lajss.org
Latin American Journal of Solids and Structures
www.lajss.org

An indentation contact problem of a stamp and a bonded thin elastic layer

Xu Zhang [a], Shihong Han [b], Kai Zhu [c]*

[a] College of Engineering, Zunyi Normal University, Zunyi 563002, China. Email: zhangxu_xaut@163.com
[b] Guangdong Provincial Freeway Limited Company, Guangzhou 510623, China. Email: hanc526308299@outlook.com
[c] School of Mechanical and Energy Engineering, Guangdong Ocean University, Yangjiang, 529500, China. Email: kaychu@gdou.edu.cn

* Corresponding author

Abstract
An indentation contact problem between a cylinder stamp and a bonded elastic layer is investigated in this paper. Based on the Navier equation and boundary conditions, the contact problem is transformed into a governing formula, which is in the form of singular integral equation of the first kind, through Fourier transform techniques. By using dimensionless parameterization tricks, a numerical scheme for inverse contact problem is built, and the contact stresses could be obtained directly through the coupled solution of governing equation and equilibrium relationship. Thereinto, the contact pressure is approximated by Chebyshev polynomials, and the governing equation is solved numerically. The reliability of the numerical results is demonstrated through comparison with existing analytical solutions, and the influences of Poisson's ratio and layer thickness on contact stresses are discussed. Results show that, contact pressure converges to a Hertzian type when the layer thickness increases, and both increasing Poisson's ratio and reducing the layer thickness can sharpen the contact stress.

Keywords
Indentation contact, Chebyshev polynomials, Singular integral equation (SIE), Elastic layer, Fourier transform techniques.

1 INTRODUCTION

Due to the appropriate size and special performance, thin layer on a substrate is widely applied in engineering manufacturing, such as protective coating in bearings, and the pavement on roadbed etc. As an attractive topic in the classic contact mechanics, the indentation problem of the thin elastic layer has been investigated by Keer [1] and other researchers. Lebdev and Ufliand [2] present harmonic functions for Papkovich-Neuber equation, and facilitate the solution about indentation problem of a circular cross-section stamp and the elastic layer. Aleksandrove et al. [3, 4] put forward three-dimensional contact problems of elastic layer, and solve the Lame equation through Fourier transform techniques to get the contact stresses. Meijer [5] fills the gape of asymptotic solutions for incompressible layer, and the work has been improved by Alblas and Kuipers [6] then. The solution of layered contact problems mentioned above, was predominantly based on Fredholm equation frameworks, although singular integral equation was employed to govern contact problem. To the best of authors' knowledge, this phenomenon was prevalent prior to the mid-20th century.

Literatures show that, in recent years, the framework has been sporadically developed and applied in contact problem of layer-substrate system. Aleksandrove [7, 8] solve a doubly periodic contact problems, and then focuses on the improvement of the asymptotic solution about calculation of the integral kernel. Scholars [9] try to build a law between the pressure and the displacement of coating in the axisymmetric indentation problem. Others [10] applies the theory to investigate the effect of thickness on the elastic behavior. However, its requirement to convert singular integral equations into Fredholm equations undoubtedly complicates the solution theory. This limitation persisted until Erdogan [11] developed a dedicated solution framework specifically for singular integral equations at 1970s. The Gauss-Jacobi quadrature [12] and Jacobi polynomials [13, 14] have been widely applied to solve various contact problem, such as static contact problem of layers supported by elastic half-plane [15] or Winkler foundation [16], and receding contact problem of elastic layer [17] etc.

On the other hand, some scholars approached the problem through simplified contact theories, with the most notable being Johnson's hypotheses on layer that "plane sections remain plane after compression" [18]. On the basis of the assumption, Jaffer[19, 20] decomposes the integral kernel of the governing equation to two parts, and presents a new asymptotic solution for thin layer. The numerical solution is extended to three-dimensional problems by Barber[21] then. Conway et al.[22] investigate a compact problem of a thin layer. Contact problem of transversely isotropic elastic layer [23] is studied, and asymptotic solution for axisymmetric contact pressure is obtained. For practical engineering problem involves thin layer, Matthewson[24] pays attention to the optimization of protective coating design in the contact problems. However, Johnson's theory inevitably fails when applied to thick-layered bodies.

The numerical solution framework from Erdogan remains mainstream, despite efforts to explore innovative algorithms, such as application of Green's function by Greenwood and Barber[25] or others [26, 27] to indentation problem of elastic layer. It should be noted that, meanwhile, Erdogan's solution framework is rooted in Riemann boundary value problem theory [28]. This poses significant challenges in engineering applications, as the theoretical foundation is overly complex. Even for fundamental problem like frictionless contact of layer-substrate system, index calculation is tedious and error-prone. Fortunately, the application of Chebyshev polynomial in Gladwell's work [29] seems to point out a new path, and a same work is from Turhan [30], which circumvent Rieman boundary value problem theory.

In literatures, layer-substrate contact problems are mostly addressed as direct problem, for which contact stress and region are determined under given loads, e.g. Argatov et al.[31, 32] devotes articular layer in biomechanics and develops asymptotic solution, or others focuses on contact problem of anisotropic linear elastic materials of the layer [33]. Inverse problems, which require solving for unknown contact tractions from prescribed deformation, are rarely presented otherwise.

By recognizing that the method from Turhan [30] is confined to a mathematical framework, this study aims to establish a systematic solution for inverse problem of frictionless contact in layer-substrate system through methodology integration. The paper is organized as follows. By applying Fourier transform, the problem is formulated by a singular integral equation of the first kind in the 2nd section. In 3rd section, dimensionless parameterization treatment for inverse problem is presented, and flow chart of the calculation process is built. Based on the orthogonality properties and the Gauss-Chebyshev formula[34], numerical solution for singular integral equation is provided in 4th section. The numerical results of contact pressure are validated in 5th section, and the influences of thickness or Poisson's ratio on contact stresses are discussed.

