

Fractional heat conduction with finite wave speed in a thermo-visco-elastic spherical shell

Abstract

This problem deals with the thermo-elastic interaction due to step input of temperature on the stress free boundaries of a homogeneous visco-elastic orthotropic spherical shell in the context of a new consideration of heat conduction with fractional order generalized thermoelasticity. Using the Laplace transformation, the fundamental equations have been expressed in the form of a vector-matrix differential equation which is then solved by eigen value approach and operator theory analysis. The inversion of the transformed solution is carried out by applying a method of Bellman et al (1966). Numerical estimates for thermophysical quantities are obtained for copper like material for weak, normal and strong conductivity and have been depicted graphically to estimate the effects of the fractional order parameter. Comparisons of the results for different theories (TEWED (GN-III), three-phase-lag model) have also been presented and the effect of viscosity is also shown. When the material is isotropic and outer radius of the hollow sphere tends to infinity, the corresponding results agree with that of existing literature.

Keywords

Generalized thermo-visco-elasticity, Three-phase-lag model, Fractional order heat equation, Eigen value approach, Vector-matrix differential equation, Step input temperatures.

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1 INTRODUCTION

Linear viscoelasticity has been an important area of research since the period of Maxwell, Boltzman, Voigt and Kelvin. Valuable information regarding linear viscoelasticity theory may be obtained in the books of Gross (1953), Staverman and Schwrzl, Alfery and Gurnee, Ferry, Bland and Lakes. Many researchers like Biot (1954, 1955), Gurtin and Sternberg, Liioushin and Pobedria, Tanner, Huilgol and Phan-Thein have contributed notably on thermoviscoelasticity. Freudenthal has pointed out that most of the solids, when subjected to dynamic loading, exhibit viscous effects.

The Kelvin-Voigt model is one of the macroscopic mechanical models often used to describe the viscoelastic behavior of a material. The model represents the delayed elastic response subjected to

stress when the deformation is time dependent but recoverable. The dynamic interaction of thermal and mechanical fields in solids has great practical applications in modern aeronautics, astronautics, Nuclear reactors and high-energy particle accelerators. Several researchers are working in this field. A problem involving Two-Temperature Magneto-Viscoelasticity with thermal Relaxation time in Perfect conducting medium have been solved by Ezzat and El-Karamany (2009). A two temperature thermo-electro-viscoelastic problem subjected to modified Ohm's and Fourier's Laws have been solved by Ezzat et al. (2012).

The classical theories of thermoelasticity involving infinite speed of propagation of thermal signals, contradict physical facts. During the last five decades, non-classical theories involving finite speed of heat transportation in elastic solids have been developed to remove the paradox. In contrast with the conventional coupled thermoelasticity theory, which involves a parabolic-type heat transport equation, these generalized theories involving a hyperbolic-type heat-transport equation are supported by experiments exhibiting the actual occurrence of wave-type heat transport in solids, called second sound effect. The first generalization to this theory is due to Lord and Shulman (1967) who formulated the generalized thermoelasticity theory involving one thermal relaxation time, which is known as extended thermo-elasticity theory (ETE). The second generalization to the coupled thermoelasticity theory due to Green and Lindsay (1972), involves two relaxation times.

The third generalization to the coupled thermoelasticity theory is known as low-temperature thermoelasticity introduced by Hetnarski and Ignaczak called the H-I theory. This model is characterized by a system of non-linear field equations.

The fourth generalization is concerned with the thermo-elasticity without energy dissipation (TEWOED) and thermoelasticity with energy dissipation (TEWED) introduced by Green and Naghdi (1991, 1992, 1993) and provide sufficient basic modifications in the constitutive equations that permit treatment of a much wider class of heat flow problems, labeled as types I, II, III. The natures of these three types of constitutive equations are such that when the respective theories are linearized, type-I is same as the classical heat equation (based on Fourier's law) whereas types II and III permit propagation of thermal signals at a finite speed. When Fourier conductivity is dominant the temperature equation reduces to classical Fourier's law of heat conduction and when the effect of conductivity is negligible, the equation has undamped thermal wave solutions without energy dissipation. Applying the above theories of generalized thermoelasticity, several problems have been solved by Mallik and Kanoria (2008), Kar and Kanoria (2009), Islam and Kanoria (2011), Ghosh and Kanoria (2010), Banik and Kanoria (2011).

Recently Roychoudhury (2007) has established a generalized mathematical model of a coupled thermoelasticity theory that includes three-phase lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. The more general model established reduces to the previous models as special cases. According to this model $\bar{q} = -\left[K \bar{\nabla} T(P, t + \tau_T) + K^* \bar{\nabla} v(P, t + \tau_v) \right]$, where $\bar{\nabla} v (\dot{v} = T)$ is the thermal displacement gradient and K^* is the additional material constant. To study some practical relevant problems, particularly in heat transfer problems involving very short time intervals and in the problems of very high heat fluxes, the hyperbolic equation gives significantly different results than the parabolic equation. According to this phenomenon the lagging behavior in the heat conduction in solid should not be ignored particularly when the elapsed times during a transient process are very small, say about 10^{-7} s or the heat flux is very much high.

Three-phase-lag model is very useful in the problems of nuclear boiling, exothermic catalytic reactions, phonon-electron interactions, phonon-scattering etc., where the delay time τ_q captures the thermal wave behavior (a small scalar response in time), the phase-lag τ_r captures the effect of phonon-electron interactions (a microscopic response in space), the other delay time τ_v is effective since, in the three-phase-lag model, the thermal displacement gradient is considered as a constitutive variable whereas in the conventional thermoelasticity theory temperature gradient is considered as a constitutive variable. Banik and Kanoria (2012) have solved the effect of three-phase-lag in an infinite medium with a spherical cavity. The magneto-thermo-elastic responses in a perfectly conducting medium under three-phase-lag model have been studied by Das and Kanoria (2012).

However, over the last few decades, anisotropic materials have been increasingly used. There are materials which have natural anisotropy such as zinc, magnesium, sapphire, wood, some rocks and crystals, and also there are artificially manufactured materials such as fiber-reinforced composite materials which exhibit anisotropic character. The advantage of composite materials over the traditional materials lies on their valuable strength, elastic and other properties (1980). A reinforced material may be regarded to some order of approximation, as homogeneous and anisotropic elastic medium having a certain kind of elastic symmetry depending on the symmetry of reinforcement. Some glass fibre reinforced plastics may be regarded as transversely isotropic. Thus, problems of solid mechanics should not be restricted to the isotropic medium only. Increasing use of an anisotropic media demand that the study of elastic problems should be extended to anisotropic medium also.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic, and this is one reason why fractional calculus has become more and more popular (1967, 1997, 1999).

Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the second half of the nineteenth century. The first application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches, and some various alternative definitions of fractional derivatives appeared in Refs. (1974, 1997, 2000). In the last few years, fractional calculus was applied successfully in various areas to modify many existing models of physical processes, e.g., chemistry, biology, modeling and identification, electronics, wave propagation and viscoelasticity (1971, 1974, 1983, 1984, 1997). One can refer to Padlubny (1999) for a survey of applications of fractional calculus.

Recently, a considerable research effort is expended to study anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation by Kimmich (2002) as follows

$$\rho c = \kappa I^{\xi} c_{,ii} \quad (1)$$

where ρ is the mass density, c is the concentration, κ is the diffusion conductivity, i the coordinate symbol, which takes the value 1, 2, 3. The notation I^{ξ} is the Riemann-Liouville fractional integral, introduced as a natural generalization of the well-known n -fold repeated integral $I^n f(t)$ written in a convolution-type form as in (2000).

Youssef (2010) introduced another formula of heat conduction in the following form

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -\kappa I^{\xi-1} \nabla T, \quad 0 < \xi \leq 2 \quad (2)$$

and a uniqueness theorem has also been proved.

Ezzat established a new model of fractional heat conduction equation by using the new Taylor series expansion of time-fractional order, developed by Jumarie (2010) as

$$q_i + \frac{\tau_0^{\xi}}{\xi!} \frac{\partial^{\xi} q_i}{\partial t^{\xi}} = -\kappa \nabla T, \quad 0 < \xi \leq 1, \quad (3)$$

El-Karamany and Ezzat (2011) introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. Uniqueness and reciprocal theorems are proved, and the convolutional variational principle is established and used to prove a uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors except symmetry conditions. For fractional thermoelasticity not involving two-temperatures, El-Karamany and Ezzat (2011) established the uniqueness, reciprocal theorems and convolution variational principle. The dynamic coupled and Green-Naghdi thermoelasticity theories result as limit cases. The reciprocity relation in case of quiescent initial state is found to be independent of the order of differentiation. Fractional order theory of a perfect conducting thermoelastic medium not involving two temperatures was investigated by Ezzat and El-Karamany (2011). Thermal wave propagation in an infinite half-space under fractional order Green-Naghdi theory was studied by Sur and Kanoria (2012).

To the authors' knowledge, under three-phase-lag effect, no solution of visco-elastic orthotropic materials for fractional heat conduction equation has been reported. With this motivation in mind the present analysis is to study the thermoelastic stresses, displacement and temperature distribution in a orthotropic hollow sphere in the context of GN-III and three-phase-lag model of generalized thermoelasticity where the heat equation consists of some non-local fractional operator signifying not only the present state, but also the previous states due to sudden temperature change on the stress-free boundaries. The governing equations are formed in Laplace transform domain which is then solved by eigen-value approach and operator theory analysis. The inversion of the transformed solution are carried out numerically applying the method of Bellman et al. A comprehensive analysis of the result have been presented for 3P model and GN-III model for both viscous and non-viscous isotropic materials. The effect of the fractional order parameter is also discussed.

2 FORMULATION OF THE PROBLEM

We consider a homogeneous orthotropic thermo-visco-elastic spherical shell of inner radius a and outer radius b in an undisturbed state and initially at uniform temperature T_0 . We introduce spherical polar coordinates (r, θ, ϕ) with the center of the cavity at the origin as shown in Figure a.

We consider spherically symmetric thermal problem so that the displacement component $u = [u(r, t), 0, 0]$ and the temperature T are assumed to be functions of r and t only.

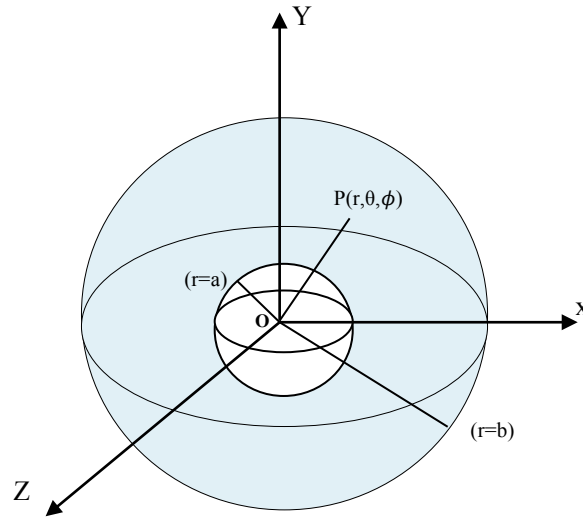


Figure a Visco-elastic spherical shell.

