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About lateral torsional buckling of steel beams – geometrically exact nonlinear theory results

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Abstract

The intention of this paper is to establish accurate values for the elastic critical moment of steel beams in several cases of loading and end-restraint conditions by using a Geometrically Exact Nonlinear Theory. The influence of warping and lateral rotation restraints is studied for four idealized support conditions. The results are compared with the ones derived from approximate theories, and in particular the Brazilian Code NBR8800:1986, the American Specification prAISC-LRFD:2003, and the European Prestandards prEN1993-1-1:2002 Stage 54 and prEN1999-1-1:2004 Stage 54. A parametrical analysis is performed for welded doubly-symmetric I-beams using the finite element program PEFSYS for the usual range in conventional structures.

Keywords: structural stability, steel structures, buckling, beams

1 Introduction

Several publications present expressions and tables for doubly-symmetric beams under usual loading and with idealized boundary conditions in order to establish the elastic critical moment. On the other hand, codes present simplified expressions that have been commonly used for these cases; few authors use a Geometrically Exact Nonlinear Theory of Bars or a Nonlinear Theory of Shells.

The lateral-torsional buckling of doubly-symmetric steel beams in elastic range is studied in this paper through a Geometrically Exact Nonlinear Theory of Bars, and the influence of the restriction to warping and to lateral rotation is analyzed. The results are compared, when possible, with recommendations of Brazilian Code [2], American Specification [9], European Prestandards [3] and [4], and the technical literature.

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2 Geometrically Exact Nonlinear Theory

The Geometrically Exact Nonlinear Theory used in this paper can be applied to any cross-section, thin or not, and is valid for structures with large displacements and rotations, without any limitations. Therefore, geometric simplifications and strain-displacement relation approximations, similar to those in the first and second order theories, are not necessary. The displacements at any point in the cross-section can be decomposed in two parts: the first corresponding to the movement of the bar, the sections remaining plane and indeformable, although not orthogonal to the axis; and the second corresponding to warping, orthogonal to the cross-section in the deformed shape.

In this theory, cross-sections do not remain orthogonal to the bar's axis, that is, Bernoulli-Euler's assumption is not valid; as a consequence, rotations are independent of declivity ($\varphi_x \neq -v^{'}$ and $\varphi_y \neq u^{'}$). The warping displacements p are dealt with as independent parameters of rotation φ_z, so that warping intensity is independent of rotation derivative ($\varphi_z^{'} \neq p$), in contrast with what is admitted in Vlasov's Theory.

The expression of the second variation of the total potential energy can be seen in [5, 6, 10]. Under matrix notation it is given by:

$$\delta^2 \cup = \int_L \left[\left(\underset{-}{D}\, \underset{-}{B}\, \underset{-}{H}\, \delta \underset{\sim}{\Delta} \right) \cdot \left(\underset{-}{B}\, \underset{-}{H}\, \delta \underset{\sim}{\Delta} \right) + \left(\underset{-}{G}\, \underset{-}{H}\, \delta \underset{\sim}{\Delta} \right) \cdot \left(\underset{-}{H}\, \delta \underset{\sim}{\Delta} \right) - \left(\underset{-}{L^e}\, \delta \underset{\sim}{\Delta} \right) . \delta \underset{\sim}{\Delta} \right] dz \quad (1)$$

where:

→ $\underset{-}{D}$ is the matrix of the cross-section's tangent rigidity coefficients, which is the constitutive part of the operator.

→ $\underset{-}{G}$ is the matrix that characterizes the geometric effect of the internal forces.

→ $\underset{-}{L^e}$ is the matrix that characterizes the geometric effect of the external forces.

→ $\underset{-}{B}$ and $\underset{-}{H}$ are auxiliary matrices.

→ $\delta \underset{\sim}{\Delta}$ are generalized virtual displacements, constituted of:

- vector $\delta \underset{\sim}{u}$ that contains three displacements in directions x, y and z;
- vector $\delta \underset{\sim}{\varphi}$ that contains three rotations;
- vector $\delta\, p$ that represents the intensity of warping.

Expression **(1)** gave origin to a computational Finite Element Program for the geometrically exact nonlinear analysis of structures called PEFSYS. Developed in the Computational Mechanics Laboratory of Polytechnic School of the University of Sao Paulo, PEFSYS is the main tool used in this paper.

3 Boundary conditions

Influence of boundary conditions at beam ends in the elastic critical moment depends on the restrained degrees of freedom, it being usual to restrain rotation φ_z, lateral displacement u, corresponding declivity u' and warping p. Figure 1 illustrates some types of end restraints.