2 PROBLEM FORMULA

A layer-substrate structure is shown in Fig.1, the layer is elastic and bonded to a rigid substrate. Indentation between a cylindrical stamp and the layer is gradually progressed under a normal load P. The layer thickness is h, and the radius of stamp is defined as R. The problem could be discussed in plane-strain frame since the stamp is long enough in the longitudinal direction. The contact region between the stamp and layer is indicated by a red arc with radius a, and indentation depth is defined as d. Displacement components in the plane coordinate (x, y) are characterized by (u, v).

From Navier's equation, the equilibrium in plane coordinate could be wrote as

$$\left(\kappa+1\right)\frac{\partial^2 u}{\partial x^2}+\left(\kappa-1\right)\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 v}{\partial x\partial y}=0 \tag{1}$$

$$\left(\kappa+1\right)\frac{\partial^2 v}{\partial y^2}+\left(\kappa-1\right)\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}=0 \tag{2}$$

where κ is Kolosov's constant with $\kappa=3-4\nu$, and ν is Poisson's ratio.

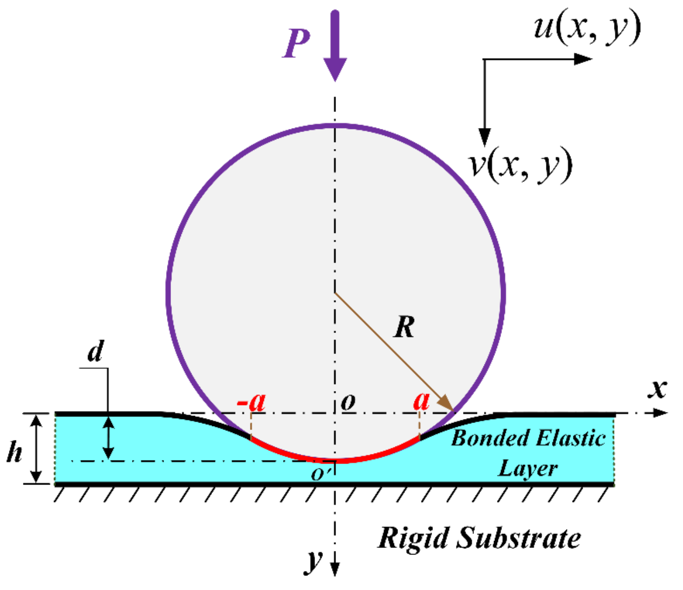

Fig.1 *The indentation contact model of a stamp and a bonded elastic layer*

By defining the Fourier transforms of displacements u and v for layer as following

$$U\left(\alpha,y\right)=\int_{-\infty}^{\infty}u\left(x,y\right)e^{-i\alpha x}dx \tag{3}$$

$$V\left(\alpha,y\right)=\int_{-\infty}^{\infty}v\left(x,y\right)e^{-i\alpha x}dx \tag{4}$$

Applying the property about the Fourier transforms of the derivatives of a function below

$$F\left[f^{(n)}\left(x\right)\right]=\left(i\alpha\right)^n F\left(\alpha\right) \tag{5}$$

where F means the Fourier transform.

Eq. (1) and (2) thus could be transformed to

$$-\left(\kappa+1\right)\alpha^2 U+\left(\kappa-1\right)\frac{\partial^2 U}{\partial y^2}+2i\alpha\frac{\partial V}{\partial y}=0 \tag{6}$$

$$\left(\kappa+1\right)\frac{\partial^2 V}{\partial y^2}-\left(\kappa-1\right)\alpha^2 V+2i\alpha\frac{\partial U}{\partial y}=0 \tag{7}$$

where U, V is the abbreviation of $U\left(\alpha, y\right)$, $V\left(\alpha, y\right)$ respectively.

Solving the second-order differential equations above, one could obtain

$$U=\left(A+By\right)e^{-\alpha y}+\left(C+Dy\right)e^{\alpha y} \tag{8}$$

$$V=i\left(A+\kappa\alpha^{-1}B+By\right)e^{-\alpha y}-i\left(C-\kappa\alpha^{-1}D+Dy\right)e^{\alpha y} \tag{9}$$

where A, B, C, and D are unknown functions of α, which would be determined from boundary conditions.

According to the elastic equations in plane frame,

$$\sigma_x=\lambda\frac{1-\nu}{\nu}\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y} \tag{10}$$

$$\sigma_y=\lambda\frac{1-\nu}{\nu}\frac{\partial v}{\partial y}+\lambda\frac{\partial u}{\partial x} \tag{11}$$

$$\tau_{xy}=G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \tag{12}$$

where λ is Lamé constant, and G is the shear modulus. The stresses for the layer could be obtained as

$$\frac{T_x}{2G}=\left[A+\left(y+\frac{\kappa-3}{2\alpha}\right)B\right]i\alpha e^{-\alpha y}+\left[C+\left(y-\frac{\kappa-3}{2\alpha}\right)D\right]i\alpha e^{\alpha y} \tag{13}$$

$$\frac{T_y}{2G}=-\left[A+\left(y+\frac{\kappa+1}{2\alpha}\right)B\right]i\alpha e^{-\alpha y}-\left[C+\left(y-\frac{\kappa+1}{2\alpha}\right)D\right]i\alpha e^{\alpha y} \tag{14}$$

$$\frac{T_{xy}}{2G}=\left[-A+\left(\frac{1-\kappa}{2\alpha}-y\right)B\right]\alpha e^{-\alpha y}+\left[C+\left(\frac{1-\kappa}{2\alpha}+y\right)D\right]\alpha e^{\alpha y} \tag{15}$$

where T_x, T_y and T_{xy} are abbreviation of $T_x\left(\alpha, y\right)$, $T_y\left(\alpha, y\right)$ and $T_{xy}\left(\alpha, y\right)$, which are the Fourier transforms of $\sigma_x\left(x, y\right)$, $\sigma_y\left(x, y\right)$, $\tau_{xy}\left(x, y\right)$.