The stress-strain-temperature relations in the present problem are (Kelvin-Voigt type)

$$\tau_{rr} = C_{33} \left(1 + t_0 \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial r} + (C_{13} + C_{23}) \left(1 + t_0 \frac{\partial}{\partial t} \right) \frac{u}{r} - \beta_r T, \tag{4}$$

$$\tau_{\theta\theta} = C_{13} \left(1 + t_0 \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial r} + (C_{11} + C_{12}) \left(1 + t_0 \frac{\partial}{\partial t} \right) \frac{u}{r} - \beta_\theta T, \tag{5}$$

$$\tau_{\phi\phi} = C_{23} \left(1 + t_0 \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial r} + (C_{12} + C_{22}) \left(1 + t_0 \frac{\partial}{\partial t} \right) \frac{u}{r} - \beta_\phi T, \tag{6}$$

and the generalized heat conduction equation for fractional order three-phase-lag model is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left\{ K_r^* I^{\alpha-1} \frac{\partial T}{\partial r} + \tau_v^* I^{\alpha-1} \frac{\partial \dot{T}}{\partial r} + K_r \tau_r I^{\alpha-1} \frac{\partial \ddot{T}}{\partial r} \right\} \right] = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2} \right) \times \frac{\partial^2}{\partial t^2} \left[\rho C_e T + T_0 \left\{ \beta_r \frac{\partial u}{\partial r} + (\beta_\theta + \beta_\phi) \frac{u}{r} \right\} \right] \tag{7}$$

where $\tau_{ij}(i, j = r, \theta, \phi)$ are the stress tensor, T is the temperature increase over the reference temperature T_0 , $C_{ij}(i, j = 1, 2, 3)$ are the elastic constants, $\beta_i(i = r, \theta, \phi)$ are the thermal moduli, K_r is the coefficient of thermal conductivity along the radial direction, K_r^\star is the additional material constant along the radial direction, ρ is the mass density, C_e is the specific heat of the solid at constant strain, t_0 is the mechanical relaxation time, τ_T and τ_q are the phase-lag of temperature gradient and the phase-lag of the heat flux respectively. Also $\tau_v^\star = K_r + \tau_v K_r^\star$ where τ_v is the phase-lag of thermal displacement gradient.

In the case $K_r = 0$ and $\tau_T = \tau_q = \tau_v = 0$, we arrive at the thermo-elasticity equations with energy dissipation (TEWED(GN-III)).

The stress equation of motion in spherical polar co-ordinate is given by

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}(2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi}) = \rho \frac{\partial^2 u}{\partial t^2} \tag{8}$$

Introducing the following non-dimensional quantities

$$U = \frac{C_{33}}{a\beta_r T_0} u, \quad (R, S) = \left(\frac{r}{a}, \frac{b}{a} \right), \quad (\sigma_r, \sigma_\theta, \sigma_\phi) = \frac{1}{\beta_r T_0} (\tau_{rr}, \tau_{\theta\theta}, \tau_{\phi\phi}),$$

$$\Theta = \frac{T}{T_0}, \quad \eta = \frac{Gt}{a}, \quad G^2 = \frac{C_{33}}{\rho}, \quad (\tau'_q, \tau'_v, \tau'_T) = \frac{G}{a} (\tau_q, \tau_v, \tau_T).$$

Equations (4)-(8) become

$$\sigma_r = \left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) \frac{\partial U}{\partial R} + \frac{C_{13} + C_{23}}{C_{33}} \left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) \frac{U}{R} - \Theta, \tag{9}$$

$$\sigma_\theta = \frac{C_{13}}{C_{33}} \left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) \frac{\partial U}{\partial R} + \frac{C_{11} + C_{12}}{C_{33}} \left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) \frac{U}{R} - \frac{\beta_\theta}{\beta_r} \Theta, \tag{10}$$

$$\sigma_\phi = \frac{C_{23}}{C_{33}} \left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) \frac{\partial U}{\partial R} + \frac{C_{12} + C_{22}}{C_{33}} \left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) \frac{U}{R} - \frac{\beta_\phi}{\beta_r} \Theta, \tag{11}$$

$$\left[C_T^2 I^{\alpha-1} + (C_K^2 + \tau'_v C_T^2) I^{\alpha-1} \frac{\partial}{\partial \eta} + \tau'_T C_K^2 I^{\alpha-1} \frac{\partial^2}{\partial \eta^2} \right] \left(\frac{\partial^2 \Theta}{\partial R^2} + \frac{2}{R} \frac{\partial \Theta}{\partial R} \right) =$$

$$\left(1 + \tau'_q \frac{\partial}{\partial \eta} + \frac{1}{2} \tau_q'^2 \frac{\partial^2}{\partial \eta^2} \right) \left\{ \frac{\partial^2 \Theta}{\partial \eta^2} + \varepsilon \frac{\partial^2}{\partial \eta^2} \left(\frac{\partial U}{\partial R} + M \frac{U}{R} \right) \right\} \tag{12}$$

and

$$\left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta}\right) \left\{ \frac{\partial^2 U}{\partial R^2} + \frac{2}{R} \frac{\partial U}{\partial R} - \frac{AU}{R^2} \right\} = \frac{\partial \Theta}{\partial R} + \frac{\partial^2 U}{\partial \eta^2}. \quad (13)$$

where

$$A = \frac{C_{11} + 2C_{12} + C_{22} - (C_{13} + C_{23})}{C_{33}}, \quad (14)$$

and

$$M = \frac{\beta_\theta + \beta_\phi}{\beta_r}. \quad (15)$$

Also, $C_T^2 = \frac{K_r^*}{\rho C_e G^2}$, $C_K^2 = \frac{K_r}{a \rho C_e G}$ and $\varepsilon = \frac{\beta_r^2 T_0}{\rho C_e C_{33}}$ are dimensionless constants, ε being the thermoelastic coupling constant. Where C_T is the non-dimensional thermal wave velocity and C_K is the damping co-efficient.

The boundary conditions are given by

$$\sigma_R = 0 \text{ on } R=1, S \quad \eta \mathbf{P} 0 \quad (16)$$

$$\Theta = \chi_1 (H(\eta) - H(\eta - \eta_1^0)) \text{ on } R=1, \eta > 0, \quad (17)$$

$$= \chi_2 H(\eta) \text{ on } R=S, \eta > 0. \quad (18)$$

where χ_1 and χ_2 are dimensionless constants, and $H(\eta)$ is the Heaviside unit step function. The above condition indicate that for time $\eta \mathbf{P} \eta_1^0$ there is no temperature ($\Theta=0$) on the inner boundary and for $\eta \neq 0$ there is no temperature ($\Theta=0$) on the outer boundary. Thermal shocks are given on the boundaries of the shell ($R=1, S$). Thermal stresses in the elastic medium due to the application of these thermal shocks are calculated. We assume that the medium is at rest and undisturbed initially.

The initial and the regularity conditions can be written as

$$U = \frac{\partial U}{\partial \eta} = \frac{\partial^2 U}{\partial \eta^2} = \frac{\partial^3 U}{\partial \eta^3} = 0 \text{ and } \Theta = \frac{\partial \Theta}{\partial \eta} = \frac{\partial^2 \Theta}{\partial \eta^2} = \frac{\partial^3 \Theta}{\partial \eta^3} = 0 \text{ at } \eta=0, \mathbf{R P} 0 \quad (19)$$

$$U = \Theta = \frac{\partial U}{\partial \eta} = \frac{\partial \Theta}{\partial \eta} = 0 \text{ when } R \rightarrow \infty. \tag{20}$$

3 METHOD OF SOLUTION

Let

$$\{\bar{U}(R, p), \bar{\Theta}(R, p)\} = \int_0^\infty \{U(R, \eta), \Theta(R, \eta)\} e^{-p\eta} d\eta$$

with $\text{Re}(p) > 0$ denote the Laplace transform of U and Θ respectively.

Since we have

$$L\{I^\alpha f(t)\} = \frac{1}{p^\alpha} L\{f(t)\}.$$

On taking Laplace transform, equations (12) and (13) reduce to

$$\frac{d^2 \bar{\Theta}}{dR^2} + \frac{2}{R} \frac{d\bar{\Theta}}{dR} = a_3 \left[\bar{\Theta} + \varepsilon \left(\frac{d\bar{U}}{dR} + \frac{2\bar{U}}{R} \right) \right], \tag{21}$$

and

$$\frac{d^2 \bar{U}}{dR^2} + \frac{2}{R} \frac{d\bar{U}}{dR} - \frac{2U}{R^2} = a_4 \left(\frac{d\bar{\Theta}}{dR} + p^2 \bar{U} \right), \tag{22}$$

where $a_3 = \frac{p^{\alpha+1} \left(1 + \tau'_q p + \frac{1}{2} \tau_q'^2 p^2 \right)}{(1 + \tau'_v p) C_T^2 + p(1 + \tau'_t p) C_K^2}$, $a_4 = \frac{a}{a + t_0 p G}$ and assuming $M = 2$.

Differentiating equation (21) with respect to R and using equation (22), we get

$$\frac{d^2}{dR^2} \left(\frac{d\bar{\Theta}}{dR} \right) + \frac{2}{R} \left(\frac{d\bar{\Theta}}{dR} \right) - \frac{2}{R^2} \left(\frac{d\bar{\Theta}}{dR} \right) = a_3 \left[\varepsilon p^2 a_4 \bar{U} + (1 + \varepsilon a_4) \frac{d\bar{\Theta}}{dR} \right]. \tag{23}$$

Equations (22) and (23) can be written in the form

$$L(\bar{U}) = a_4 p^2 \bar{U} + a_4 \frac{d\bar{\Theta}}{dR}, \tag{24}$$

and

$$L\left(\frac{d\bar{\Theta}}{dR}\right) = \varepsilon a_3 a_4 p^2 \bar{U} + a_3 (1 + \varepsilon a_4) \frac{d\bar{\Theta}}{dR}, \quad (25)$$

where, we assume that $A=2$, ε is the thermo-elastic coupling constant and

$$L \equiv \frac{d^2}{dR^2} + \frac{2}{R} \frac{d}{dR} - \frac{2}{R^2}. \quad (26)$$

From equations (24) and (25), we have the vector-matrix differential equation as follows

$$L\tilde{V} = \tilde{A}\tilde{V}, \quad (27)$$

where

$$\tilde{V} = \begin{bmatrix} \bar{U} & \frac{d\bar{\Theta}}{dR} \end{bmatrix}^T, \quad \tilde{A} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \quad (28)$$

and $D_{11} = a_4 p^2$, $D_{12} = a_4$, $D_{21} = \varepsilon a_3 a_4 p^2$, $D_{22} = a_3 (1 + \varepsilon a_4)$.