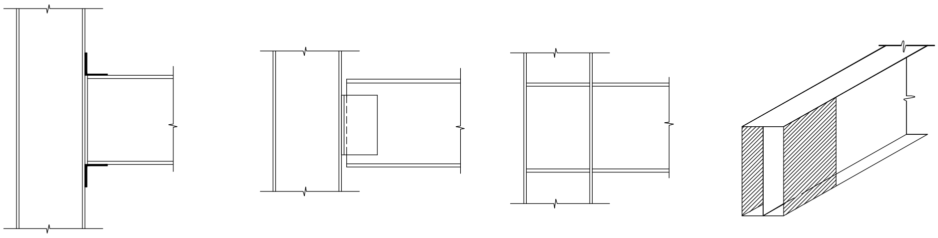


Figure 1: Usual restraints to lateral-torsional buckling of beams

Besides boundary conditions, bending moment variation over the span also affects significantly the critical moment. The most unfavorable situation is one in which the bending moment is constant over the span, because all cross-sections are subjected to the maximum value of moment. The elastic critical moment for this situation in simply supported beams that are prevented from lateral deflection and twisting, but free to rotate laterally and to warp at both ends, is called basic critical moment $M_{0\,cr}$. It is usual to present the elastic critical moment as being this value multiplied by coefficients that take into account loading and boundary conditions, thus obtaining the critical moment M_{cr}.

For each case of loading and in-plane boundary conditions studied in this paper, the results of the analysis are presented separately as a function of the restriction to out-of-plane bending and warping.

Each above-mentioned boundary condition is associated to an effective length, here denominated $k_y L$ and $k_\omega L$ respectively. Coefficients k vary from 0.5 (both ends totally restrained from out-of-plane bending or warping) to 1.0 (both ends simply supported or with no warping restriction).

Four cases of boundary conditions at beam ends are considered for each loading case, as follows:

→ Support condition type I: $k_\omega = 1.0$ and $k_y = 1.0$

→ Support condition type II: $k_\omega = 0.5$ and $k_y = 1.0$

→ Support condition type III: $k_\omega = 1.0$ and $k_y = 0.5$

→ Support condition type IV: $k_\omega = 0.5$ and $k_y = 0.5$

For the attainment of the critical moment in elastic-linear range, some authors like [1,11,13] use the same expressions stated in the European Standards, presented here as follows:

$$M_{cr} = C_1 \frac{\pi^2 E I_y}{(k_y L)^2} \left\{ \sqrt{\left(\frac{k_y}{k_\omega}\right)^2 \frac{I_\omega}{I_y} + \frac{(k_y L)^2 G I_t}{\pi^2 E I_y} + [C_2 (e_y - y_C) + C_3 r_{0y}]^2} \right\} - \\ - C_1 \frac{\pi^2 E I_y}{(k_y L)^2} [C_2 (e_y - y_C) + C_3 r_{0y}] \tag{2}$$

C_1, C_2 and C_3 are coefficients that depend mainly on the loading and end restraint conditions. For doubly-symmetric I-beams ($C_3 = 0$) and for loads applied at the shear centre ($e_y - y_C = 0$), expression (2) can be rewritten as:

$$M_{cr} = C_1 \left(\frac{\pi}{k_y L}\right) \sqrt{G I_t E I_y} \sqrt{1 + \left(\frac{\pi}{k_\omega L}\right)^2 \frac{E I_\omega}{G I_t}} \tag{3}$$

On the other hand, other authors like [7, 8, 14] who also consider the four above-mentioned boundary conditions do not present the results in the form of expression (2), as they directly provide the ratio between the critical moment and the basic critical moment, here called C_b, implying:

$$M_{cr} = C_b \frac{\pi}{L} \sqrt{G I_t E I_y} \sqrt{1 + \left(\frac{\pi}{L}\right)^2 \frac{E I_\omega}{G I_t}} \quad ; \quad C_b = \frac{M_{cr}}{M_{0cr}} \tag{4}$$

In particular for doubly-symmetric I-beams with loads and end restraints applied at the shear centre, and for $k_\omega = k_y = 1.0$, $C_1 = C_b$.

4 Numerical analysis

For each case, a critical load obtained from PEFSYS is converted into a critical moment, called *reference moment*. It is stipulated that this moment is the maximum bending moment over the entire span of the beam, regardless of whether it occurs in mid-span or near the supports.

A parametric analysis of usual cases of lateral-torsional buckling in elastic-linear range using Finite Element Program PEFSYS is performed, comparing the results with those found in the technical literature (see *References*). To accomplish this analysis, only doubly-symmetric welded I-beams with *height-to-width* relation varying between two and four, usual in Brazil, are adopted.