The boundary conditions for the problem could be defined as follows

$$u\left(x, h\right)=v\left(x, h\right)=0 \tag{16}$$

$$\sigma_y\left(x, 0\right)=\begin{cases}-p\left(x\right) & x\le\left|a\right| \\ 0 & x>\left|a\right|\end{cases} \tag{17}$$

$$\tau_{xy}\left(x,0\right) = 0 \tag{18}$$

$$v\left(x,0\right) = \frac{a^2 - x^2}{2R} \tag{19}$$

By applying (16), the unknown constants could be obtained from Eqs. (6) and (7) as

$$A = -\left(\frac{\kappa}{2\alpha} + h\right)B - \frac{\kappa}{2\alpha}De^{2\alpha h} \tag{20}$$

$$C = \frac{\kappa}{2\alpha}Be^{-2\alpha h} + \left(\frac{\kappa}{2\alpha} - h\right)D \tag{21}$$

Substituting equations above into the Eq. (14) and (15), we could obtain stress components at surface of the layer as

$$\frac{P_y}{2G} = \left(h\alpha - \frac{1}{2} - \frac{\kappa}{2}e^{-2\alpha h}\right)Bi + \left(\frac{\kappa}{2}e^{2\alpha h} + h\alpha + \frac{1}{2}\right)Di \tag{22}$$

$$\frac{Q}{2G} = \left(h\alpha + \frac{1}{2} + \frac{\kappa}{2}e^{-2\alpha h}\right)B + \left(\frac{\kappa}{2}e^{2\alpha h} - h\alpha + \frac{1}{2}\right)D \tag{23}$$

where $\lim_{y\to 0} T_y = P_y$, and $\lim_{y\to 0} T_{xy} = Q$.

Thus, the gradient of the displacements at the surface of the layer could be expressed with respect to x giving

$$\lim_{y\to 0}\frac{\partial v\left(x,y\right)}{\partial x} = \lim_{y\to 0} F^*\left[i\alpha V, \alpha \to x\right] \tag{24}$$

$$\lim_{y\to 0}\frac{\partial u\left(x,y\right)}{\partial x} = \lim_{y\to 0} F^*\left[i\alpha U, \alpha \to x\right] \tag{25}$$

where F^* means the inverse Fourier transform.

Applying the equations (13) to (15), Eq. (24) and Eq. (25) could be rewritten as

$$\lim_{y\to 0} i\alpha V = \frac{Q}{2G} + \frac{-1-\kappa}{2}B + \frac{-1-\kappa}{2}D \tag{26}$$

$$\lim_{y\to 0} i\alpha U = -\frac{P_y}{2G} - \frac{1+\kappa}{2}iB + \frac{1+\kappa}{2}iD \tag{27}$$

Substituting equations (22) and (23) into (26), then the Eq. (24) could be rewritten as

$$\frac{\partial v\left(x,0\right)}{\partial x} = F^*\left\{\frac{Q}{2G} + i\frac{P_y}{2G}\frac{1+\kappa}{2}\left[N_2\left(\alpha\right) - 1\right]sign\left(\alpha\right) - \frac{Q}{2G}\frac{1+\kappa}{2}\left[N_1\left(\alpha\right) + 1\right], \alpha \to x\right\} \tag{28}$$

where

$$\left[N_2(\alpha) - 1\right] sign(\alpha) = \frac{4h\alpha - \kappa\left(e^{2\alpha h} - e^{-2\alpha h}\right)}{1 + \kappa^2 + 4h^2\alpha^2 + \kappa\left(e^{2\alpha h} + e^{-2\alpha h}\right)} \tag{29}$$

$$N_1(\alpha) = \frac{2 + \kappa\left(e^{2\alpha h} + e^{-2\alpha h}\right)}{1 + \kappa^2 + 4h^2\alpha^2 + \kappa\left(e^{2\alpha h} + e^{-2\alpha h}\right)} - 1 \tag{30}$$

Expanding Eq. (28), and applying equations (17) and (18), we could obtain the governing formula of the problem as

$$H(x) = \frac{1}{\pi}\int_{-a}^{a} p(t) k_1(x-t)\, dt + \frac{1}{\pi}\int_{-a}^{a} \frac{p(t)}{t-x}\, dt \tag{31}$$

where

$$H(x) = \frac{4G}{1+\kappa}\frac{\partial v(x,0)}{\partial x} \tag{32}$$

$$k_1(x,t) = \int_0^{\infty}\left[\frac{4h\alpha - \kappa\left(e^{2\alpha h} - e^{-2\alpha h}\right)}{1 + \kappa^2 + 4h^2\alpha^2 + \kappa\left(e^{2\alpha h} + e^{-2\alpha h}\right)} + 1\right]\left[\sin\alpha(x-t)\right] d\alpha \tag{33}$$

The equilibrium condition for the contact pressure could be expressed as

$$\int_{-a}^{a} p(t) dt = P \tag{34}$$

and the contact stress σ_x at the layer surface can be calculated through the following equation, which is obtained directly with Fourier transform techniques,

$$\sigma_x(x,0) = -p(x) - \frac{1}{\pi}\int_{-a}^{a} p(t) K_2(x-t)\, dt \tag{35}$$

where

$$K_2(x-t) = \int_0^{\infty}\frac{2 - 8h^2\alpha^2 - 2\kappa^2}{1 + 4h^2\alpha^2 + \kappa^2 + 2\kappa\cosh 2\alpha h}\cos\alpha(x-t)\, d\alpha \tag{36}$$