4 EIGEN VALUE APPROACH

Let

$$\tilde{V} = \tilde{X}(m)\omega(R, m), \quad (29)$$

where m is a scalar, \tilde{X} is a vector depending on R and $\omega(R, m)$ is a non-trivial solution of the scalar differential equation

$$L\omega = m^2\omega. \quad (30)$$

Let $\omega = R^{-1/2}\omega_1$. Therefore, from equation (30) we have

$$\frac{d^2\omega_1}{dR^2} + \frac{1}{R} \frac{d\omega_1}{dR} - \left(\frac{9}{4R^2} + m^2\right)\omega_1 = 0. \quad (31)$$

The solution of equation (30) is

$$\omega = [A_1 I_{3/2}(mR) + B_1 K_{3/2}(mR)] / \sqrt{R}, \quad (32)$$

Using equation (29) and (30) into equation (27) we get

$$\tilde{A}\tilde{X} = m^2\tilde{X}, \tag{33}$$

where $\tilde{X}(m)$ is the eigen vector corresponding to the eigen value m^2 .

The characteristic equation corresponding to \tilde{A} can be written as

$$m^4 - (D_{11} + D_{22})m^2 + (D_{11}D_{22} - D_{12}D_{21}) = 0. \tag{34}$$

The roots of the characteristic equation (37) are of the form $m^2 = m_1^2$ and $m^2 = m_2^2$, where

$$m_1^2 + m_2^2 = D_{11} + D_{22}, \quad m_1^2 m_2^2 = D_{11}D_{22} - D_{12}D_{21}. \tag{35}$$

Equation (34) can be written as

$$m^4 - (a_3 + a_4 p^2 + \epsilon a_3 a_4)m^2 + a_3 a_4 p^2 = 0. \tag{36}$$

Therefore, the positive roots of the equation (36) are

$$m_1, m_2 = \frac{1}{2}(\sqrt{\alpha} \pm \sqrt{\beta}), \tag{37}$$

where

$$\alpha, \beta = (\sqrt{a_3} \pm \sqrt{a_4 p})^2 + \epsilon a_3 a_4. \tag{38}$$

Therefore, m_1 and m_2 are real positive quantities.

The eigen vectors $X(m_j), j=1,2$ corresponding to the eigen values $m_j^2, j=1,2$ can be calculated as

$$\tilde{X}(m_j) = \begin{bmatrix} X_1(m_j) \\ X_2(m_j) \end{bmatrix} = \begin{bmatrix} D_{12} \\ -(D_{11} - m_j^2) \end{bmatrix}, j=1,2. \tag{39}$$

Therefore, from equation (29) and using equation (28) we get

$$\bar{U} = \sum_{i=1,2} D_{12} [A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R)] / \sqrt{R}, \tag{40}$$

and

$$\frac{d\bar{\Theta}}{dR} = -\sum_{i=1,2} (D_{11} - m_i^2) [A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R)] / \sqrt{R}, \quad (41)$$

where $I_{3/2}(m_i R)$ and $K_{3/2}(m_i R)$ are the modified Bessel functions of order 3/2 of first and second kind respectively. A_i 's and B_i 's ($i=1,2$) are independent of R but dependent of p and are to be determined from the boundary conditions.

Using the recurrence relations of modified Bessel functions we obtain from equation (41)

$$\bar{\Theta} = \sum_{i=1,2} \frac{(D_{11} - m_i^2)}{m_i} [A_i I_{1/2}(m_i R) + B_i K_{1/2}(m_i R)] / \sqrt{R}, \quad (42)$$

since

$$\frac{1}{R^{1/2}} P_{3/2}(mR) = -\frac{d}{dR} \left[\frac{P_{1/2}(mR)}{mR^{1/2}} \right], \quad (43)$$

where $P = I, K$. Taking Laplace transform on the equations (9), (10) and (11) we get

$$\bar{\sigma}_R = \sum_{i=1,2} \frac{A_i}{R^{3/2}} \left[a_5 I_{3/2}(m_i R) + \left(2m_i R - \frac{a_4 p^2}{m_i} R \right) I_{1/2}(m_i R) \right] + \sum_{i=1,2} \frac{B_i}{R^{3/2}} \left[a_5 K_{3/2}(m_i R) - \frac{a_4 p^2}{m_i} R K_{1/2}(m_i R) \right] \quad (44)$$

$$\bar{\sigma}_\theta = \sum_{i=1,2} \frac{A_i}{R^{3/2}} \left[\left(\frac{C_{11} + C_{12} - 2C_{13}}{C_{33}} \right) I_{3/2}(m_i R) + \left\{ \frac{\beta_\theta}{\beta_r} \frac{a_4 p^2 - m_i^2}{m_i} - \frac{C_{13}}{C_{33}} m_i \right\} R I_{1/2}(m_i R) \right] + \sum_{i=1,2} \frac{B_i}{R^{3/2}} \left[\left(\frac{C_{11} + C_{12} - 2C_{13}}{C_{33}} \right) K_{3/2}(m_i R) - \left\{ \frac{\beta_\theta}{\beta_r} \frac{a_4 p^2 - m_i^2}{m_i} + \frac{C_{13}}{C_{33}} m_i \right\} R K_{1/2}(m_i R) \right] \quad (45)$$

$$\bar{\sigma}_\phi = \sum_{i=1,2} \frac{A_i}{R^{3/2}} \left[\left(\frac{C_{12} + C_{22} - 2C_{23}}{C_{33}} \right) I_{3/2}(m_i R) + \left\{ \frac{\beta_\phi}{\beta_r} \frac{a_4 p^2 - m_i^2}{m_i} - \frac{C_{23}}{C_{33}} m_i \right\} R I_{1/2}(m_i R) \right] + \sum_{i=1,2} \frac{B_i}{R^{3/2}} \left[\left(\frac{C_{12} + C_{22} - 2C_{23}}{C_{33}} \right) K_{3/2}(m_i R) - \left\{ \frac{\beta_\phi}{\beta_r} \frac{a_4 p^2 - m_i^2}{m_i} + \frac{C_{23}}{C_{33}} m_i \right\} R K_{1/2}(m_i R) \right] \quad (46)$$

where $a_5 = \left(-3 + \frac{C_{13} + C_{23} + C_{33}}{C_{33}}\right)$. Using the boundary conditions $\bar{\sigma}_R = 0$ on $R = 1, R = S$ and $\bar{\Theta} = \frac{\chi_1'}{p}$ on $R = 1$, where $\chi_1' = \chi_1(1 - e^{-pm^0})$, $\bar{\Theta} = \frac{\chi_2}{p}$ on $R = S$.

Using the recurrence relations (Watson, 1980) from equations (42) and (44) we obtain

$$\begin{aligned}
 A_1W_{11} + A_2W_{12} + B_1W_{13} + B_2W_{14} &= 0, \\
 A_1W_{21} + A_2W_{22} + B_1W_{23} + B_2W_{24} &= 0, \\
 A_1W_{31} + A_2W_{32} + B_1W_{33} + B_2W_{34} &= \frac{\chi_1'}{p}, \\
 A_1W_{41} + A_2W_{42} + B_1W_{43} + B_2W_{44} &= \frac{\chi_2}{p},
 \end{aligned}
 \tag{47}$$

$$\begin{aligned}
 W_{1i} &= a_5 I_{3/2}(m_j) + \left(2m_j - \frac{a_4 p^2}{m_j}\right) I_{1/2}(m_j), & i, j = 1, 2 \\
 W_{2i} &= a_5 I_{3/2}(m_j S) + \left(2m_j - \frac{a_4 p^2}{m_j}\right) S I_{1/2}(m_j S), & i, j = 1, 2 \\
 W_{3i} &= a_5 K_{3/2}(m_j) - \frac{a_4 p^2}{m_j} K_{1/2}(m_j), & i = 3, 4; j = 1, 2 \\
 W_{4i} &= a_5 K_{3/2}(m_j S) - \frac{a_4 p^2}{m_j} S I_{1/2}(m_j S), & i = 3, 4; j = 1, 2
 \end{aligned}
 \tag{48}$$

and

$$\begin{aligned}
 W_{3i} &= \frac{a_4 p^2 - m_j^2}{m_j} P_{1/2}(m_j), \\
 W_{4i} &= \frac{a_4 p^2 - m_j^2}{m_j S^{1/2}} P_{1/2}(m_j S),
 \end{aligned}$$

where $P = I$ for $i, j = 1, 2$; $P = K$ for $i = 3, j = 1$ and $i = 4, j = 2$.

From (44), the values of A_1, A_2, B_1 and B_2 are given as

$$\begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \frac{\chi_1'}{p} \\ \frac{\chi_2}{p} \end{pmatrix}.
 \tag{49}$$

5 SPECIAL CASES

For the homogeneous and transversely isotropic material $C_{11} = C_{22}$, $C_{13} = C_{23}$ and $\beta_r = \beta_\theta = \beta_\phi$. Therefore from (9) and (10), $\sigma_\theta = \sigma_\phi$. Hence, from (45) and (46), we can write

$$\bar{\sigma}_R = \sum_{i=1,2} \frac{A_i}{R^{3/2}} \left[\left(-2 + \frac{2C_{13}}{C_{33}} \right) I_{3/2}(m_i R) + \left(\frac{2m_i^2 - a_4 p^2}{m_i} \right) R I_{1/2}(m_i R) \right] + \sum_{i=1,2} \frac{B_i}{R^{3/2}} \left[\left(-2 + \frac{2C_{13}}{C_{33}} \right) K_{3/2}(m_i R) - \frac{a_4 p^2}{m_i} R K_{1/2}(m_i R) \right] \tag{50}$$

$$\bar{\sigma}_\theta = \sum_{i=1,2} \frac{A_i}{R^{3/2}} \left[\left(\frac{C_{11} + C_{12} - 2C_{13}}{C_{33}} \right) I_{3/2}(m_i R) + \left(\frac{a_4 p^2 - m_i^2}{m_i} - \frac{C_{13}}{C_{33}} m_i \right) R I_{1/2}(m_i R) \right] + \sum_{i=1,2} \frac{B_i}{R^{3/2}} \left[\left(\frac{C_{11} + C_{12} - 2C_{13}}{C_{33}} \right) K_{3/2}(m_i R) - \left(\frac{a_4 p^2 - m_i^2}{m_i} + \frac{C_{13}}{C_{33}} m_i \right) R K_{1/2}(m_i R) \right] \tag{51}$$

