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Explicit solution of the large amplitude transverse vibrations of a flexible string under constant tension

Abstract

This paper presents the analytical simulation of string with large amplitudes using the Variational Iteration Method (VIM) and Hamiltonian Approach (HA). In order to verify the precision of the presented methods, current results were compared with He's Variational Approach and Runge-Kutta 4^{th} order. It has been found that these methods are well suited for a range of parameters and the approximate frequencies and periodic solutions show a good agreement with other techniques. The results show that both methods can be easily extended to other nonlinear oscillations and it can be predicted that both methods can be found widely applicable in engineering.

Keywords

Variational Iteration Method; Hamiltonian Approach; Nonlinear oscillation; Non-linear vibration string

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1 INTRODUCTION

Study of nonlinear problems in strings with large amplitude is a very important research area in many fields of physics and engineering. Long cables and wires are extensively used in many mechanical systems and structures such as cranes, ships, offshore platforms and bridges. The study of nonlinear vibration of strings (cables and wires) with large amplitude is in practice very complicated. Primarily, the force data like air flow speed, direction and turbulence characteristic is, as a rule, difficult to definitely specify. Further, the mechanical characteristic of cables like bending stiffness and damping effects depend on a number of parameters (Omran et al., 2013).

On the other hand, with the rapid development of nonlinear science, there appears an ever-increasing interest of scientists in the analytical asymptotic techniques for nonlinear problems, and several analytical approximation methods have been developed. such as: variational iteration method (He, 2007a; Barari et al., 2011, Saadati et al., 2009), homotopy perturbation method (Torabi and Yaghoobi, 2011; Torabi et al., 2011; Saravi et al., 2013), homotopy analysis method (Liao, 2003; Khan et al., 2012)and some other methods (Bagheri et al. 2014; Nikkar et al., 2012,2014; Ghasempoor et al., 2012; Akbarzade and Khan, 2012; Alinia et al., 2011; Ganji, 2012; Torabi et al., 2012; Sheikholeslami et al., 2012a; 2012b; Bayat et al., 2011;2012; Rafieipour et al., 2012; Marinca and Herisanu, 2010; Salehi et al., 2012; Hamidi et al., 2012)

The main objective of the present study is to propose two analytical methods, namely the Variational Iteration Method (VIM) and Hamiltonian Approach (HA) which were introduced by He (He, 1999; 2010) to solve the nonlinear differential equation of the large deformation of string with large amplitudes. The Effectiveness and convenience of the methods is revealed in comparisons with the other solution techniques.

The results reveal that these methods are very effective and convenient in predicting the solution of such problems, and it is predicted that the VIM and Hamiltonian Approach (HA) can find a wide application in new engineering problems.

2 THE GOVERNING EQUATION OF STRINGS WITH LARGE AMPLITUDE

Considering the differential equation of large amplitude transverse vibrations of a flexible string under constant tension as shown in Fig. 1.

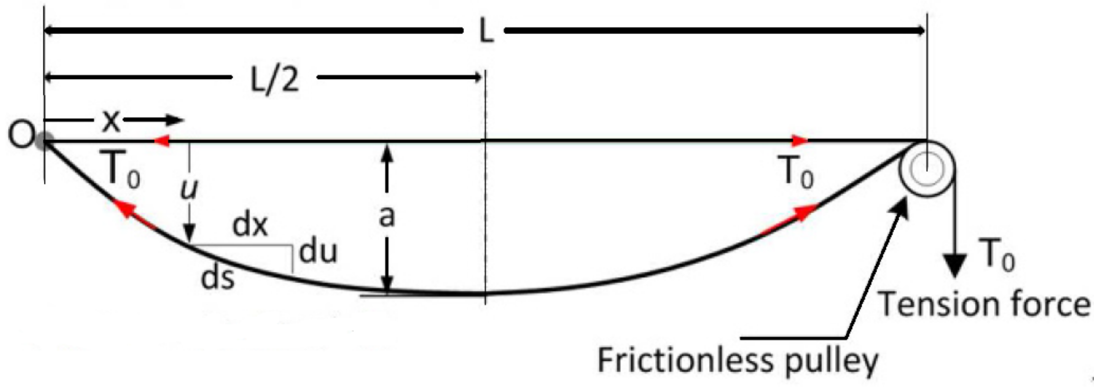

Figure 1 Schematic of a constant tension vibrating string with large amplitude (Omran et al., 2013).