3 SOLUTION FOR INVERSE PROBLEM

Defining the following non-dimensional quantities

$$r = \frac{x}{a},\ s = \frac{t}{a} \tag{37}$$

$$\phi(s) = \frac{p(s)}{P/a},\ \frac{h}{a} = \zeta,\ \frac{a}{R} = \beta,\ \frac{\tau}{a} = \alpha,\ \upsilon = \frac{G}{P/a} \tag{38}$$

Eqs. (31), (34) and (35) could be rewritten as

$$H(r) = -\frac{1}{\pi}\int_{-1}^{1}\phi(s)\,k_1(r,s)ds - \frac{1}{\pi}\int_{-1}^{1}\frac{\phi(s)}{s-r}ds \tag{39}$$

$$\int_{-1}^{1}\phi(s)\,ds = 1 \tag{40}$$

$$\bar{\sigma}_x(r,0) = -\phi(r) + \frac{1}{\pi}\int_{-1}^{1}\phi(s)K_2(r-s)ds \tag{41}$$

where

$$H(r) = \frac{4}{1+\kappa}\frac{G}{P/a}\frac{a}{R}r \tag{42}$$

$$k_1(r,s) = \int_0^\infty\left[\frac{4\tau\zeta - 2\kappa\sinh(2\tau\zeta)}{1+\kappa^2+4(\tau\zeta)^2+2\kappa\cosh(2\tau\zeta)}+1\right]\left[\sin\tau(r-s)\right]d\tau \tag{43}$$

$$K_2(r-s) = \int_0^\infty\frac{-2+8\zeta^2\tau^2+2\kappa^2}{1+4\zeta^2\tau^2+\kappa^2+2\kappa\cosh 2\zeta\tau}\cos\tau(r-s)d\tau \tag{44}$$

$$\bar{\sigma}_x(r,0) = \frac{\sigma_x(r,0)}{P/a} \tag{45}$$

The non-dimensional contact pressure can be approximated with the second kind Chebyshev polynomials $U_i(s)$ as

$$\phi(s) \approx \phi_n(s) = \sqrt{1-s^2}\sum_{i=0}^{n-1}\alpha_i U_i(s) \tag{46}$$

where α_i are unknown coefficients to be determined. For inverse problems, the contact pressure is determined from prescribed deformation, and it is reasonable to remove the non-dimensional force $G/(P/a)$ on the right hand side of Eq. (46) therefore. The contact pressure Eq. (46) is furtherly rewritten as

$$\tilde{\phi}(s) \approx \tilde{\phi}_n(s) = \sqrt{1-s^2}\sum_{i=0}^{n-1}\tilde{\alpha}_i U_i(s) \tag{47}$$

where

$$\tilde{\alpha}_i = \frac{P/a}{G}\alpha_i \tag{48}$$

The governing Eq. (39) is modified to

$$\tilde{H}(r) = -\frac{1}{\pi}\int_{-1}^{1}\tilde{\phi}(s)\,k(r,s)ds - \frac{1}{\pi}\int_{-1}^{1}\frac{\tilde{\phi}(s)}{s-r}ds \tag{49}$$

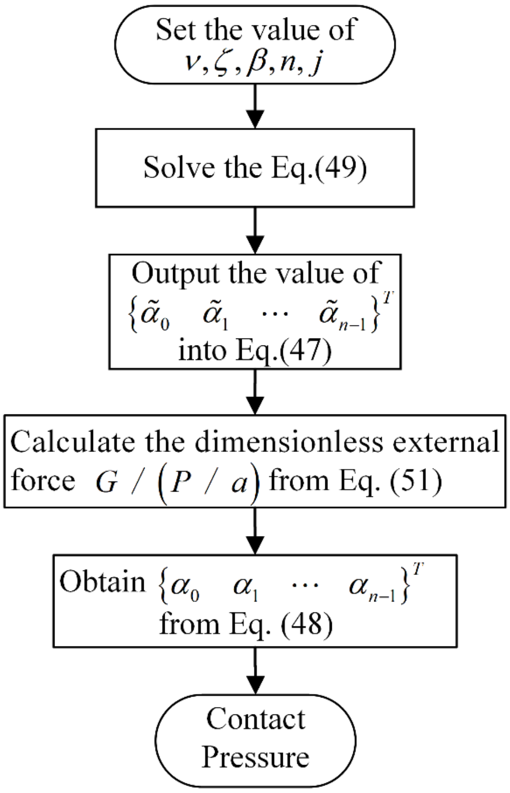

***Fig.2** Flow chart of the calculation process*

where

$$\tilde{H}(r) = \frac{4}{1+\kappa}\frac{a}{R}r \tag{50}$$

The equilibrium Eq. (40) is furtherly rewritten as

$$\int_{-1}^{1} \tilde{\phi}(s)\, ds = \frac{P / a}{G} \tag{51}$$

As different from the direct problem, the iterative scheme is not necessary anymore here. The computational process is therefore remarkably simple, and proceeding through a streamlined sequence of steps as also shown in Fig. 2: (1) the Poisson's ratio ν, dimensionless parameters ζ and β are input as known quantities, with the computational term count n in Eq. (46) predefined; (2) based on the numerical tool in section 4, Eq. (49) is discretized into a system of linear equations, and the unknown coefficients $\tilde{\alpha}_i$ are determined through least-squares method under the assumption that $j = n$; (3) the contact pressure Eq. (47) is substituted into the equilibrium Eq. (51) to obtain the dimensionless external force $G / (P / a)$; (4) coefficients α_i are obtained from Eq. (48) and the final expression for contact pressure $\phi(s)$ are determined.