Also for an isotropic material, $C_{33} = \lambda + 2\mu$, $C_{13} = \lambda$ and $C_{11} + C_{12} = 2(\lambda + \mu)$ and for a non-viscous material, we have $t_0 = 0$. Hence, $a_4 = 1$. Thus, for an isotropic material, equations (45) and (46) reduce to

$$\bar{\sigma}_R = \sum_{i=1,2} A_i \left[-\frac{4\mu}{\lambda + 2\mu} I_{3/2}(m_i R) + \left(2m_i - \frac{a_4 p^2}{m_i} \right) R I_{1/2}(m_i R) \right] / R^{3/2} + \sum_{i=1,2} B_i \left[-\frac{4\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{a_4 p^2}{m_i} R K_{1/2}(m_i R) \right] / R^{3/2} \tag{52}$$

$$\bar{\sigma}_\theta = \sum_{i=1,2} A_i \left[\frac{2\mu}{\lambda + 2\mu} I_{3/2}(m_i R) + \frac{\lambda m_i^2 - (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R I_{1/2}(m_i R) \right] / R^{3/2} + \sum_{i=1,2} B_i \left[\frac{2\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R K_{1/2}(m_i R) \right] / R^{3/2} \tag{53}$$

Moreover, for large value of b i.e., for large value of S , $K_0(m_i S)$ and $K_1(m_i S)$ tend to zero. Thus we have

$$\bar{\sigma}_R = \bar{\sigma}_R(I) + \bar{\sigma}_R(K), \tag{54}$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\theta(I) + \bar{\sigma}_\theta(K). \tag{55}$$

Hence for large value of b , the asymptotic expressions of $\bar{\sigma}_R(I)$ and $\bar{\sigma}_\theta(I)$ are given as

$$\bar{\sigma}_r(t) = \frac{\chi_2 S}{p R^2} \times \frac{e^{-m_2(S-R)} \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_1 S} \right) - \left(2m_1 - \frac{p^2}{m_1} \right) S \right] \times \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_2 R} \right) - \left(2m_2 - \frac{p^2}{m_2} \right) R \right] - e^{-m_1(S-R)} \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_2 S} \right) - \left(2m_2 - \frac{p^2}{m_2} \right) S \right] \times \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_1 R} \right) - \left(2m_1 - \frac{p^2}{m_1} \right) R \right]}{\frac{p^2 - m_1^2}{m_1} \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_2 S} \right) - \left(2m_2 - \frac{p^2}{m_2} \right) S \right] - \frac{p^2 - m_2^2}{m_2} \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_1 S} \right) - \left(2m_1 - \frac{p^2}{m_1} \right) S \right]} \quad (56)$$

→ 0 as $S \rightarrow \infty$

and

$$\bar{\sigma}_\theta(t) = \frac{\chi_2 S}{p R^2} \times \frac{e^{-m_2(S-R)} \left[\frac{2\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_1 R} \right) + \frac{\lambda m_1^2 - (\lambda+2\mu)(p^2 - m_1^2)}{(\lambda+2\mu)m_1} R \right] \times \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_2 S} \right) - \left(2m_2 - \frac{p^2}{m_2} \right) S \right] - e^{-m_1(S-R)} \left[\frac{2\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_2 R} \right) + \frac{\lambda m_2^2 - (\lambda+2\mu)(p^2 - m_2^2)}{(\lambda+2\mu)m_2} R \right] \times \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_1 S} \right) - \left(2m_1 - \frac{p^2}{m_1} \right) S \right]}{\frac{p^2 - m_1^2}{m_1} \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_2 S} \right) - \left(2m_2 - \frac{p^2}{m_2} \right) S \right] - \frac{p^2 - m_2^2}{m_2} \left[\frac{4\mu}{\lambda+2\mu} \left(1 - \frac{1}{m_1 S} \right) - \left(2m_1 - \frac{p^2}{m_1} \right) S \right]} \quad (57)$$

→ 0 as $S \rightarrow \infty$

Therefore, for an infinitely extended body

$$\bar{\sigma}_r(K) = \sum_{i=1,2} B_i \left[-\frac{4\mu}{\lambda+2\mu} K_{3/2}(m_i R) - \frac{p^2}{m_i} R K_{1/2}(m_i R) \right] / R^{3/2}, \quad (58)$$

$$\bar{\sigma}_\theta(K) = \sum_{i=1,2} B_i \left[\frac{2\mu}{\lambda+2\mu} K_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda+2\mu)(p^2 - m_i^2)}{(\lambda+2\mu)m_i} R K_{1/2}(m_i R) \right] / R^{3/2}. \quad (59)$$

where

$$B_1 = -\frac{\chi_1}{p} \times \frac{(1 - e^{-pR^0}) m_1 [4\mu m_2 K_{3/2}(m_2) + (\lambda+2\mu) p^2 K_{1/2}(m_2)]}{4\mu [(p^2 - m_2^2) m_1 K_{3/2}(m_1) K_{1/2}(m_2) - (p^2 - m_1^2) m_2 K_{3/2}(m_2) K_{1/2}(m_1)] + (\lambda+2\mu) p^2 (m_1^2 - m_2^2) K_{1/2}(m_1) K_{1/2}(m_2)}, \quad (60)$$

$$B_2 = \frac{\chi_1}{p} \times \frac{(1 - e^{-pR^0}) m_2 [4\mu m_1 K_{3/2}(m_1) + (\lambda+2\mu) p^2 K_{1/2}(m_1)]}{4\mu [(p^2 - m_2^2) m_1 K_{3/2}(m_1) K_{1/2}(m_2) - (p^2 - m_1^2) m_2 K_{3/2}(m_2) K_{1/2}(m_1)] + (\lambda+2\mu) p^2 (m_1^2 - m_2^2) K_{1/2}(m_1) K_{1/2}(m_2)}. \quad (61)$$

The results agree with those of Kar and Kanoria (2007) for GN III model.

6 OPERATOR THEORY ANALYSIS

Equations (12) and (13) can be expressed in the following form

$$\left[C_T^2 I^{\alpha-1} + (C_K^2 + \tau'_v C_T^2) I^{\alpha-1} \frac{\partial}{\partial \eta} + \tau'_r C_K^2 I^{\alpha-1} \frac{\partial^2}{\partial \eta^2} \right] (D_1 D) \Theta = \left(1 + \tau'_q \frac{\partial}{\partial \eta} + \frac{1}{2} \tau_q'^2 \frac{\partial^2}{\partial \eta^2} \right) \left\{ \frac{\partial^2 \Theta}{\partial \eta^2} + \varepsilon \frac{\partial^2}{\partial \eta^2} (D_1 U) \right\}, \quad (62)$$

with $M = 2$ and

$$\left(1 + \frac{t_0 G}{a} \frac{\partial}{\partial \eta} \right) (DD_1) U = D\Theta + \frac{\partial^2 U}{\partial \eta^2}, \quad (63)$$

where $A = 2$, $D \equiv \frac{\partial}{\partial R}$ and $D_1 \equiv \frac{\partial}{\partial R} + \frac{2}{R}$.

Taking the Laplace transform, we have

$$(DD_1 - a_3) \bar{\Theta} = \varepsilon a_3 D_1 \bar{U} \quad (64)$$

and

$$(DD_1 - a_4 p^2) \bar{U} = a_4 D \bar{\Theta}. \quad (65)$$

Where $a_3 = \frac{p^{\alpha+1} \left(1 + \tau'_q p + \frac{1}{2} \tau_q'^2 p^2 \right)}{(1 + \tau'_v p) C_T^2 + p(1 + \tau'_r p) C_K^2}$ and $a_4 = \frac{a}{a + t_0 p G}$.

Operating $D_1 D$ on (64) and using (65) we have

$$\left\{ M^2 - (m_1^2 + m_2^2) M + m_1^2 m_2^2 \right\} \bar{\Theta} = 0. \quad (66)$$

Similarly, operating DD_1 on (65) and using (64) we have

$$\left\{ L^2 - (m_1^2 + m_2^2) L + m_1^2 m_2^2 \right\} \bar{U} = 0, \quad (67)$$

where $L \equiv DD_1$ and $M \equiv D_1 D$ are the two operators and m_1^2 and m_2^2 are the roots of the quadratic equation in m^2 given by

$$m^4 - (a_3 + \epsilon a_3 a_4 + a_4 p^2) + a_3 a_4 p^2 = 0. \tag{68}$$

As the solution of equation (66) and (67) we have

$$\bar{U} = \sum_{i=1,2} \left[A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R) \right] / \sqrt{R}, \tag{69}$$

and

$$\bar{\Theta} = \sum_{i=1,2} \left[C_i I_{1/2}(m_i R) + D_i K_{1/2}(m_i R) \right] / \sqrt{R}. \tag{70}$$

Where $I_j(m_i R)$ and $K_j(m_i R)$ are the modified Bessel functions of order j of first and second kind respectively; A_i, B_i, C_i and D_i are independent of R but dependent on p .

Therefore, substituting the expressions of \bar{U} and $\bar{\Theta}$ in equation (65), we get

$$C_i = \frac{a_4 p^2 - m_i^2}{a_4 m_i} A_i \tag{71}$$

and

$$D_i = \frac{a_4 p^2 - m_i^2}{a_4 m_i} B_i. \tag{72}$$

Therefore

$$\bar{U} = \sum_{i=1,2} a_4 \left[A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R) \right] / \sqrt{R}, \tag{73}$$

$$\bar{\Theta} = \sum_{i=1,2} \frac{a_4 p^2 - m_i^2}{m_i} \left[C_i I_{1/2}(m_i R) + D_i K_{1/2}(m_i R) \right] / \sqrt{R}. \tag{74}$$

Equations (73) and (74) are the same as that of equations (40) and (42) (i.e., the solutions obtained by Eigen-value approach).