The results are presented in Figures 2 to 11, which express the relation between the ratio M_{cr}/M_{0cr} and the parameter $\mu = G I_t L^2 / E I_\omega$.

5 Discussion of the results

5.1 Simply supported beam under moment gradient (Figure 2)

The results obtained from a Geometrically Exact Theory show that the value of the relation M_{cr}/M_{0cr} is strongly influenced not only by the ratio of end moments (ψ), as usually stated,

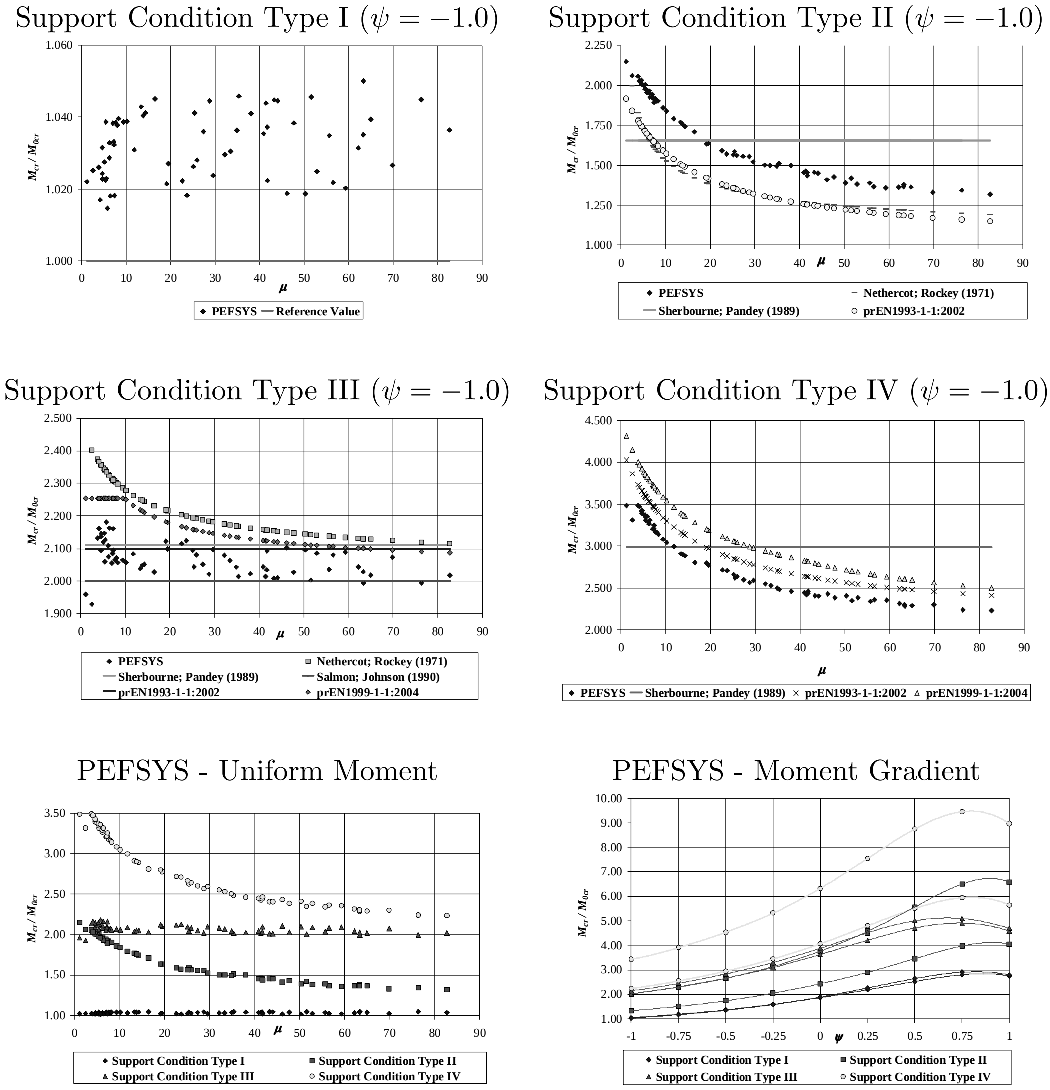


Figure 2: Simply supported beam under moment gradient ($\psi = M_1/M_2$)