The contact stress $\bar{\sigma}_x(r, 0)$ could be obtained after substitution of $\phi(s)$ into Eq. (41). The kernel could be calculated through the Gauss-Chebyshev formula., and the contact pressure is only available inside the contact region (-1, 1).

4 NUMERICAL TOOL

The known function $\tilde{H}(r)$ is defined by the first kind Chebyshev polynomial $T_i(r)$ as

$$\tilde{H}(r) \approx H_n(r) = \sum_{i=0}^{n}{}' H_i T_i(r) \tag{52}$$

where

$$H_i \approx \frac{2}{n+1} \sum_{m_1=1}^{n+1} H(x_{m1}) T_i(x_{m1}) \tag{53}$$

and the dash in $\sum{}'$ means the first term in the sum is to be halved.

Substituting Eq. (46) and (52)into Eq. (39), and using the following relationship

$$\int_{-1}^{1} \frac{\sqrt{1-s^2} U_i(s)}{s-r} ds = -\pi T_{i+1}(r) \tag{54}$$

The governing equation could be processed to

$$\sum_{i=0}^{n-1} \alpha_i T_{i+1}(r) - \frac{1}{\pi} \sum_{i=0}^{n-1} \alpha_i \eta_i(r) = \sum_{i=0}^{n}{}' H_i T_i(r) \tag{55}$$

where

$$\eta_i(r) = \int_{-1}^{1} \sqrt{1-s^2} k_1(r,s) U_i(s) ds \tag{56}$$

Using the relationship in Eq. (52) as Turhan[30] did, one has

$$\eta_i(r) \approx \sum_{j=0}^{n}{}' \lambda_{ij} T_j(r) \tag{57}$$

where

$$\lambda_{ij} \approx \frac{2}{n+1} \sum_{m_1=1}^{n+1} \eta_i(x_{m1}) T_j(x_{m1}) \tag{58}$$

By using the Gauss-Chebyshev formula[34], $\eta_i(x_{m1})$ could be calculated by

$$\eta_i(x_{m1}) = \frac{\pi}{n+1} \sum_{m_2=1}^{n} \left(1 - t_{m_2}^2\right) k_1(x_{m1}, t_{m2}) U_i(t_{m2}) \tag{59}$$

The integral kernel $k(x_{m1}, t_{m2})$ could be handled numerically as oscillatory integral, and collocation points x_{m1} and t_{m2} are zeros of Chebyshev polynomials of the first and second kinds

$$x_{m1} = \cos\frac{2m_1 - 1}{2n+2}\pi,\ t_{m2} = \cos\frac{m_2}{n+1}\pi \tag{60}$$

Thus, the Eq. (55) could be rewritten as

$$\sum_{k=0}^{n-1} \alpha_k T_{k+1}(r) - \frac{1}{\pi} \sum_{k=0}^{n} {}' \sum_{i=0}^{n-1} \alpha_i \lambda_{ik} T_k(r) = \sum_{k=0}^{n} {}' H_k T_k(r) \tag{61}$$

Coefficients α_k are determined by expanding the Eq. (61) as the following system of linear algebraic equations

$$\left(\widehat{\mathbf{I}}_{\mathbf{n+1}} - \frac{1}{\pi} \boldsymbol{\Delta} \right) \boldsymbol{\alpha} = \mathbf{H} \tag{62}$$

where

$$\boldsymbol{\alpha} = \{ \alpha_0 \quad \alpha_1 \quad \cdots \quad \alpha_{n-1} \}^T \tag{63}$$

$$\mathbf{H} = \{ H_0' \quad H_1 \quad \cdots \quad H_n \}^T \tag{64}$$

$$\widehat{\mathbf{I}}_{\mathbf{n+1}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \tag{65}$$

$$\boldsymbol{\Delta} = \begin{bmatrix} \eta_{00}'(r) & \eta_{10}'(r) & \ldots & \eta_{n-1,0}'(r) \\ \eta_{01}(r) & \eta_{11}(r) & \cdots & \eta_{n-1,1}(r) \\ \eta_{02}(r) & \eta_{12}(r) & \cdots & \eta_{n-1,2}(r) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{0n}(r) & \eta_{1n}(r) & \cdots & \eta_{n-1,\ n}(r) \end{bmatrix} \tag{66}$$

5 RESULTS DISCUSSION

The flow chart of numerical solution is manipulated in MATHEMATICA software, and the initial number of n is set to 10 with $j = n$. In the computational process, the tedious part is about the numerical calculation of kernel $k(x_{m1}, t_{m2})$ due to the integrand involving sine-type function.

Table 1 Numerical result of non-dimensional contact pressure $\phi(s)$ with ν=0.3, ζ=0.5 and β=0.1.

x/a	Untruncated (UT)	Truncated (T)	Erdogan
-0.9	0.222031154907394	0.222031154907394	0.222031154907402
-0.6	0.486736741503531	0.486736741503531	0.486736741503523
-0.3	0.643346471150379	0.643346471150379	0.643346471150392
0.3	0.643346471150379	0.643346471150379	0.643346471150392
0.6	0.486736741503531	0.486736741503531	0.486736741503523
0.9	0.222031154907394	0.222031154907394	0.222031154907402

To speed the calculation, ExtrapolatingOscillatory is ulitized for the numerical integration, and the integral interval $(0, \infty)$ is discretized into several parts. Results show that the integrand in the kernel converges to zero dramatically at the points near 8000, so we truncate the integration and minimize the interval to $(0, 10000)$, the contact pressure $\phi(s)$ obtained are recorded in Table 1 and shown in Fig.3. It is obvious that errors of $\phi(s)$ calculated through truncated and untruncated integration are negligible.