The governing differential equation of large amplitude vibrations of a constant tension string of this form is written as:

$$c^2 \frac{\partial u^2}{\partial x^2} = (1+\frac{\partial u^2}{\partial x^2})^2 \frac{\partial u^2}{\partial t^2} \tag{1}$$

where u(x,t) is the transverse displacement amplitude relative to the spatial x and temporal t coordinates, $c = \tau_0 / \rho_0$ is the velocity of transverse wave with τ_0 and ρ_0 being the tension and the mass per unit length, respectively. Let's also consider that the transverse displacement is expressed as:

$$u(x,t) = V(t)\sin(\frac{\pi}{L}x) \tag{2}$$

Substituting Eq. (2) into Eq. (1) and averaging over the string length L (i.e. a Galerkin procedure is performed to take the multiplication of Eq. (2) by Eq. (1)) and integrating such equation over x from 0 to L results in an ordinary second order differential equation as:

$$\frac{d^2\varphi}{dt^2}+\frac{\alpha\varphi}{1+\frac{\varphi^2}{4}+\frac{\varphi^4}{8}}=0 \tag{3}$$

With initial conditions:

$$\varphi(0)=A, \quad \dot{\varphi}(0)=0 \tag{4}$$

where φ and t are generalized dimensionless displacement and time variables. Also we have:

$$\varphi(t)=\frac{\pi}{L}V(t), \qquad \alpha=(\frac{\pi c}{L})^2 \tag{5}$$

3 DESCRIPTION OF THE VARIATIONAL ITERATION METHOD

To clarify the idea of the proposed method for solution of the large deformation of string with large amplitudes, the basic concept of Variational Iteration Method is firstly treated. We consider the following general differential equation,

$$Lu+Nu=g(t) \tag{6}$$

Where, L is a linear operator, and N a nonlinear operator, g(t) an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correct functional as follows:

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(Lu_n(\tau)+N\tilde{u}_n(\tau)-g(\tau))d\tau \tag{7}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n=0$.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In this method, the problems are initially approximated with possible unknowns and it can be applied in non-linear problems without linearization or small parameters. The approximate solutions obtained by the proposed method rapidly converge to the exact solution.

4 IMPLEMENTATION OF VARIATIONAL ITERATION METHOD

Now, we can solve the Eq. (3). Assume that the angular frequency of the system (3) is ω, we have the following linearized equation:

$$\ddot{\varphi}+\omega^2\varphi=0 \tag{8}$$

So we can rewrite Eq. (3) in the form:

$$\ddot{\varphi}+\omega^2\varphi+f(\varphi)=0 \tag{9}$$

Where

$$f(\varphi)=-\omega^2\varphi+\frac{\alpha\varphi}{1+\frac{\varphi^2}{4}+\frac{\varphi^4}{8}}=0 \tag{10}$$

Applying the proposed method, the following iterative formula is formed as:

$$\varphi_{n+1}(t)=\varphi_n(t)+\int_0^t \lambda(\ddot{\varphi}_n(\tau)+\omega^2\varphi_n(\tau)+f(\varphi(\tau))d\tau \tag{11}$$

Its stationary conditions can be obtained as follows:

$$\begin{aligned} &1-\lambda'\big|_{\tau=t}=0,\\ &\lambda\big|_{\tau=t}=0,\\ &\lambda''+\omega^2\lambda\big|_{\tau=t}=0. \end{aligned} \tag{12}$$