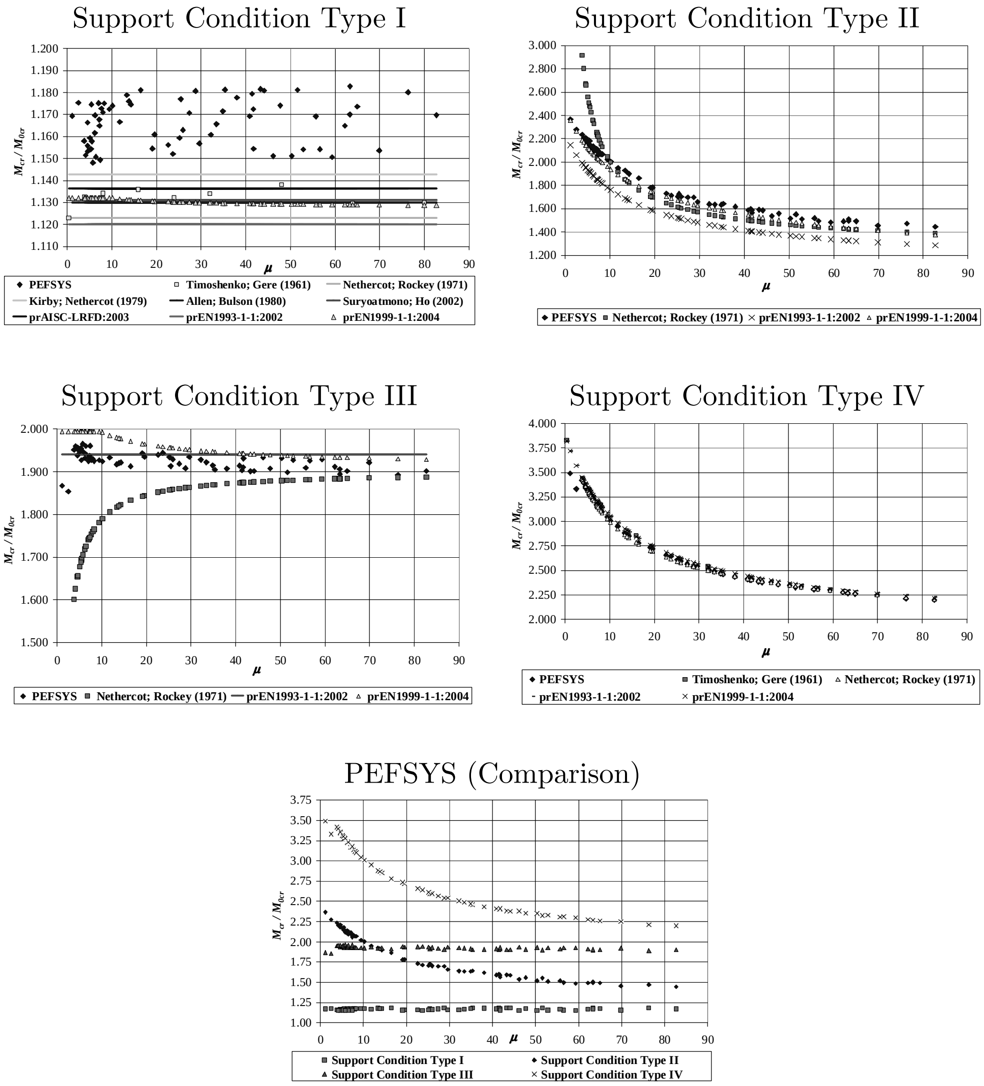


Figure 3: Simply supported beam with uniform load

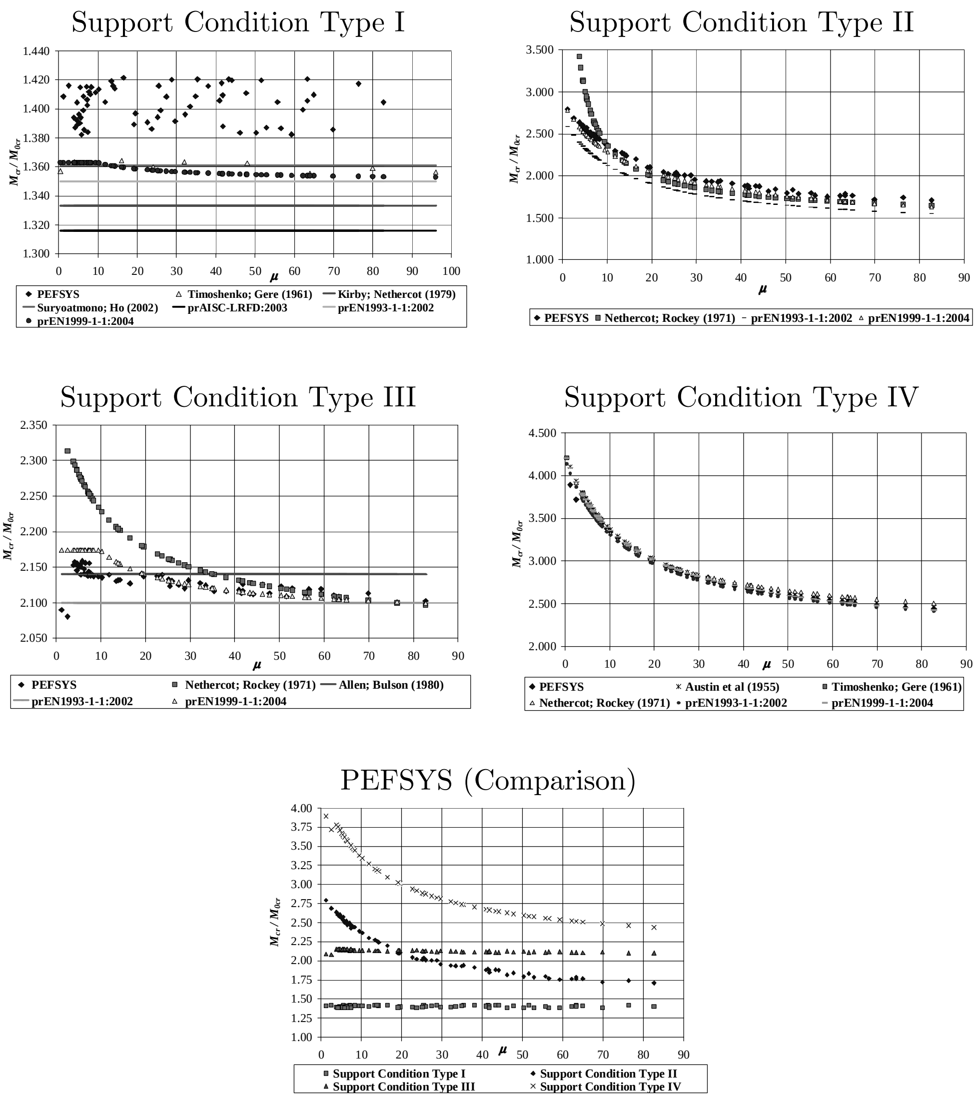


Figure 4: Simply supported beam with concentrated load at mid-span