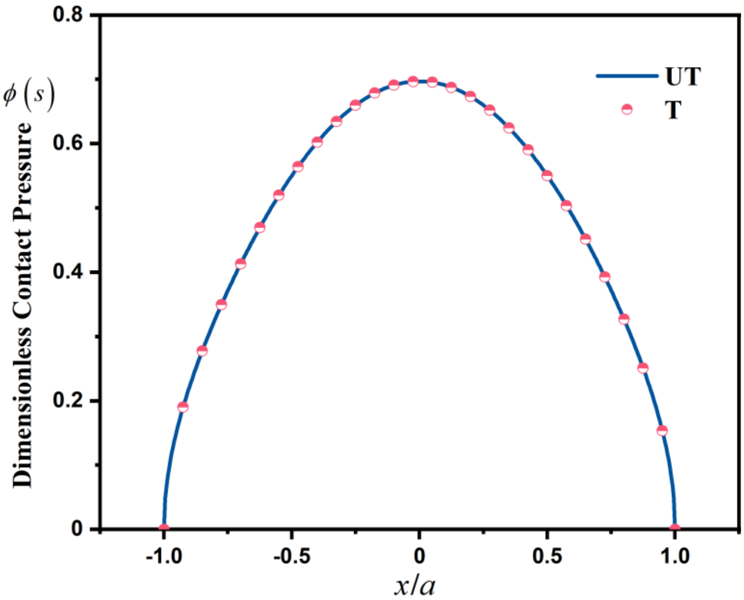

Fig. 3 Distribution of non-dimensional contact pressure $\phi(s)$ for truncated and untruncated integral interval.

Moreover, to ensure the precision and availability of the numerical solution used in this paper, the method from Erdogan and Gupta[11] is programmed for comparison. From Table 1 and Fig.4, it is apparent that results of non-dimensional contact pressure obtained from the two methods match well.

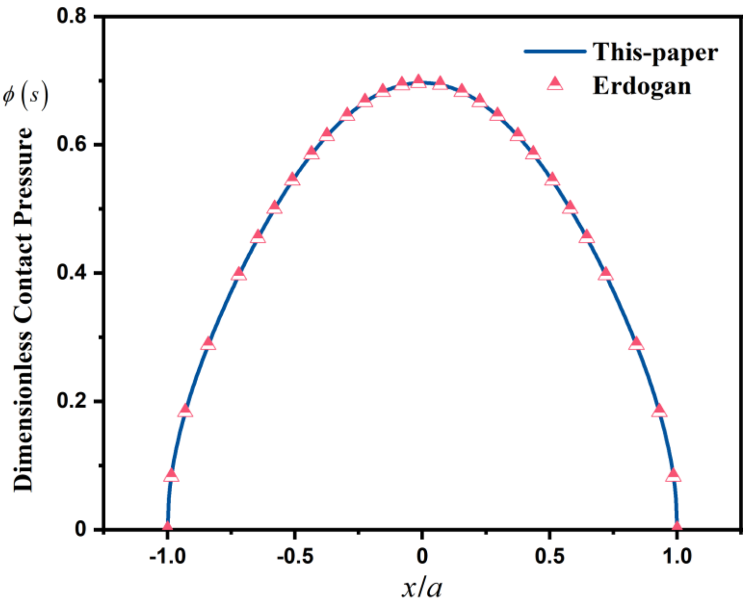

Fig. 4 Comparison for $\phi(s)$ obtained from the present solution and method of Erdogan when $\nu=0.3$, $\zeta=0.5$ and $\beta=0.1$.

Table 2 Comparison for numerical results of non-dimensional contact pressure $\phi(s)$ when $n=10$ and $n=20$, with $\nu=0.3$, $\zeta=0.5$ and $\beta=0.1$.

x/a	$n=10$	$n=20$
-0.9	0.222031154907394	0.22203614187814
-0.6	0.486736741503531	0.486741347850588
-0.3	0.643346471150379	0.643339927493522
0.3	0.643346471150379	0.643339927493522
0.6	0.486736741503531	0.486741347850588
0.9	0.222031154907394	0.22203614187814

As shown in Table 2, the term n is increased from 10 to 20, it is found that the influence for the result is ignorable, and thus the numerical solution method presented in this paper exhibits good robustness.

Fig.5 shows the variation of $\phi(s)$ with the dimensionless layer thickness ζ. For thin layer ($\zeta=0.5$), contact pressure mainly concentrates on the head of the indenter as a bullet, and distribution of the pressure is more uniform as the ζ increases. It indicates that, when contact radius remains constant under the same indenter, a thinner layer leads to a sharper distribution of contact pressure. This phenomenon may be attributed to the increased constraint imposed by the rigid substrate, from which the resistance in the bonded layer would be more pronounced as its thickness decreases, and therefore more concentrated contact pressure is required at the tip of indenter to displace the layer materials. Meanwhile, the $\phi(s)$ approximates the Hertz contact pressure when ζ approaches to 5, and the layer indented could be considered as a half-space. Results demonstrate the reliability of the numerical solution presented in this paper again.