7 NUMERICAL RESULTS AND DISCUSSIONS

To get the solutions for displacement, temperature distribution and stresses in the space-time domain we have to apply Laplace inversion formula to the equations (40), (42), (44) and (45) respectively, which have been done numerically using the method of Bellman et al. (1966) for fixed value of the space variable and for $\eta = \eta_i, i = 1(1)7$, where η_i 's are computed from roots of the shifted

Legendre polynomial of degree 7 (see Appendix) with $S = 4$. The computations for the state variables are carried out for different values of $R(r)$ and values of $\eta_i = 0.0257750, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612, 3.67119$. The materials chosen for numerical evaluation are copper material. The physical data for orthotropic material are (2009)

$$\begin{aligned}\rho &= 6.96 \times 10^3 \text{ kg/m}^3, \quad \varepsilon = 0.0186, \quad T_0 = 20^\circ\text{C}, \\ C_{11} &= 1.544 \times 10^{11} \text{ N/m}^2, \quad C_{12} = 0.617 \times 10^{11} \text{ N/m}^2, \\ C_{13} &= 0.597 \times 10^{11} \text{ N/m}^2, \quad C_{22} = 1.747 \times 10^{11} \text{ N/m}^2, \\ C_{23} &= 0.496 \times 10^{11} \text{ N/m}^2, \quad C_{33} = 1.716 \times 10^{11} \text{ N/m}^2, \\ \chi_1 &= 4, \quad \chi_2 = 3, \quad C_T = 2, \quad C_K = 1.2,\end{aligned}$$

and the hypothetical values of the relaxation time parameters are taken as

$$t_0 = 1.0 \times 10^{-7} \text{ sec}, \quad \tau_q = 2.0 \times 10^{-7} \text{ sec}, \quad \tau_T = 1.5 \times 10^{-7} \text{ sec}, \quad \tau_v = 1.0 \times 10^{-7} \text{ sec}.$$

Here, in this article we have considered three-phase-lag model. Now, for this model, the solution of heat conduction is stable if $K^* \tau_q < \tau_v^* < \frac{2K\tau_T}{\tau_q}$, where $\tau_v^* = K + K^* \tau_v$ i.e., the stability condition of Quintanilla and Racke is verified (2008).

Also, for an isotropic material, the physical data are taken as (www.matweb.com).

$$\begin{aligned}\rho &= 8.96 \text{ gm/cm}^3, \quad \varepsilon = 0.0186, \quad T_0 = 20^\circ\text{C}, \\ \lambda &= 1.387 \times 10^{12} \text{ dy/cm}^2, \quad \mu = 0.448 \times 10^{12} \text{ dy/cm}^2, \\ C_e &= 0.23 \text{ cal/gm}^\circ\text{C}, \quad K = 0.92 \text{ cal}, \quad \eta_1^0 = 0.1.\end{aligned}$$

In case of GN theory, K^* is an additional material constant depending on the material. For copper like material, we take $K^* = \frac{C_e(\lambda + 2\mu)}{4}$.

The results of the numerical evaluation of the thermo-elastic stress variations and temperature distribution are illustrated in figures 1-8 for both large time ($\eta = 1.21$) and small time ($\eta = 0.026$) for weak conductivity ($\alpha = 0.5$), normal conductivity ($\alpha = 1.0$) and strong conductivity ($\alpha = 1.2$) respectively for 3P and GN III models. In these figures, the magnitudes of the variation of stresses and temperature are observed for viscous material when the step-input temperatures are applied on the inner boundary $R = 1$ and outer boundary $S = 4$ of the hollow sphere. Figures 1 and 2 depict the variation of the radial stress against the radial distance R of the sphere. From the figures it is observed that the radial stress (σ_R) vanishes at the inner boundary ($R = 1$) and the outer boundaries ($R = 4$) of the shell which satisfy our theoretical boundary conditions. The magnitude of the radial stress is maximum near $R = 2.1$ for a strong conductive material and for GN III model. Also, for three-phase-lag model, the oscillatory nature is observed. This is due to the presence of the oscilla-

tion term in the heat equation of three-phase-lag model. For weak conductivity ($\alpha = 0.5$), the oscillatory nature is also seen for GN III model.

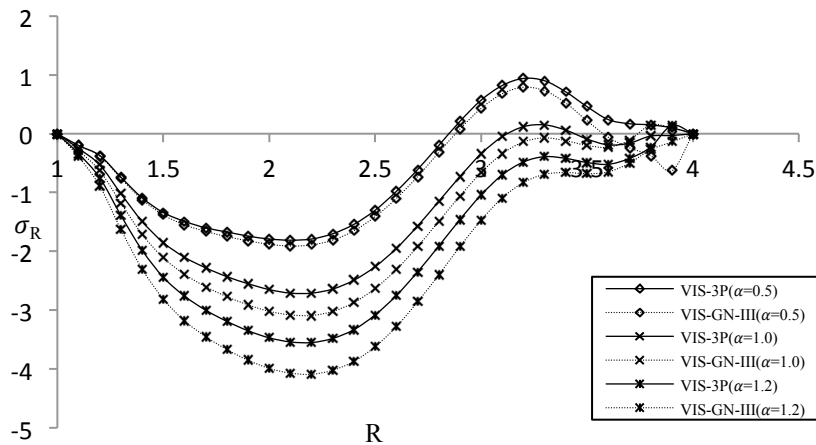


Figure 1 σ_R versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 1.21$.

Figure 2 represents the variation of σ_R for $\eta = 0.026$ and $\alpha = 0.5, 1.0, 1.2$ respectively. It is seen that σ_R vanishes at the boundaries of the shell where there are thermal sources which agree with our theoretical boundary conditions. As may be seen from the figure, the stress wave is compressive in nature near both the boundaries. Also, at earlier stage of wave propagation, both the models give close results, whereas with advancement of time time, the stress wave is propagating with different speeds. For $\alpha = 0.5$, the effect of σ_R is very prominent inside the shell, whereas, for $\alpha = 1.0$ and $\alpha = 1.2$, the radial stress vanishes for $2 < R < 3$ and $1.5 < R < 3.5$ respectively, which is physically plausible.

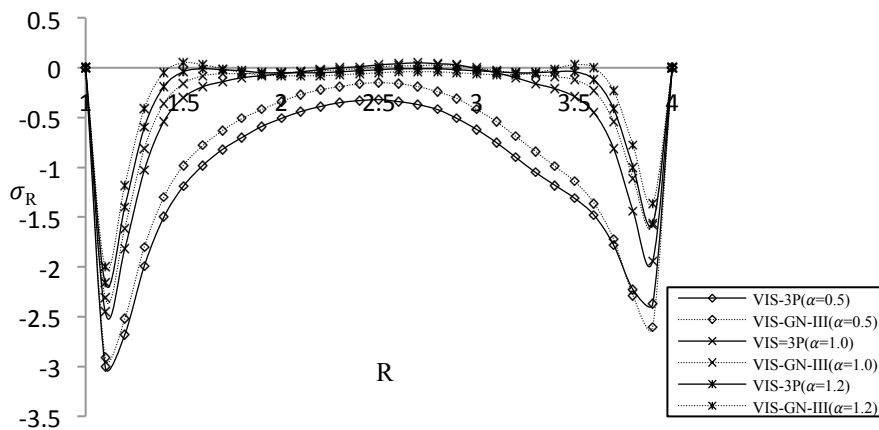


Figure 2 σ_R versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 0.026$.

Figures 3 and 4 are plotted to show the variation of the stress σ_θ along the radius of the sphere for different values of the non-local fractional parameter α and for $\eta = 1.21, 0.026$ respectively. In

figure 3, variation of σ_θ is shown for larger time $\eta=1.21$ for same set of parameters. As seen from the figure, σ_θ attains the maximum magnitude near the inner boundary of the shell. The magnitude of σ_θ for $\alpha=1.2$ is larger than that of $\alpha=1.0$ which is again larger than that of $\alpha=0.5$ when $1 < R < 3$ and the rate of decay in magnitude of σ_θ for $\alpha=1.2$ is faster than that of $\alpha=1.0$ which is again faster than that obtained for $\alpha=0.5$.

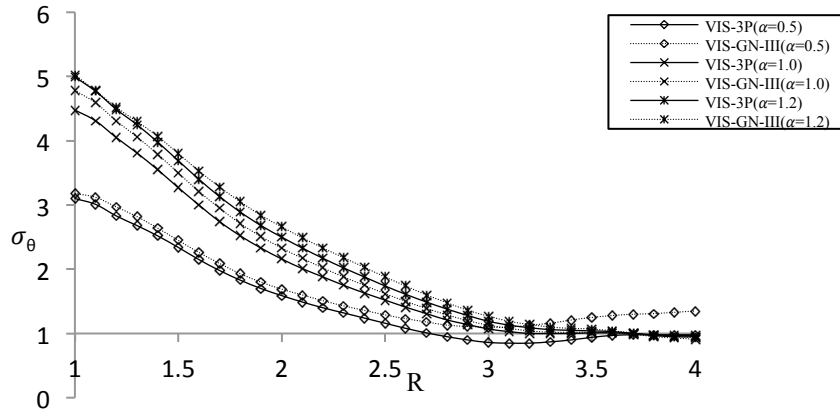


Figure 3 σ_θ versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 1.21$.

In both cases, the stress corresponding to each model and for different nonlocal fractional parameter α , the stress propagates for $1 < R < 4$. For small time, the stress is compressive in nature near the inner boundary of the shell. Here also, for $\alpha=1.2$, the stress component almost disappears for $2.2 < R < 3.5$ and for $\alpha=1.0$, it vanishes for $2.3 < R < 3$. For $R=2.5$, i.e., the region equidistant from the boundaries σ_θ vanishes in earlier situations for normal conductivity and strong conductivity of the materials.

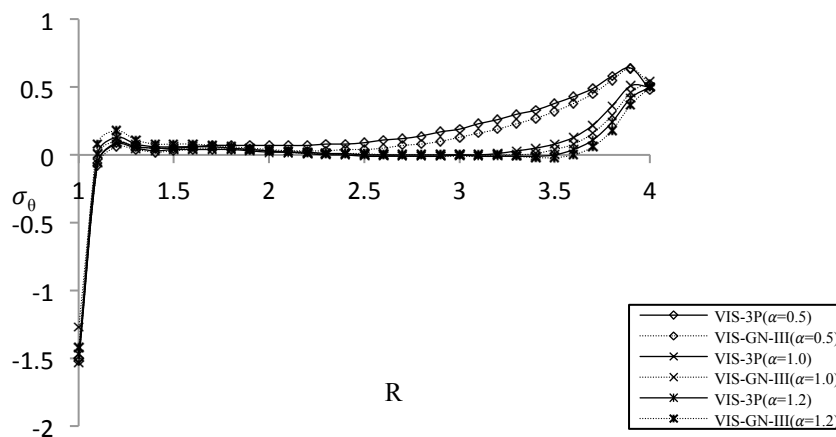


Figure 4 σ_θ versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 0.026$.

Figures 5 and 6 are plotted to show the variation of the stress component σ_ϕ for different fractional parameter α . From the figure 5, it is seen that when $\eta=1.21$, the magnitude of σ_ϕ is maximum near the inner boundary of the shell. Also it is observed that increase in the nonlocal fractional parameter α also increases the magnitude of the stress component σ_ϕ . For $\alpha=1.0$ and $\alpha=1.2$, the decay in magnitude of σ_ϕ is more rapid compared to that of $\alpha=0.5$. As may be seen from the figure, σ_ϕ is compressive near the inner boundary of the shell and the similar qualitative behavior is seen in the variation of σ_ϕ as that in figure 4.