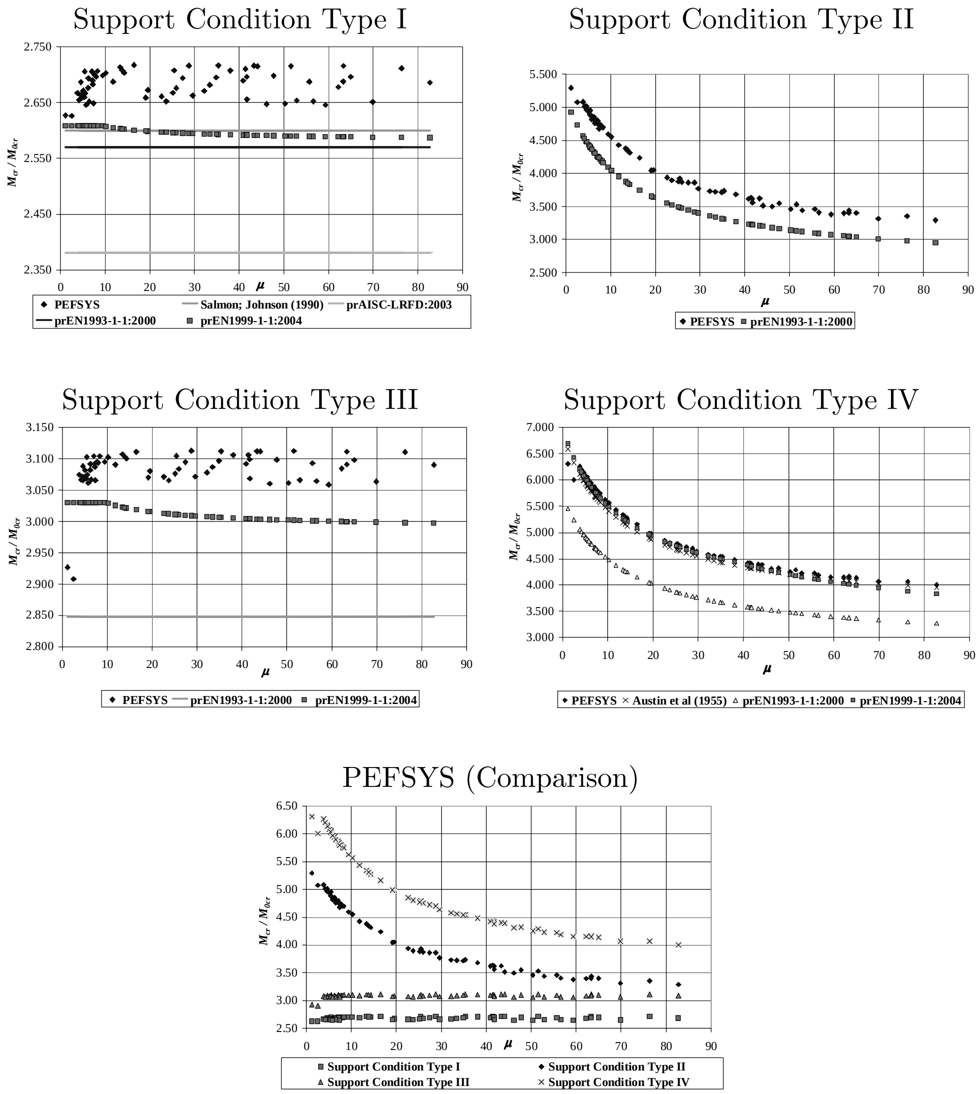


Figure 5: Fully fixed beam with uniform load

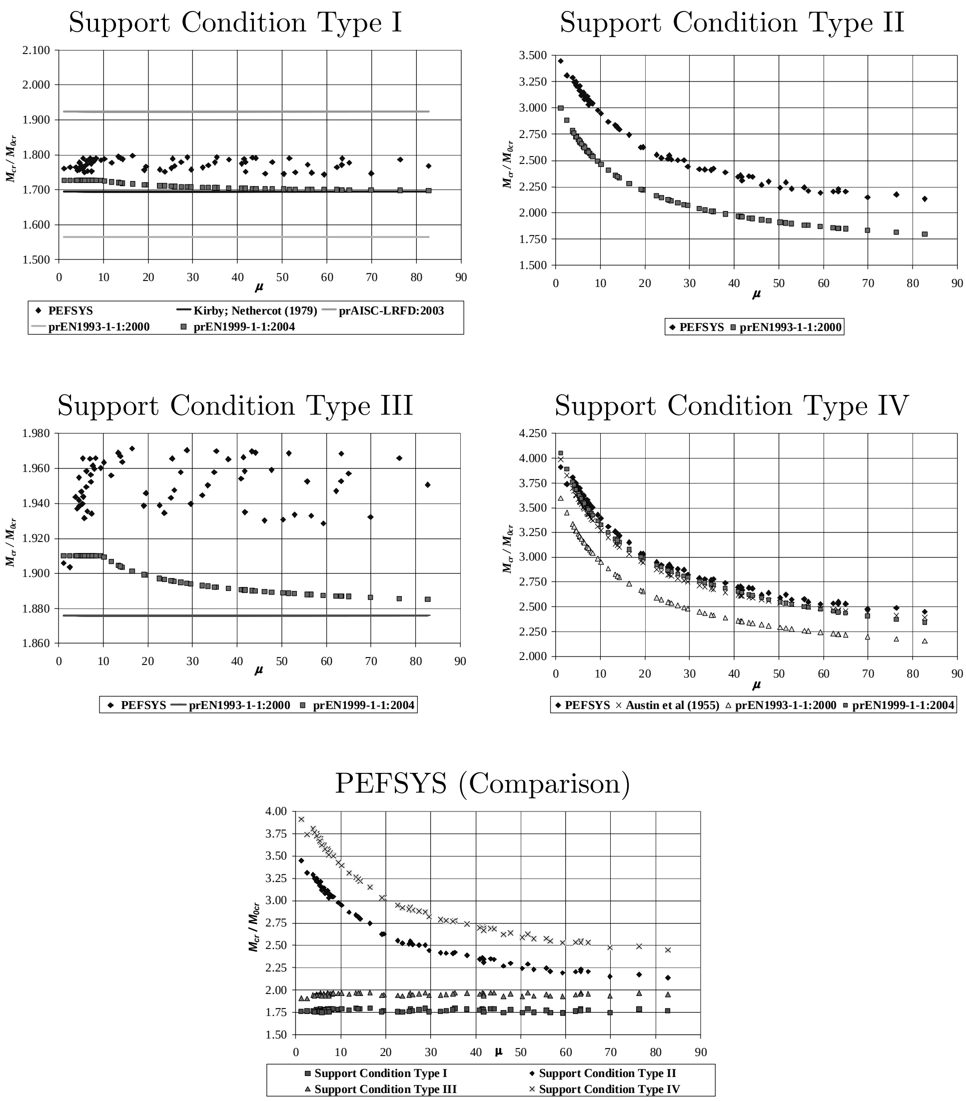


Figure 6: Fully fixed beam with concentrated load at mid-span