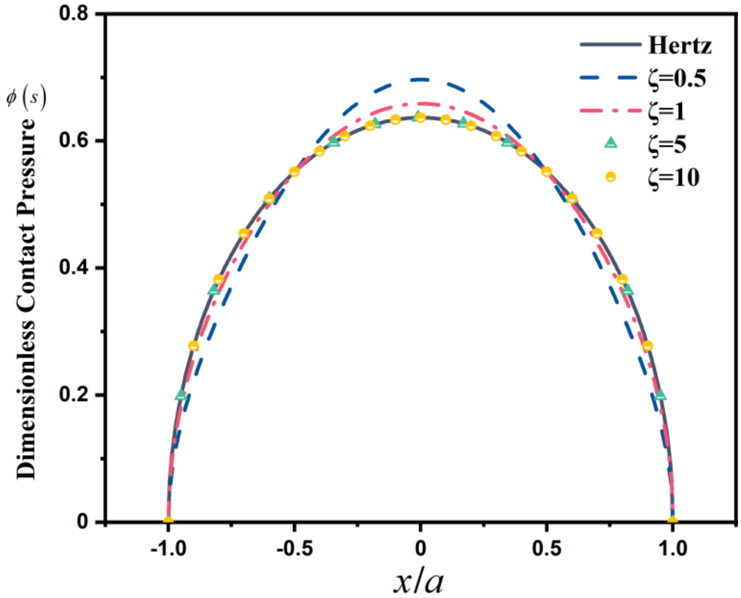

Fig. 5 *The variation of non-dimensional contact pressure* $\phi(s)$ *with layer thickness* ζ *when* $\nu=0.3$ *and* $\beta=0.1$.

Fig.6 shows the variation of $\phi(s)$ with the Poisson's ratio ν. As Poisson's ratio increases, the shape of pressure distribution sharps as a bullet. The results also show that $\phi(s)$ can be efficiently obtained when Poisson's ratio approaches to 0.5, it means that the present method is available for incompressible materials problems. As illustrated in Fig.3 and Fig.4, an increase in Poisson's ratio could also enhance the resistance originated from the layer-substrate structure.

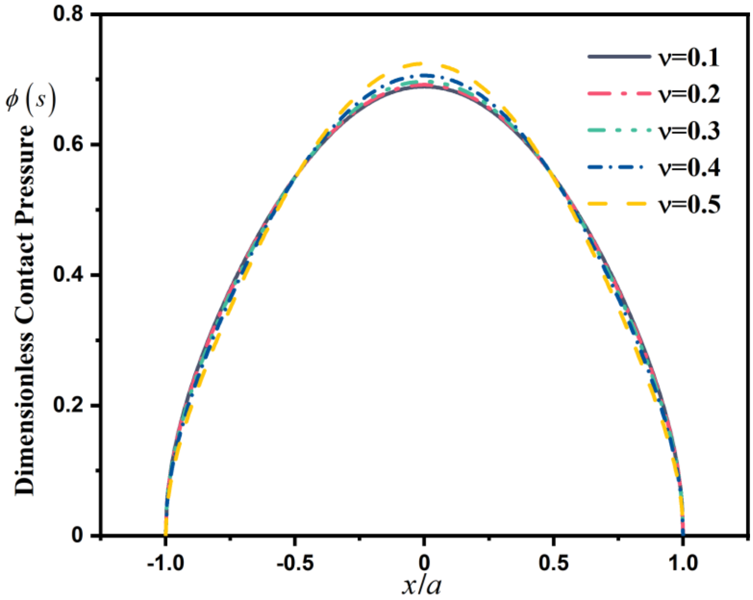

***Fig. 6** The variation of non-dimensional contact pressure* $\phi(s)$ *with the Poisson's ratio* ν *when* ζ=0.**5** and β=0.1.

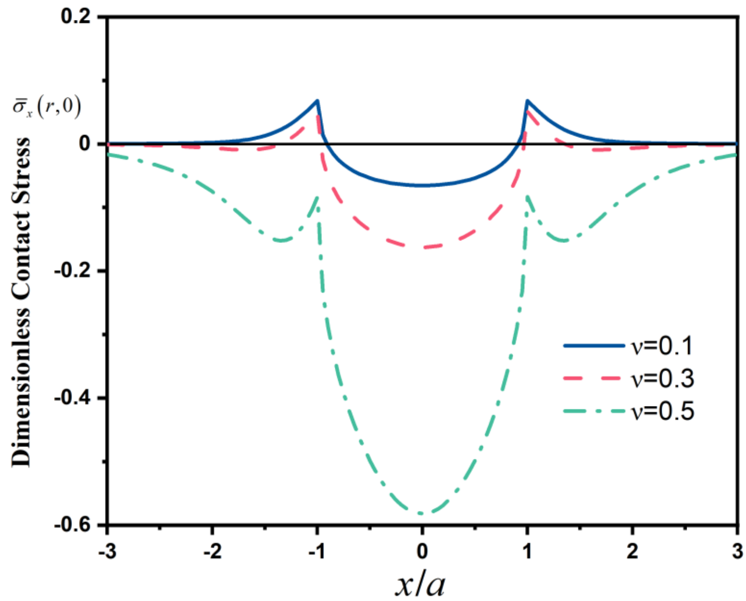

***Fig. 7** The variation of non-dimensional contact stress* $\bar{\sigma}_x(r, 0)$ *with the Poisson's ratio* ν *when* ζ=0.**5** and β=0.1.

The variation of dimensionless contact stress σ_x with Poisson's ratio at the layer surface is delineated at Fig. 7. It is obvious that the tensile peak is located at the contact edge, and the maximum value would decrease as the Poisson's ratio increases. The compressive stresses beyond the contact region may indicate that the materials would pile up during the indenting process.

6 CONCLUSIONS

An indentation contact problem of a rigid cylinder stamp and a bonded elastic thin layer is investigated. Singular integral equation of the first kind for the contact problem is obtained by solving the Navier equations in Fourier domain. A concise numerical solution scheme based on Chebyshev polynomials is presented, and truncation algorithm for the calculation of the kernel is also proposed. Results show that the numerical method in this paper is reliable and precise.