The Lagrange multiplier, therefore, can be identified as;

$$\lambda=\frac{1}{\omega}\sin\omega(\tau-t) \tag{13}$$

By substituting this identified multiplier into Eq. (11), we come to:

$$\varphi_{n+1}(t)=\varphi_n(\tau)+\frac{1}{\omega}\int_0^t \sin\omega(\tau-t)\left(\frac{d^2\varphi_n(\tau)}{dt^2}+\frac{\alpha\varphi_n(\tau)}{1+\frac{\varphi_n(\tau)^2}{4}+\frac{\varphi_n(\tau)^4}{8}}\right)d\tau \tag{14}$$

Assuming its initial approximate solution has the form:

$$\varphi_0(t) = A\cos(\omega t) \tag{15}$$

And substituting Eq. (15) into Eq. (3) leads to the following residual:

$$R_0(t) = -A\omega^2\cos(\omega t) + \frac{\alpha A\cos(\omega t)}{1 + \frac{A^2\cos(\omega t)^2}{4} + \frac{A^4\cos(\omega t)^4}{8}} \tag{16}$$

By the formulation (14), we can obtain

$$\varphi_1(t) = \varphi_0(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)(R_0(\tau))d\tau \tag{17}$$

In the same manner, the rest of the components of the iteration formula can be obtained. In order to ensure that no secular terms appear in u_1, resonance must be avoided. To do so, the coefficient of $\cos(\omega t)$ in Eq. (16) requires being zero, i.e.,

$$\omega_{VIM} = \frac{2}{A}\sqrt{2\alpha\arctan(A + \frac{1}{2}A^2) - 2\alpha\arctan(A + \frac{1}{4}A^4)} \tag{18}$$

5 BASIC CONCEPT OF HAMILTONIAN APPROACH

Previously, He (He, 2002) had introduced the energy balance method based on collocation and Hamiltonian. Recently, in 2010 it was developed into the Hamiltonian approach (He, 2010). This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator:

$$\ddot{u} + f(u,\dot{u},\ddot{u}) = 0 \tag{19}$$

With initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{20}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (19), which reads;

$$J(u) = \int_0^{\frac{T}{4}} (-\frac{1}{2}\dot{u}^2 + F(u))dt \tag{21}$$

Where T is period of the nonlinear oscillator and $\frac{\partial F}{\partial u} = f$

In the Eq. (21), $\dot{u}^2/2$ is kinetic energy and *F (u)* potential energy, so the Eq. (21) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads:

$$H(u) = -\frac{1}{2}\dot{u}^2 + F(u) = CONSTANT \tag{22}$$

From Eq. (22), we have;

$$\frac{\partial H}{\partial A} = 0. \tag{23}$$

Introducing a new function, $\overline{H}(u)$, defined as:

$$\overline{H}(\varphi) = \int_0^{\frac{T}{4}} (-\frac{1}{2}\dot{u}^2 + F(u))dt = \frac{1}{4}TH \tag{24}$$

Eq. (23) is, then, equivalent to the following one;

$$\frac{\partial}{\partial A}(\frac{\partial \overline{H}}{\partial T}) = 0. \tag{25}$$

Or

$$\frac{\partial}{\partial A}(\frac{\partial \overline{H}}{\partial (\frac{1}{\omega})}) = 0. \tag{26}$$

From Eq. (26) we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator.

6 THE APPLICATION OF HAMILTONIAN APPROACH

To illustrate the basic procedure of the present method, the Hamiltonian of Eq. (5) can be written in the form:

$$H(\varphi) = -\frac{1}{2}\dot{\varphi}^2 + 2\alpha \arctan(1 + \frac{1}{2}\varphi^2) \tag{27}$$

Introducing a new function, $\overline{H}(\varphi)$. Integrating Eq. (27) with respect to t from 0 to T/4, we obtain:

$$\overline{H}(\varphi) = \int_0^{\frac{T}{4}} (-\frac{1}{2}\dot{\varphi}^2 + 2\alpha \arctan(1 + \frac{1}{2}\varphi^2))dt \tag{28}$$

Assuming that the solution can be expressed as $\varphi = A\cos(\omega t)$ and substituting it into Eq. (28) yields:

$$\overline{H}(\varphi) = \int_0^{\frac{T}{4}} (-\frac{1}{2}A^2\omega^2 \sin(\omega t)^2 + 2\alpha \arctan(1 + \frac{1}{2}A^2\cos(\omega t)^2))dt \tag{29}$$

Setting

$$\frac{\partial}{\partial A}(\frac{\partial \overline{H}}{\partial(\frac{1}{\omega})}) = 0. \tag{30}$$

So the frequency can be approximated as:

$$\omega_{HA} = \frac{2}{A}\sqrt{2\alpha \arctan(A + \frac{1}{2}A^2) - 2\alpha \arctan(A + \frac{1}{4}A^4)} \tag{31}$$

Note: One can expand $\arctan(A + \frac{1}{2}A^2)$ into Taylor series for small A, and compare the result with that obtained by perturbation method.

6 RESULTS AND DISSCUSIONS

A mathematical model describing the process of large amplitude transverse vibrations of a flexible string under constant tension is proposed. Two efficient analytical methods are applied to solve the dynamic model of the large amplitude non-linear oscillation equation. In order to verify the precision of the methods, current results were compared with Runge-Kutta 4^{th} order method in Table 1 and 2. It is observable that our results are in excellent agreement with the results provided by Runge-Kutta 4^{th} order. The behavior of $\varphi(A,t)$ obtained by VIM and HA at $\alpha=2$ is shown in Figs. 2 and 3. To further illustrate the accuracy of presented methods, the plot of the results are presented and compared in Figs. 4-6. All plots are done for a constant value of $\alpha = 1$ and varying amplitude values A = 0.2, 1 and 5.

Table 1 Comparison of response of the system $\varphi(A,t)$ obtained by the presented methods with numerical results when $t = 0.5(s)$, $\alpha=1$

A	VIM	HA	RKF	ERROR(VIM)	ERROR(HA)
1	0.8899905311	0.914508356	0.90951228723	0.01952175613	-0.00499606877
2	1.8878586871	2.040047936	1.93700826192	0.04914957482	-0.10303967408
3	2.7467435860	3.029905311	2.97185868708	0.22511510108	-0.05804662392
4	3.8690203289	4.017121304	3.98646608228	0.11744575338	-0.03065522172
5	4.9267435859	5.010179315	4.99267435859	0.06593077269	-0.01750495641
10	9.9690826192	10.00162672	9.99902032887	0.02993770967	-0.00260639113

Table 2 Comparison of response of the system $\varphi(A,t)$ obtained by the presented methods with numerical results when $t = 1(s)$, α=2

A	VIM	HA	RKF	$ERROR(VIM)$	$ERROR(HA)$
1	0.265794960	0.382722526	0.31822973907	0.05243477906	-0.06449278693
2	2.116832133	2.327908002	2.23752825750	0.12069612450	-0.09037974449
3	2.612042483	3.242032233	2.96872183773	0.35667935473	-0.27331039527
4	3.427675192	4.137655192	3.89055794960	0.46288275760	-0.24709724240
5	4.701229739	5.081628040	4.94111051404	0.23988077504	-0.14051752596
10	9.510341463	10.01301620	9.99216000549	0.48181854249	-0. 02085619451