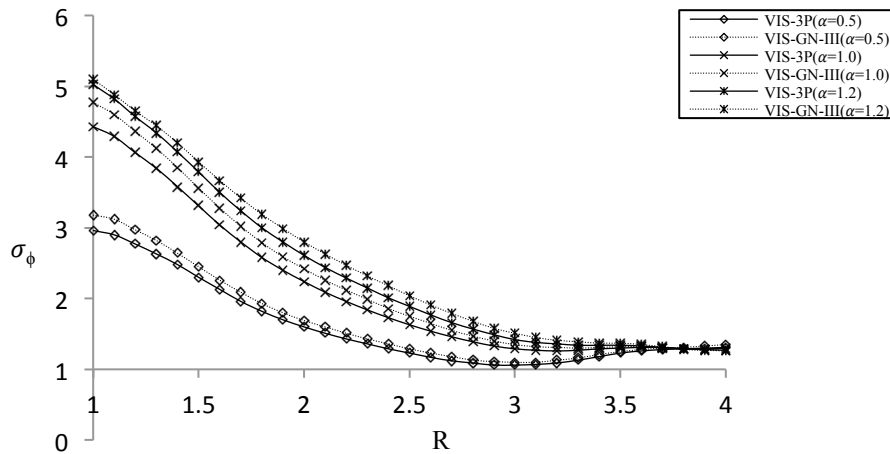


Figure 5 σ_ϕ versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 1.21$.

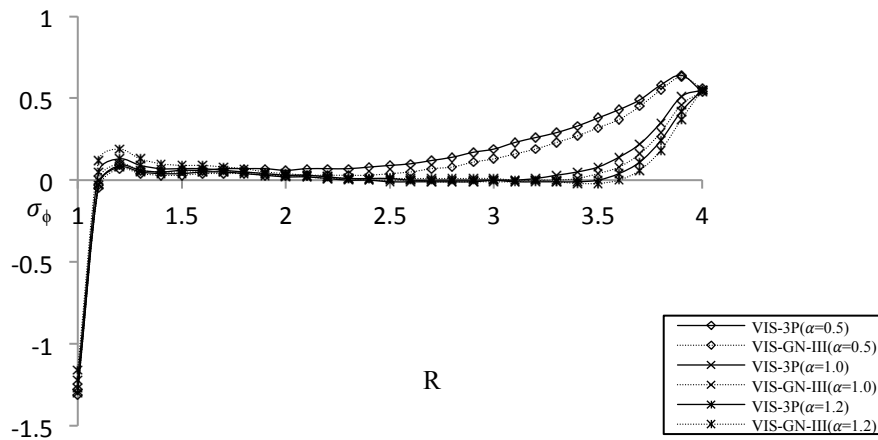


Figure 6 σ_ϕ versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 0.026$.

Figures 7 and 8 depict the variation of the temperature Θ along the radius of the sphere R for different values of α . It is seen that whenever $\eta > 0.1$, (i.e., for $\eta = 1.21$) the inner boundary of the shell is kept at zero temperature whereas for $\eta = 0.026$, the inner boundary is maintained the fixed

temperature value $\chi'_1 = 4.18$ while in both situations, outer boundary maintains the same step-input-temperature $\chi_2 = 3$. For larger time, Θ attains the maximum magnitude near $R = 2.3$ for GN III model. Whereas in the earlier situations, the magnitude of Θ decays sharply near the inner boundary of the shell for $\alpha = 1.2$ compared to that of $\alpha = 0.5$ and $\alpha = 1.0$. The rise in magnitude near the outer boundary is rapid also. For $\alpha = 1.2$ and for $\eta = 0.026$, the magnitude of the temperature almost disappears for $1.8 < R < 3.4$.

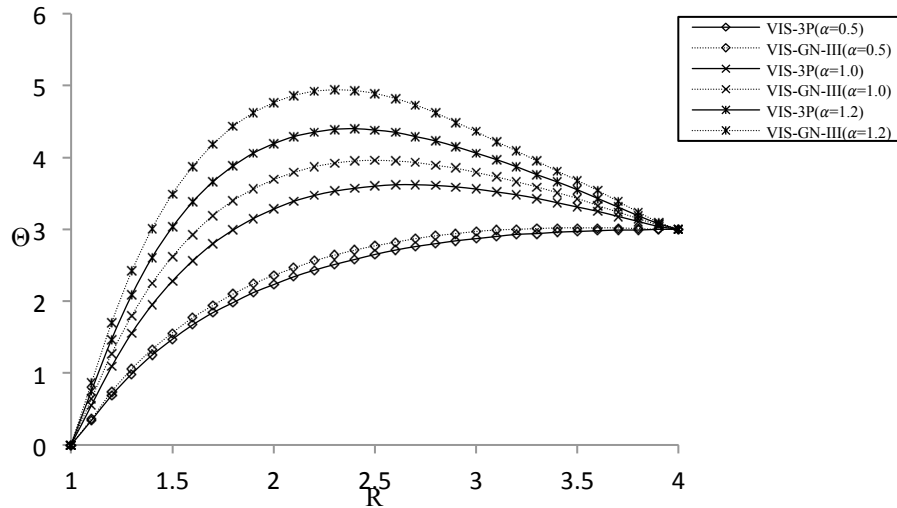


Figure 7 Θ versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 1.21$.

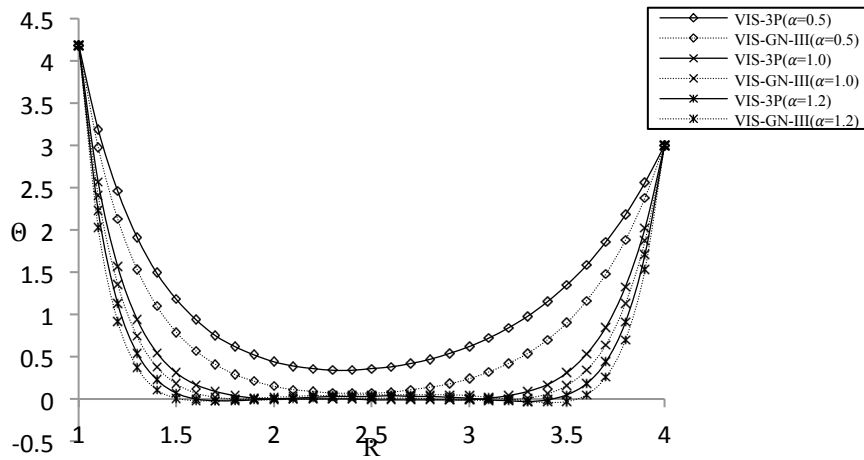


Figure 8 Θ versus R for $\alpha = 0.5, 1.0, 1.2$ and $\eta = 0.026$.

Figures 9-16 are plotted to show the effect of viscosity for two set of times for weak conductive materials. Form figures 9-10 it is seen that σ_r satisfies our theoretical boundary conditions. As may seen from figure 10, it is seen that σ_r attains the maximum value for non-viscous material for both models near the inner boundary of the shell.

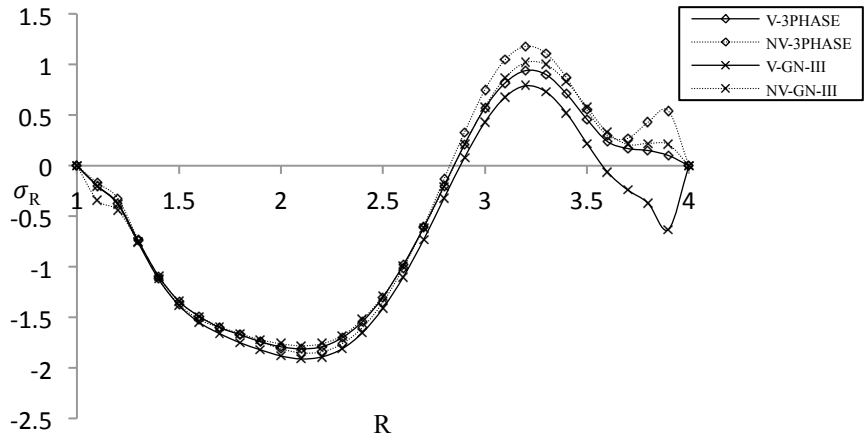


Figure 9 σ_R versus R for $\alpha = 0.5$ and $\eta = 1.21$.

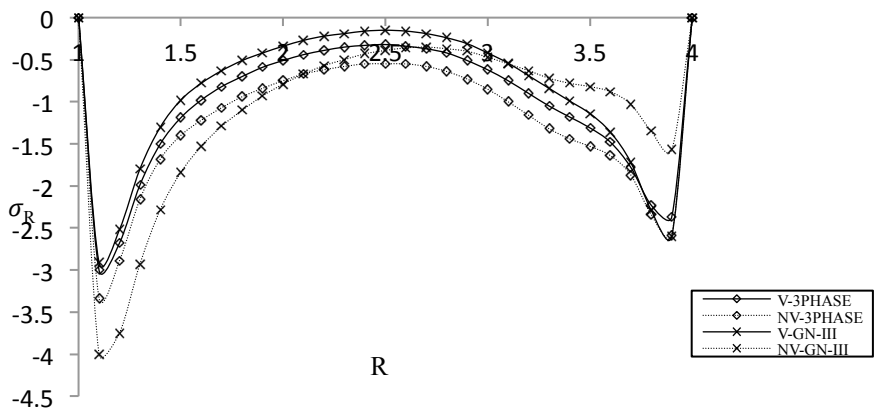


Figure 10 σ_R versus R for $\alpha = 0.5$ and $\eta = 0.026$.

Figures 11-12 are plotted to show the variation of σ_θ versus R . from these figures it is seen that the effect of viscosity is more prominent in GN III model compared to that of 3P lag model for a large time when $1 < R < 4$. Whereas, for earlier situations, the effect of viscosity for GN III model is very prominent near the boundaries of the shell compared to the interior of the shell.

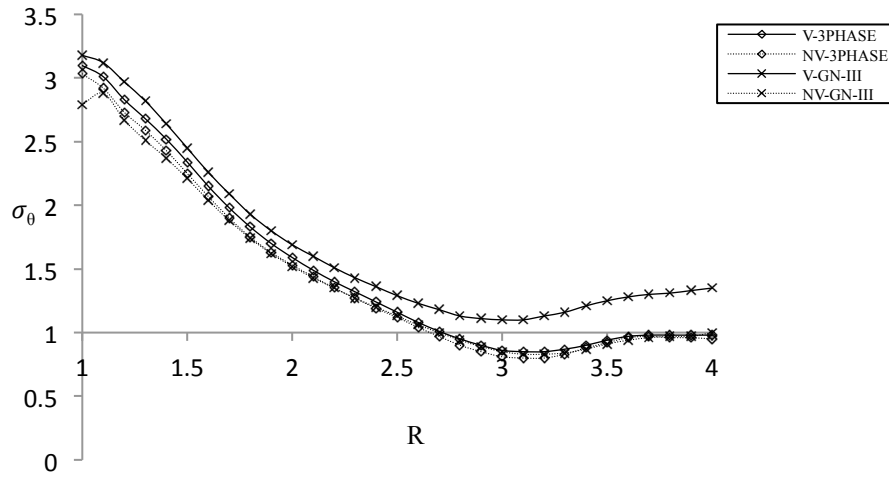


Figure 11 σ_θ versus R for $\alpha = 0.5$ and $\eta = 1.21$.

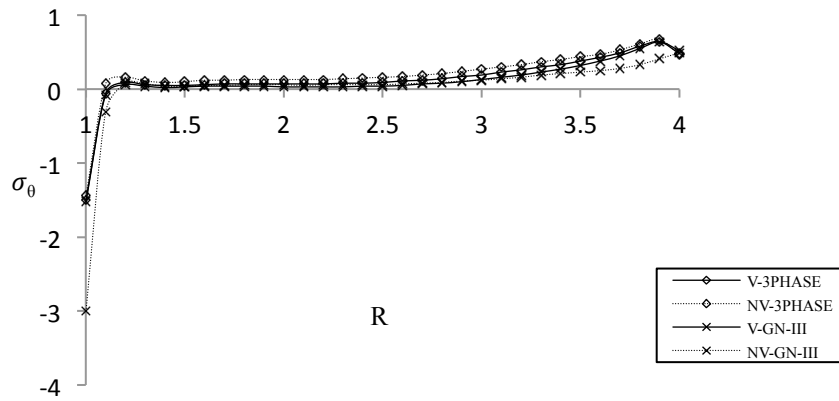


Figure 12 σ_θ versus R for $\alpha = 0.5$ and $\eta = 0.026$.

From figures 13-14, the similar qualitative nature is seen in the variation of σ_ϕ for both viscous and non-viscous materials.

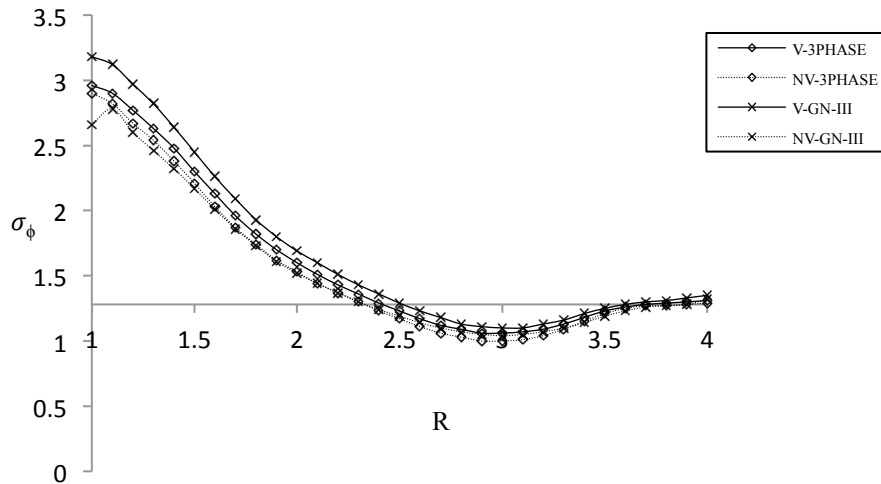


Figure 13 σ_ϕ versus R for $\alpha = 0.5$ and $\eta = 1.21$.

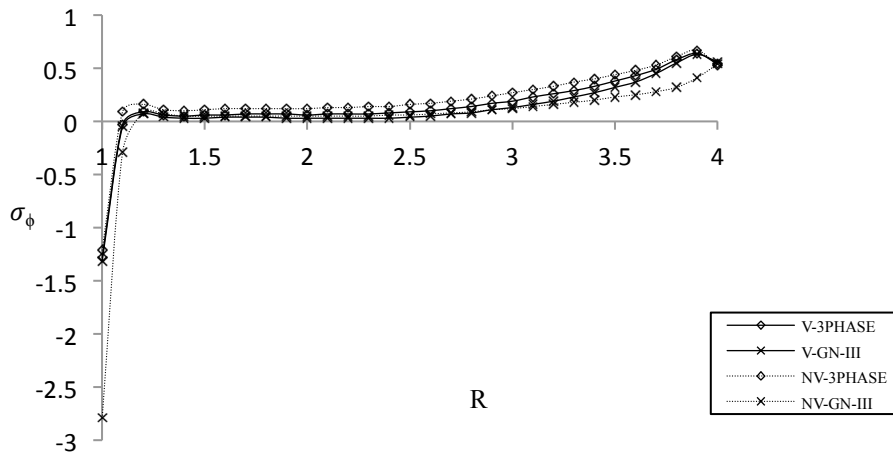


Figure 14 σ_ϕ versus R for $\alpha = 0.5$ and $\eta = 0.026$.

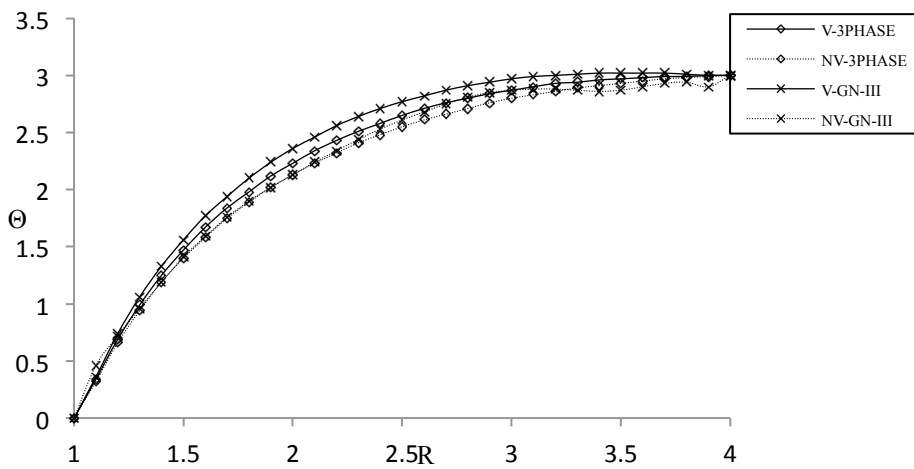


Figure 15 Θ versus R for $\alpha = 0.5$ and $\eta = 1.21$.

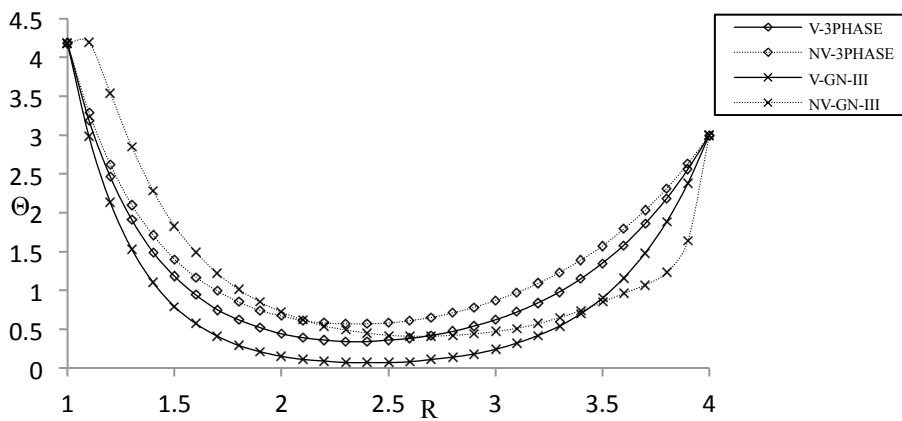


Figure 16 Θ versus R for $\alpha = 0.5$ and $\eta = 0.026$.

Figures 15 and 16 are plotted to show the effect of viscosity on temperature Θ for two sets of time. For both viscous and non-viscous material, the temperature satisfies our thermal boundary conditions. Also, the effect of viscosity is very prominent in earlier situations than latter. As may be seen from the figures, when $\eta = 1.21$, for 3P lag model, the magnitude of Θ is larger for viscous material compared to the non-viscous material. Whereas for $\eta = 0.026$, the magnitude is larger for non-viscous material compared to the viscous material.

Figures 17-19 are plotted to show the variations of σ_r , σ_θ and σ_ϕ respectively against the time η whenever $R=1.4$ and $\alpha=0.5$. From these figures, it is seen that at the beginning of time, oscillatory natures are seen in the propagation of the stress components. Finally they reach to a steady state which supports the physical fact.

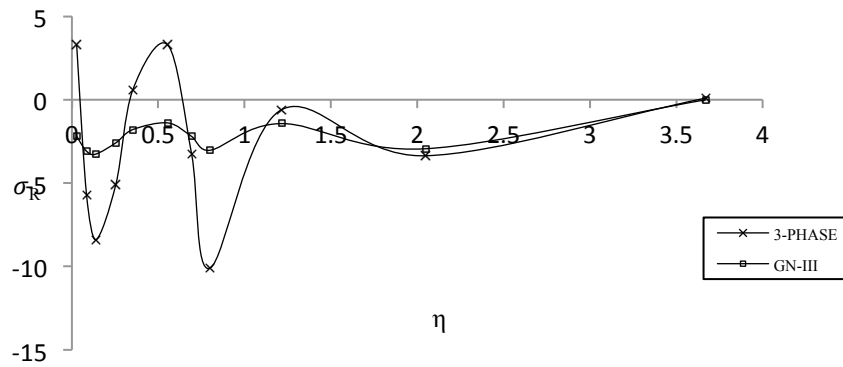


Figure 17 σ_r versus η for $R = 1.4$ and $\alpha = 0.5$.

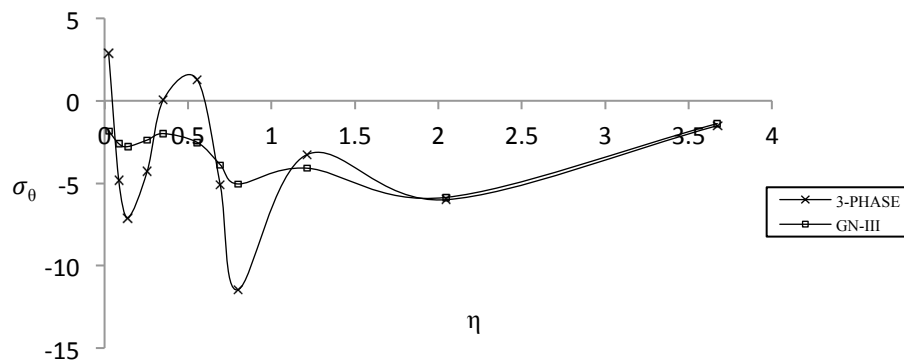


Figure 18 σ_θ versus η for $R = 1.4$ and $\alpha = 0.5$.

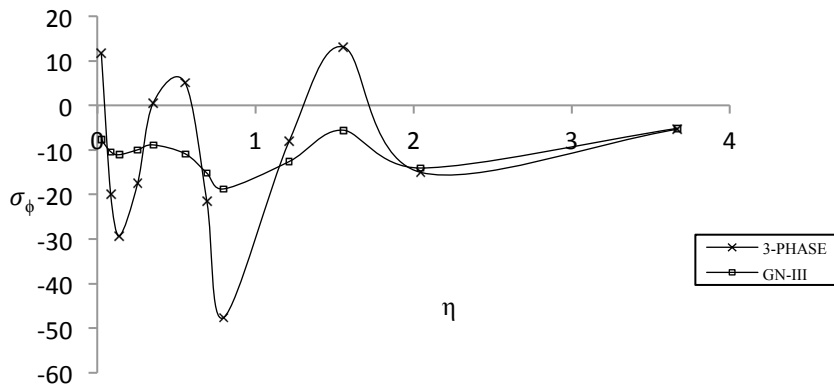


Figure 19 σ_ϕ versus η for $R = 1.4$ and $\alpha = 0.5$.

Figures 20-22 are plotted to draw the comparison between isotropic and orthotropic material for $\alpha = 0.5, 1.0$ and for $\eta = 0.026$ for viscous material. From figure 20, it is seen that for orthotropic material, the stress waves are reflected from either boundary whereas for isotropic material, the propagation of each of the waves are found to occur. Also, amplitude of σ_R decreases with the increase of the non-local fractional parameter α .

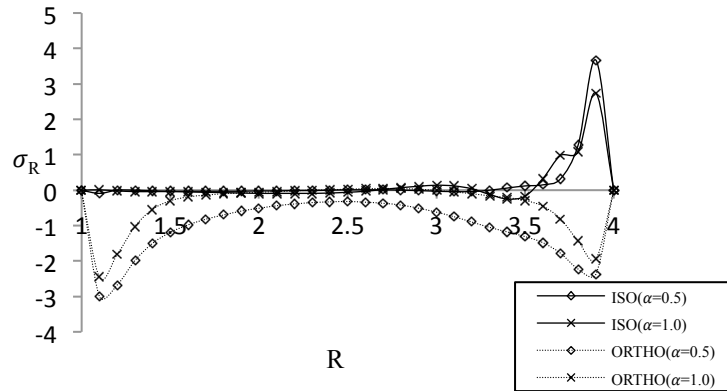


Figure 20 σ_R versus R for $\eta = 0.026$ and $\alpha = 0.5, 1.0$.

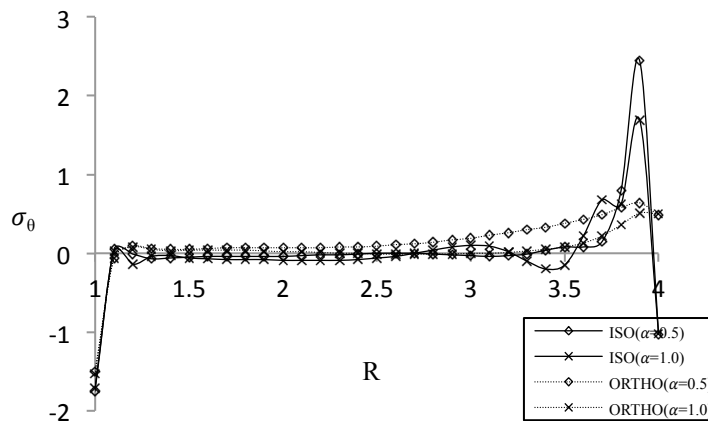


Figure 21 σ_θ versus R for $\eta = 0.026$ and $\alpha = 0.5, 1.0$.

Figure 21 depicts the variation of σ_θ versus R for isotropic and orthotropic materials. As may be seen from the figure that for an isotropic material, the oscillatory nature is observed due to the reflection as mentioned earlier. However, the magnitude of σ_θ is maximum near the outer boundary of the shell for an isotropic material.

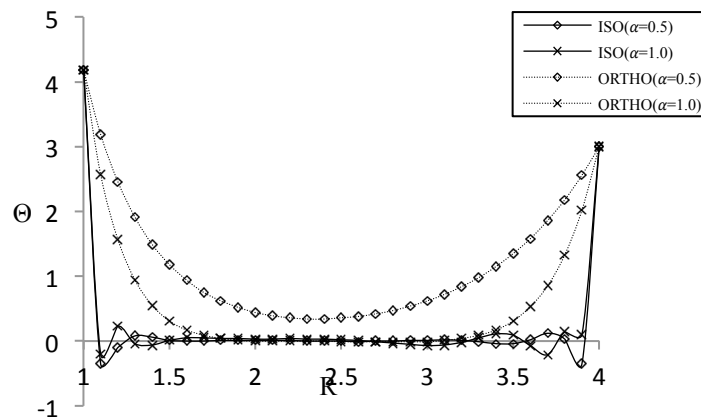


Figure 22 Θ versus R for $\eta = 0.026$ and $\alpha = 0.5, 1.0$.

Figure 22 is plotted to show the variation of Θ versus R for two different materials. For both the materials, Θ satisfies the thermal boundary conditions. The magnitude of Θ is larger for $\alpha = 0.5$ than that of $\alpha = 1.0$ for an orthotropic material. As may be seen from the figure, oscillatory behavior is seen near the boundaries for an isotropic material. It is seen that for isotropic material, when $R = 2.5$, i.e., at the surface equidistant from the boundaries, Θ almost disappears at the primary stage of thermal load application.

8 CONCLUSIONS

The problem of investigating the radial stress, hoop stress, temperature in a homogeneous isotropic viscoelastic spherical shell is studied in the light of three-phase-lag model and GN-III model in the context of space-fractional heat conduction equation. The method of Laplace Transform is used to write the basic equations in the form of a vector-matrix differential equation which is then solved by eigen-value approach. The numerical inversion of Laplace Transform is computed by the method of Bellmen. The analysis of the result permits some concluding remarks:

- (i) When the time is small, ($\eta = 0.026$), i.e., at early stage of wave propagation, both the models give close results, whereas for comparatively large time ($\eta = 1.21$), significant differences are observed for weak, normal and strong conductivities ($\alpha = 0.1, 1.0, 1.2$) respectively. Also, in the earlier situations, maximum magnitude occurs for weak conductivity whereas for large time, magnitudes are maximum when conductivity is high inside the body.

- (ii) It is observed that maximum magnitude of stresses will occur for viscous material and for strong conductivity ($\alpha=1.2$).
- (iii) For an isotropic material, the maximum temperature occurs near the boundaries of the shell and it almost disappears in the interior of the shell.
- (iv) The effect of σ_r is more prominent near the inner boundary for orthotropic material compared to that of an isotropic material.

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Appendix

Let the Laplace transform of $\sigma_j(R, \eta)$ be given by

$$\bar{\sigma}_j(R, p) = \int_0^\infty e^{-p\eta} \sigma_j(R, \eta) d\eta. \tag{A.1}$$

We assume that $\sigma_j(R, \eta)$ is sufficiently smooth to permit the use of the approximate method to apply.

Substituting $x = e^{-\eta}$ in equation (A.1) we obtain

$$\bar{\sigma}_j(R, p) = \int_0^1 x^{p-1} g_j(R, x) dx, \tag{A.2}$$

where

$$g_j(R, x) = \sigma_j(R, -\log x). \tag{A.3}$$

Applying the Gaussian quadrature rule to (A.2) we obtain the approximate relation

$$\sum_{i=1}^n W_i x_i^{p-1} g_j(R, x_i) = \bar{\sigma}_j(R, p), \tag{A.4}$$

where x_i 's ($i = 1, 2, 3, K, n$) are the roots of the shifted Legendre polynomial and W_i 's ($i = 1, 2, 3, K, n$) are the corresponding weights and $p = 1(1)n$.

Thus, we have

$$\begin{aligned} W_1 g_j(R, x_1) + W_2 g_j(R, x_2) + L + W_n g_j(R, x_n) &= \bar{\sigma}_j(R, 1) \\ W_1 x_1 g_j(R, x_1) + W_2 x_2 g_j(R, x_2) + L + W_n x_n g_j(R, x_n) &= \bar{\sigma}_j(R, 2) \\ \quad L \quad \quad L \quad \quad L \quad \quad L \quad \quad L \\ \quad L \quad \quad L \quad \quad L \quad \quad L \quad \quad L \\ W_1 x_1^{n-1} g_j(R, x_1) + W_2 x_2^{n-1} g_j(R, x_2) + L + W_n x_n^{n-1} g_j(R, x_n) &= \bar{\sigma}_j(R, n) \end{aligned}$$

Therefore

$$\begin{pmatrix} g_j(R, x_1) \\ g_j(R, x_2) \\ M \\ g_j(R, x_n) \end{pmatrix} = \begin{pmatrix} W_1 & W_2 & L & W_n \\ W_1 x_1 & W_2 x_2 & L & W_n x_n \\ M & M & O & M \\ W_1 x_1^{n-1} & W_2 x_2^{n-1} & L & W_n x_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} \bar{\sigma}_j(R, 1) \\ \bar{\sigma}_j(R, 2) \\ M \\ \bar{\sigma}_j(R, n) \end{pmatrix} \tag{A.5}$$

As the matrix is the product of $diag\{W_i\}$ multiplied by Vander monde matrix, it can be shown that the matrix is non-singular.

Hence, $g_j(R, x_1), g_j(R, x_2), K, g_j(R, x_n)$ are known. For $n = 7$ we have

<u>Roots of Shifted Legendre Polynomial</u>	<u>Corresponding Weights</u>
2.5446043828620886E-2	6.4742483084434816E-2
1.2923440720030282E-1	1.3985269574463828E-1
2.9707742431130145E-1	1.9091502525255938E-1
5.0000000000000000E-1	2.0897959183673466E-1
7.0292257568869853E-1	1.9091502525255938E-1
8.7076559279969706E-1	1.3985269574463828E-1
9.7455395617137909E-1	6.4742483084434816E-2

From equations in (A.5) we can calculate the discrete values of $g_j(R, x_i)$ i.e., $\sigma_j(R, \eta_i)$; ($i = 1, 2, K, 7$) and finally using interpolation, we obtain the stress components $\sigma_i(R, \eta)$; ($i = R, \theta$).