From the analysis, the contact pressure varies smoothly to a Hertz type as the layer thickness increases. It should be noted that meanwhile, the method is available for both compressible and incompressible material. Increasing Poisson's ratio appears to have the same effect on contact pressure distribution as decreasing the layer thickness.

ACKNOWLEDGEMENT

This work is supported by Zunyi Municipal Science and Technology Program Project (Grant No. HZ-2023-149) and Zunyi Normal University Local Industry Revolution Service Project (Grant No. CXY [2023]06).

Author's Contribuitions: Funding acquisition & Writing - original draft, Xu Zhang; Conceptualization & Writing - review, Shihong Han; Supervision & Writing - review & editing, Kai Zhu.

Data Availability: No research data was used

Editor: Pablo Andrés Muñoz Rojas

References

[1] L. Keer. The Contact Stress Problem for an Elastic Sphere Indenting an Elastic Layer. 1964: 143.
[2] N. N. Lebedev, I. S. Ufliand. Axisymmetric contact problem for an elastic layer. 1958: 442-450.
[3] V. M. Aleksandrov, I. I. Vorovich. Contact problems for the elastic layer of small thickness. 1964: 425–427.
[4] V. M. Aleksandrov, V. A. Babeshko, V. A. Kucherov. Contact problems for an elastic layer of slight thickness. 1966: 148-172.
[5] P. Meijers. The contact problem of a rigid cylinder on an elastic layer. 1968: 353-383.
[6] J. B. Alblas, M. Kuipers. On the two dimensional problem of a cylindrical stamp pressed into a thin elastic layer. 1970: 292-311.
[7] V. Aleksandrov. Doubly periodic contact problems for an elastic layer. 2002: 297-305.
[8] V. Aleksandrov. Asymptotic solution of the axisymmetric contact problem for an elastic layer of incompressible material. 2003: 589-593.
[9] L. La Ragione, F. Musceo, A. Sollazzo. Axisymmetric indentation of a rigid cylinder on a layered compressible and incompressible halfspace. 2008: 1499-1520.
[10] F. Yang. Thickness effect on the indentation of an elastic layer. 2003: 226-232.
[11] F. Erdogan, G. D. Gupta, T. S. Cook. Numerical solution of singular integral equations. 1973: 368-425.
[12] S. Krenk. On quadrature formulas for singular integral equations of the first and second kind [J]. Quarterly of Applied Mathematics, 1975, 33: 225-232.
[13] F. Erdogan, M. A. Guler. Contact Mechanics of FGM Coatings [M]. Bethlehem (PA): Leigh University. 2000.
[14] F. Erdogan, G. D. Gupta, T. S. Cook. Numerical solution of singular integral equations [M]. Methods of Analysis and Solutions of Crack Problems. Dordrecht: Springer. 1973: 368-425.
[15] F. Çakıroğlu, M. Çakıroğlu, R. Erdöl. Contact Problems for Two Elastic Layers Resting on Elastic Half-Plane. 2001: 113-118.
[16] A. Birinci, G. Adıyaman, M. Yaylacı, E. Öner. Analysis of Continuous and Discontinuous Cases of a Contact Problem Using Analytical Method and FEM [J]. Latin American Journal of Solids and Structures, 2015, 12(9): 1771-1789.
[17] M. R. Gecit, F. Erdogan. Frictionless contact problem for an elastic layer under axisymmetric loading. 1978: 771-785.
[18] K. L. Johnson. Contact mechanics [M]. Cambridge university press. 1987.
[19] M. J. Jaffar. A numerical solution for axisymmetric contact problems involving rigid indenters on elastic layers. 1988: 401-416.
[20] M. J. Jaffar. Asymptotic behaviour of thin elastic layers bonded and unbonded to a rigid foundation. 1989: 229-235.
[21] J. Barber. Contact problems for the thin elastic layer. 1990: 129-132.
[22] H. Conway, H. Lee, R. Bayer. The Impact Between a Rigid Sphere and a Thin Layer. 1970: 159-162.

[23] X. Ning, M. Lovell, W. S. Slaughter. Asymptotic solutions for axisymmetric contact of a thin, transversely isotropic elastic layer. 2006: 693-698.
[24] M. J. Matthewson. The effect of a thin compliant protective coating on Hertzian contact stresses. 1982: 237-249.
[25] J. A. Greenwood, J. R. Barber. Indentation of an elastic layer by a rigid cylinder. 2012: 2962-2977.
[26] T. S. Vasu, T. K. Bhandakkar. A Study of the Contact of an Elastic Layer–Substrate System Indented by a Long Rigid Cylinder IncorporatingSurface Effects. 2016: 0610091-0610098.
[27] W. Yuan, G. Wang. Cylindrical indentation of an elastic bonded layer with surface tension. 2019: 597-613.
[28] F. D. Gakhov. Boundary value problems [M]. Courier Corporation. 1990.
[29] G. Gladwell. On Some Unbonded Contact Problems in Plane Elasticity Theory. 1976: 263-267.
[30] İ. Turhan, H. Oğuz, E. Yusufoğlu. Chebyshev polynomial solution of the system of Cauchy-type singular integral equations of the first kind. 2012: 944-954.
[31] I. I. Argatov. Depth-sensing indentation of a transversely isotropic elastic layer: Second-order asymptotic models for canonical indenters. 2011: 3444-3452.
[32] I. Argatov. Contact problem for a thin elastic layer with variable thickness: Application to sensitivity analysis of articular contact mechanics. 2013: 8383-8393.
[33] R. C. Batra, W. Jiang. Analytical solution of the contact problem of a rigid indenter and an anisotropic linear elastic layer. 2008: 5814-5830.
[34] J. C. Mason, D. C. Handscomb. Chebyshev polynomials. 2003: