \frac{1}{2} \zeta_F \left(v_{ne}^2 - v_{nh}^2 \right) \tag{43}$$

where ζ_F either ζ_F^s or ζ_F^{ns} in Eq. (36) to Eq. (43). Utilizing Eq. (8), Eq. (11), and Eq. (30) can give the work done by the frictional component of impact as

$$W_{te} = \frac{1}{2} \mu_D \sigma_I \epsilon_I v_{ti}^2 + \frac{1}{2} \mu_D \sigma_F \epsilon_F v_{te}^2 \tag{44}$$

It is valuable to identify the following quantities:

$$\zeta_I^s = \left(a - \mu_D c \sigma_I \right)^{-1}, \qquad \zeta_F^s = \left(a - \mu_D c \sigma_F \right)^{-1}, \qquad \zeta_F^{ns} = \left(a - b^{-1} c^2 \right)^{-1} \tag{45}$$

$$\epsilon_I^s = \left(c - \mu_D b \sigma_I \right)^{-1}, \qquad \epsilon_F^s = \left(c - \mu_D b \sigma_F \right)^{-1}, \qquad \epsilon_F^{ns} = 0 \tag{46}$$

$$\mathbf{L}_I^s = \boldsymbol{M}^{-1} \boldsymbol{J}^T \left[1 \quad -\mu_D \sigma_I \right]^T, \qquad \mathbf{L}_F^s = \boldsymbol{M}^{-1} \boldsymbol{J}^T \left[1 \quad -\mu_D \sigma_F \right]^T, \qquad \mathbf{L}_F^{ns} = \boldsymbol{M}^{-1} \boldsymbol{J}^T \left[1 \quad -b^{-1} c \right]^T \tag{47}$$

Hence, the algebraic solutions for the equations of motion for the five types of motion for the planar impact can be obtained. Newton's coefficient gives:

$$v_{ne} = -e_N v_{ni} \tag{48}$$

Poisson's coefficient gives:

$$v_{ne} = -E_{1P} \left(E_{2P} v_{ni} \left(\frac{\zeta_I^s}{\zeta_F^s} \right) + v_{nh} \left(1 - \frac{\zeta_I^s}{\zeta_F} \right) \right) \tag{49}$$

Energetic coefficient gives:

$$v_{ne} = -E_{1E} \sigma_n \sqrt{ {E_{2E}}^2 v_{ni}^2 \left(\frac{\zeta_I^s}{\zeta_F^s} \right) + v_{nh}^2 \left(1 - \frac{\zeta_I^s}{\zeta_F} \right) } \tag{50}$$

where σ_n is the sign of v_n, which could be 1 or -1 and given by:

$$\sigma_n = \frac{v_n}{v_n} \tag{51}$$

The tangential velocity for the non-sliding,

$$v_{te} = 0 \tag{52}$$

and for the sliding

$$v_{te} = v_{ti} + I_{nh} \left(c - \mu_D b \sigma_I \right) + \left(I_{ne} - I_{nh} \right) \left(c - b \ \gamma_F \right) \tag{53}$$

For all types of motion

$$I_{ne} = \zeta_I^s \left(v_{nh} - v_{ni} \right) + \zeta_F \left(v_{ne} - v_{nh} \right) \quad (54)$$

$$I_{te} = -\mu_D \sigma_I I_{nh} - \gamma_F \left(I_{ne} - I_{nh} \right) \quad (55)$$

$$\Delta\dot{\mathbf{q}} = I_{nh} \mathbf{L}_{\mathbf{I}}^{\mathbf{s}} + \left(I_{ne} - I_{nh} \right) \mathbf{L}_{\mathbf{F}} \quad (56)$$

For the permanent sliding: $\zeta_F^s = \zeta_I^s$, $\gamma_F = \mu_D \sigma_I$, $\mathbf{L_F} = \mathbf{L_I^s}$, for reverse sliding: $\zeta_F = \zeta_F^s$, $\gamma_F = \mu_D \sigma_F$, $\mathbf{L_F} = \mathbf{L_F^s}$, and for non-sliding: $\zeta_F = \zeta_F^{ns}$, $\gamma_F = b^{-1} c$, $\mathbf{L_F} = \mathbf{L_F^{ns}}$. For all types of motion $E_{1P} E_{2P} = e_P$ and $E_{1E} E_{2E} = e_E$. If the tangential velocity halts in compression, then $E_{1P} = e_P$ and $E_{1E} = e_E$, and if the tangential velocity halts in restitution, then $E_{2P} = e_P$ and $E_{2E} = e_E$. The computational strategy developed in this section is shown in Fig. 1 shows. The strategy can be used to identify the type of impact and to evaluate the impact variables at the end of impact period.

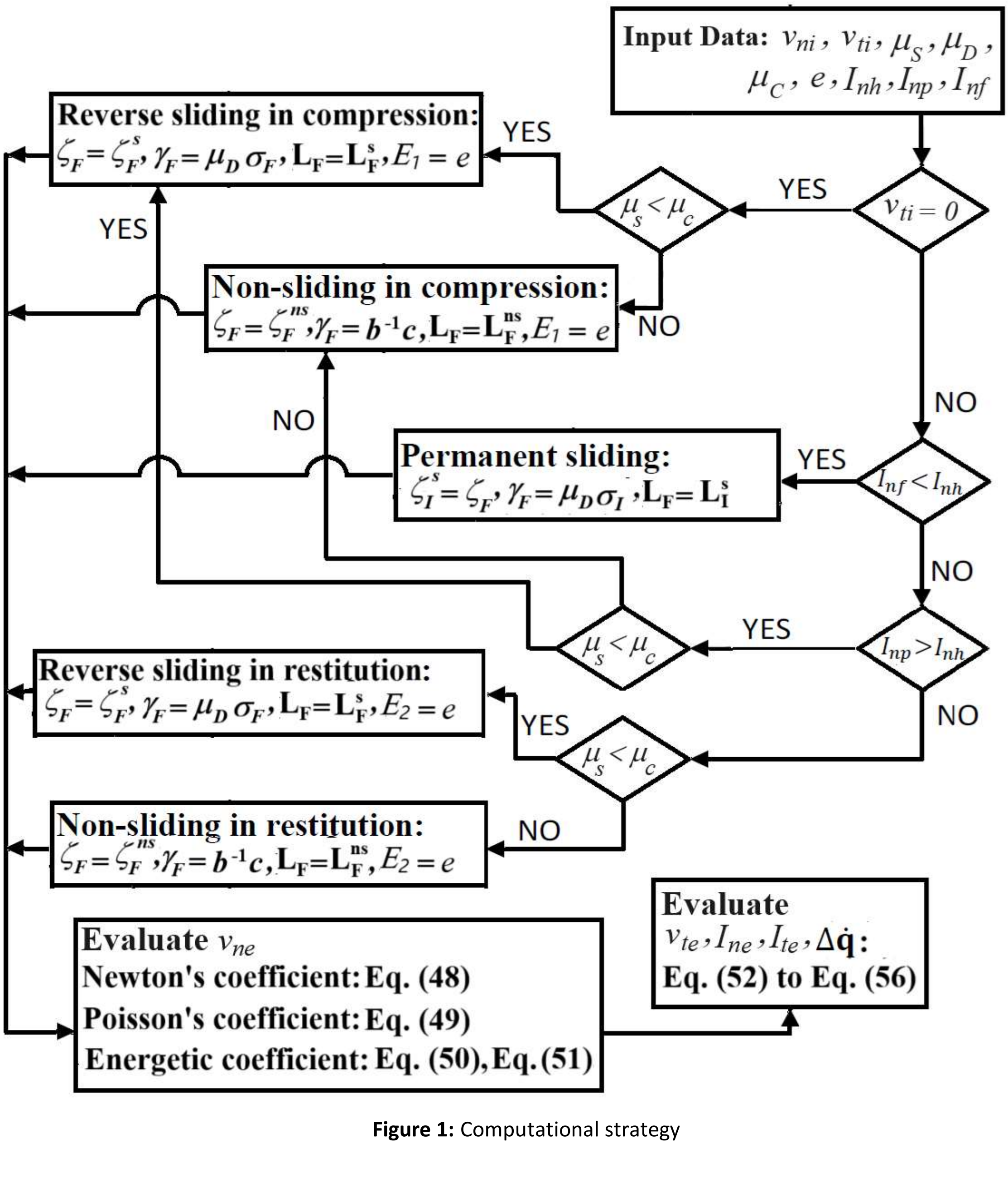


Figure 1: Computational strategy

6 NUMERICAL SIMULATIONS

The free end B of a rod shown in Fig. 2 strikes the ground while the angular velocity of rod OA is zero and the horizontal and vertical components of the velocity of the center of mass are 0.6 m/s and -1 m/s, respectively. The uniform rod has a mass of 1 kg and a length of 1 m, and the initial orientation of the rod is $\theta = 45°$. This problem has been examined before to show the paradox in mechanics when impact with friction is considered (Wang and Mason 1992). The coefficient of restitution is assumed to be $e = 0.82,$ and the coefficients of static and dynamic friction are $\mu_s = 0.73,\ \mu_D = 0.7$.

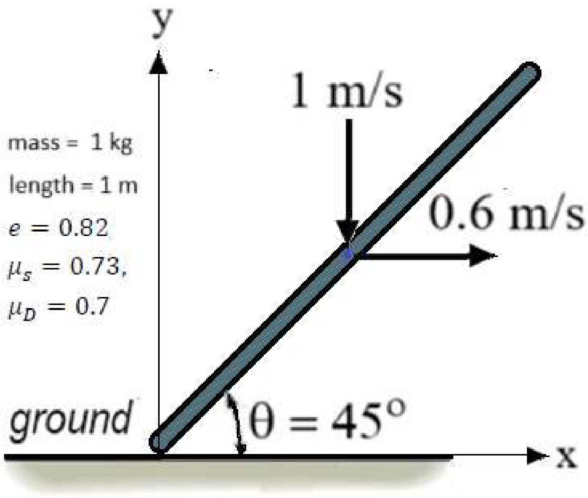


Figure 2: Impact between a rigid rod and a frictional ground floor

We solve the problem using the traditional method used in textbooks like [3,4]. In the traditional method, there are two impact modes only, continuous sticking or continuous sliding. We assume a sticking, and the solution fulfills the conditions that $\frac{I_{te}}{I_{ne}} \le \mu_s$. The change in the kinetic energy of the system $\Delta T = T_e - T_i$ is calculated, where T is given by

$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\boldsymbol{M}\dot{\mathbf{q}}$$

It has been found that $\Delta T = 0.010125 \text{ N.m}$, and the positive ΔT means that the energy increases, which is not acceptable.
To solve this energetically inconsistency, we went back to Routh's method, and tested the three definitions for the coefficient of restitution. The initial generalized joint velocities, mass matrix, the Jacobian matrix, and the Jacobian inertia matrix are given by:

$$\dot{\mathbf{q}}_{\mathrm{i}} = \begin{bmatrix}0.6 \text{ m/s} & -1 \text{ m/s} & 0 \text{ rad/s}\end{bmatrix}^T, \qquad \boldsymbol{M} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.0833\end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix}0 & 1 & -0.353553\\ 1 & 0 & 0.353533\end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix}2.5 & -1.5\\ -1.5 & 2.5\end{bmatrix}$$

the initial normal velocity, the initial tangential velocity, and the normal velocity v_{nh} if the tangential velocity vanishes during impact are:

$$v_{in} = -1 \text{ m/s}, \qquad v_{it} = 0.6 \text{ m/s}, \qquad v_{nh} = -0.344615 \text{ m/s}$$

the critical coefficient of friction is: $\mu_c = 0.6,$ the normal impulse I_{nh} if the tangential velocity vanishes during impact, the normal impulse I_{np} if the tangential velocity does not vanish up to the end of compression phase, and the normal impulse I_{nf} if the tangential velocity does not vanish up to the end of impact are:

$$I_{nh} = 0.184615 \text{ N.s}, \qquad I_{np} = 0.28169 \text{ N.s}, \text{ and } \qquad I_{nf} = 0.422535 \text{ N.s}$$

Hence

$I_{nh} < I_{np}$ and $\mu_s > \mu_c$

Therefore, the type of impact is non-sliding in compression and the parameters ζ_I^s, ζ_F^s, $\mathbf{L_I^s}$, and $\mathbf{L_F^s}$ are

$\zeta_I^s = 0.28169,$ $\zeta_F^s = 0.625$

$\mathbf{L_I^s} = [-0.7 \quad 1 \quad -7.21249]^T$, and $\mathbf{L}_F^s = [0.6 \quad 1 \quad -1.69706]^T$

The solutions for the impact variables depend upon the final velocity, which is not the same for the three impact laws. Results for the three impact laws are shown in Table 1.

Table 1 Impact variables for the three impact laws.

Impact variables	Units	Newton's	Poisson's	Energetic
v_{ne}	$\mathrm{m/s}$	0.82	0.5248	0.589
v_{te}	$\mathrm{m/s}$	0	0	0
I_{ne}	N.s	0.9125	0.728	0.7681
I_{te}	N.s	0.3075	0.1968	0.2209
$\dot{\mathbf{q}}_e$	$\begin{bmatrix} \mathrm{m/s} \\ \mathrm{m/s} \\ \mathrm{rad/s} \end{bmatrix}$	$\begin{bmatrix} 0.9075 \\ -0.0875 \\ -2.5668 \end{bmatrix}$	$\begin{bmatrix} 0.7968 \\ -0.272 \\ -2.2537 \end{bmatrix}$	$\begin{bmatrix} 0.8209 \\ -0.2319 \\ -2.3218 \end{bmatrix}$
W_{ne}	N.m	−0.0489	−0.0752	−0.0528
ΔT	N.m	0.0101	−0.1139	−0.0916

The results show that the inconsistency in the energy increase encountered in the traditional method has not been solved using Newton's coefficient of restitution with Routh's method. However, both Poisson's and energetic coefficients of restitution overcome this inconsistency. To go into details about the energy dissipation when using different coefficients of restitution, we consider $\mathrm{e} = 1$. Hence, the frictional impact component is the only energy dissipation source. The results using the three coefficients of restitution are given in Table 2.

Table 2 Impact variables when $\mathrm{e} = 1$

Impact variables	Units	Newton's	Poisson's	Energetic
$\dot{\mathbf{q}}_e$	$\begin{bmatrix} \mathrm{m/s} \\ \mathrm{m/s} \\ \mathrm{rad/s} \end{bmatrix}$	$\begin{bmatrix} 0.975 \\ -0.025 \\ -2.758 \end{bmatrix}$	$\begin{bmatrix} 0.84 \\ -0.2 \\ -2.376 \end{bmatrix}$	$\begin{bmatrix} 0.869 \\ -0.151 \\ -2.458 \end{bmatrix}$
W_{te}	N.m	−0.0388	−0.0388	−0.0388
ΔT	N.m	0.1125	−0.072	−0.0388

Hence, Newton's coefficient produces an energy increase due to impact, which violates physics laws. Poisson's coefficient does not produce an energy increase; however, $\Delta T > Wte$, which means that energy dissipation is generated due to non-frictional force, which also violates the physics laws. Contrarily, the energetic coefficient obeys the physics laws, and the dissipation of energy is only due to the frictional component of impact.

7 CONCLUSIONS

A mathematical model and a simulation technique for planar multibody systems subjected to impact with friction have been presented. Formulation of the equations of motion has been developed. To correctly apply Coulomb friction law, Routh incremental method has been used. The equations of motion for both sticking and sliding have been developed with the normal impulse at collision point as the independent variable The plasticity of the collision in the normal direction has been introduced using the coefficient of restitution. The three definitions for the coefficient of restitution have been considered, namely the Newton's, the Poisson's, and the energetic coefficients. The sliding that starts could continue until the end of collision or sticking point could be reached. A critical value of the coefficient of friction has been determined that relies only on system configuration. If the coefficient of friction is at least equals to that critical value, the non-sliding mode will continue until the end of collision once it has been reached. Otherwise, the sliding will resume in the reverse direction. Hence, five types of impact-contact modes have been identified, and the conditions leading to each of them have being specified. These conditions depend upon multibody system inertia and orientation, initial velocity, and coefficient of restitution. Algebraic solutions have been obtained for all the types of impact.

An illustration example has been considered to demonstrate the capabilities of the proposed formulation and analysis. The considered case has been solved with traditional method used in standard textbooks, and the results show an increase in the kinetic energy which contradicts the physics laws. Hence, the problem has been solved using the algebraic solutions obtained for Routh incremental method, and the solutions using the three definitions for the coefficient of restitution have been obtained, analyzed, and compared. A particular case with no dissipation of energy in the normal direction, i.e. $e = 1$, has been investigated. Only, the energetic coefficient has obeyed the physics laws. Newton's coefficient could lead to energy increase, and Poisson's coefficient could lead to energy dissipation in the normal direction even if $e = 1$. The results and analysis have brought to light the influence of different behaviors of the sliding of the contact point during impact period on impact variables and on the global motion of the system. The method presented can be used for operation support, performance analysis, as well as design and verification of planar multibody systems. The results obtained confirmed the applicability and efficiency of the formulation and the associated algebraic solutions.

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