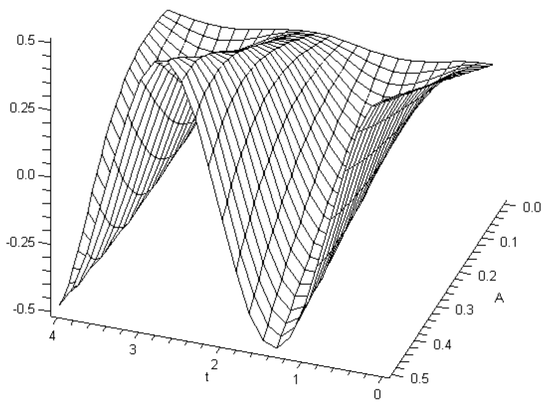

Figure 2 VIM deflection at α=2

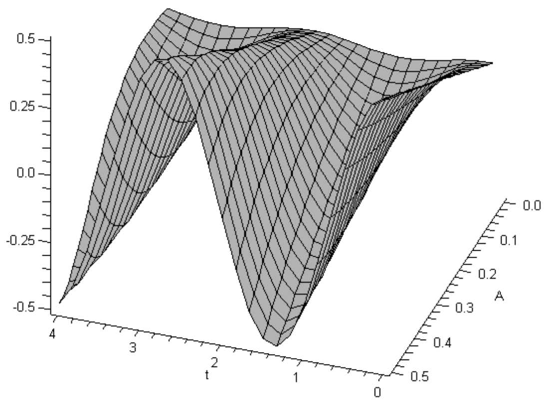

Figure 3 HA deflection at α= 2

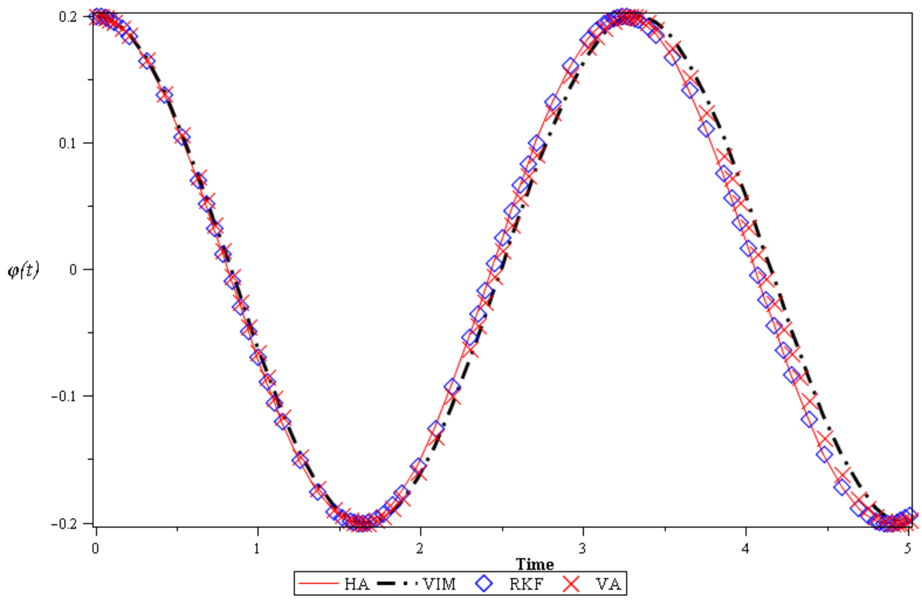

Figure 4 Comparison between Variational Iteration (VIM), Hamiltonian Approach (HA) and Variational Approach (VA) periodic solutions versus time with Runge-Kutta 4th order for α=1 and A = 0.2.

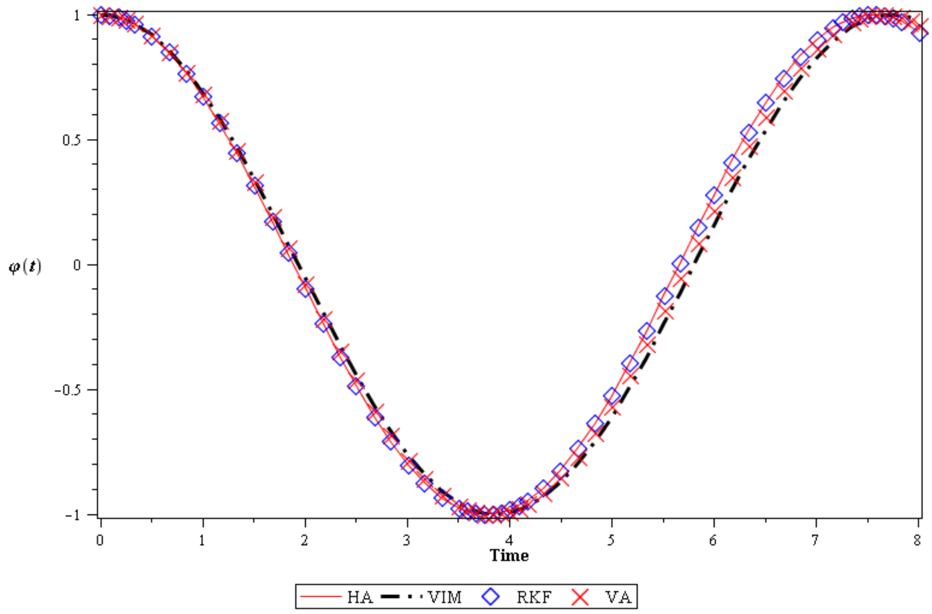

Figure 5 Comparison between Variational Iteration (VIM), Hamiltonian Approach (HA) and Variational Approach (VA) periodic solutions versus time with Runge-Kutta 4^{th} order for α=1 and A = 1.

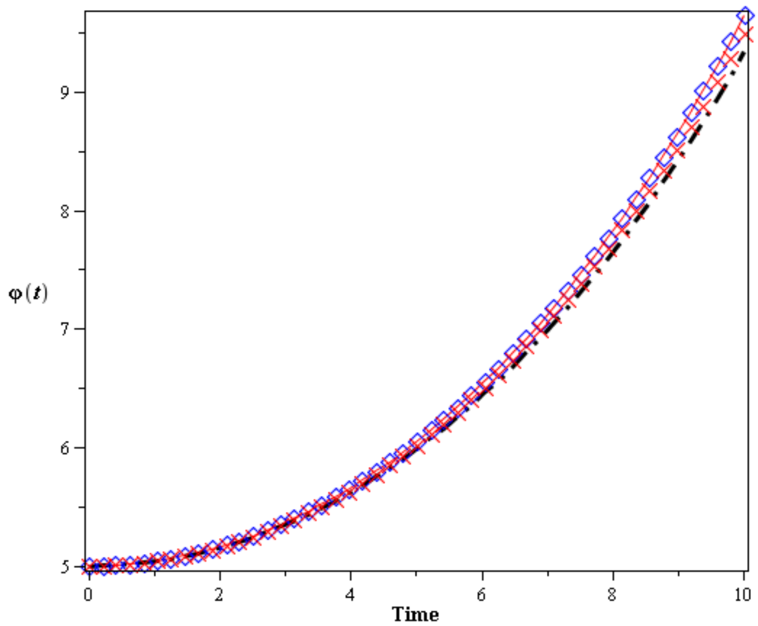

Figure 6 Comparison between Variational Iteration (VIM), Hamiltonian Approach (HA) and Variational Approach (VA) periodic solutions versus time with Runge-Kutta 4^{th} order for α=1 and A = 5.

4 CONCLUSIONS

In this work, we proposed a mathematical model describing the process of large amplitude transverse vibrations of a flexible string under constant tension. Hamiltonian approach and efficient approximate method (VIM) is employed to derive the nonlinear vibration of constant-tension string. The frequency of both methods is exactly the same. Comparing with numerical Results, it is shown that the approximate analytical solutions are in very good agreement with the corresponding solutions and presented methods are powerful mathematical tools, very effective, convenient and adequately accurate for study of nonlinear oscillators in physics and engineering problems.

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