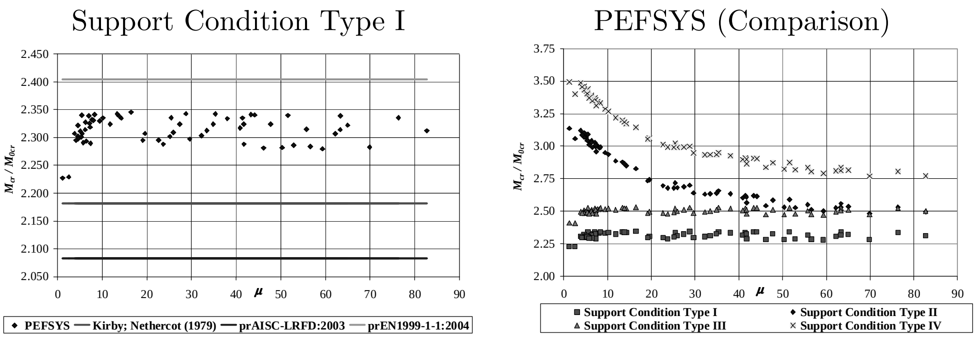


Figure 7: Beam fixed at one end and simply supported at the other end with uniform load

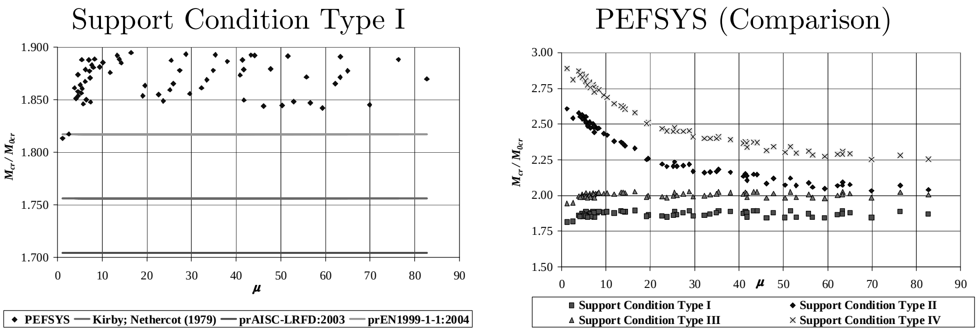


Figure 8: Beam fixed at one end and simply supported at the other end with concentrated load

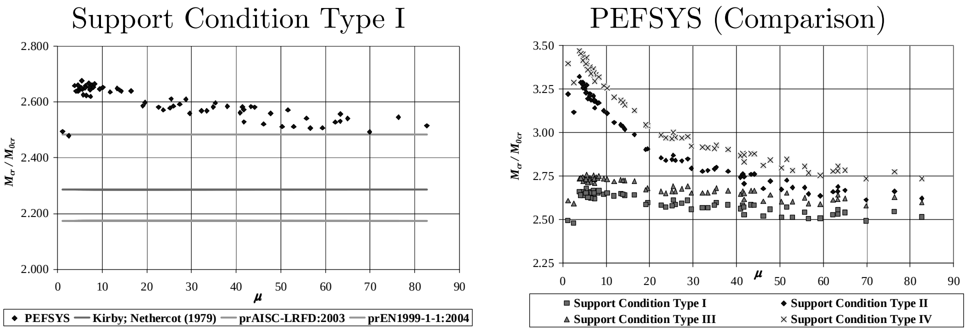


Figure 9: Beam fixed at one end and simply supported at the other end with moment applied at the simply supported end

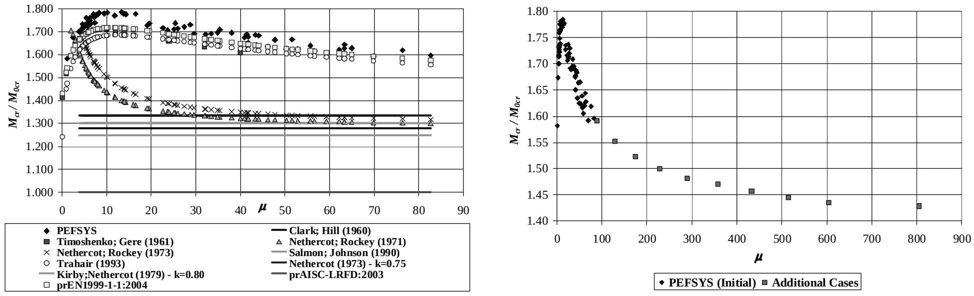


Figure 10: Cantilever with concentrated load at free end

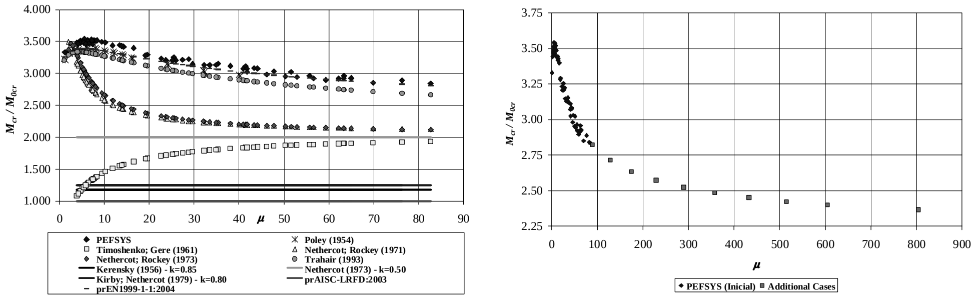


Figure 11: Cantilever with uniform load

but also by the value of the parameter μu. The influence of μ is more important for support conditions with warping restriction (II and IV), for which the relation M_{cr}/M_{0cr} can be *60%* higher for low values of μug his relation remains approximately constant for the other two support conditions (see Figure 2).

Regarding the case of uniform bending moment ($\psi = -1.00$psg most of the existing recommendations are very close to PEFSYS results for the four support conditions. In particular, for support condition type I, the difference to the basic critical moment is negligible.

In turn, considering other values of ψ technical literature results (including the Brazilian Code [2] and the American Specification [9]) are in general very conservative. It is also observed that these results present higher discrepancies for support conditions II and IV and for approaching the value $+1.00$.

Additionally, the use of support condition type I results for other kinds of supports leads to a conservative design.

Taking into account low values of μ, some points of the graph are far from the average PEFSYS curve, presenting significantly inferior values. This behavior is mainly a consequence of the shearing force effect.

5.2 Simply supported beam with uniform load (Figure 3) and concentrated load at mid-span (Figure 4)

Regarding support condition type I, PEFSYS results (varying from 1.14 to 1.19 for uniform load, and from 1.38 to 1.42 for concentrated load) are less than *5%* higher than the usual value (1.13 and 1.35, respectively). In this condition, European Prestandard [4] expressions are adequate, because the average results are the same as those obtained from a Geometrically Exact Theory.

PEFSYS relation M_{cr}/M_{0cr} for others support conditions is either higher or lower than the technical literature values, although the differences can be considered negligible for high values of μ.

prEN1999-1-1:2004 [4] values present differences to PEFSYS values lower than *5%* for the four support conditions and for any values of μ.

For support conditions type *II* and *III*, [8] results are not adequate for low values of μ, although they are recommended for μgreater than four. Support condition type *III* results are in the conservative side and for type *II*, in the nonconservative side, especially for low values of μ.

The difference between support conditions type I and IV results are as high as *150%* for low values of the parameter μ.

Finally, for support conditions *I* and *III*, M_{cr}/M_{0cr} value is practically constant, but for conditions *II* and IV, the above-mentioned relation is inversely proportional to the parameter μ.

5.3 Fully fixed beam with uniform load (Figure 5) and with concentrated load at mid-span (Figure 6)

The presence of a fixed end in the bending plan considerably increases M_{cr}/M_{0cr} value, when compared to the case of simply supported beams.

PEFSYS values are higher than technical literature recommendations for the four analyzed support conditions, but the differences between PEFSYS and [4] C_b values are lower than *5%* for all support conditions, although the 2000 version of the European Prestandard gives the worst results.

The American Specification, valid only for support condition type I, is excessively conservative for uniform load, presenting values *15%* lower than the Geometrically Exact Theory. On the other hand, for concentrated load, [9] values are approximately *10%* higher than the Geometrically Exact Theory ones, indicating that they are unsafe for this case.

The difference between PEFSYS results for support conditions type I and IV vary from *30%* to *140%*, being greater for low values of μ.

It is evident that, opposite to the case of simply supported beams, warping restriction (support condition type II) gives better results than lateral rotation restriction (support condition type III). Moreover, the effect of shear force is again important for low values of μ.

5.4 Beam fixed at one end and simply supported at the other end with uniform load (Figure 7), concentrated load (Figure 8) and with moment applied at the simply supported end (Figure 9)

Figures 7, 8 and 9, always take into account support condition type I for the simply supported end, varying only the support condition at fixed end.

Regarding support condition type *I*, **prAISC-LRFD:2003** [9] recommendations are conservative, with C_b values 10 to *20%* lower than PEFSYS.

For European Prestandard [4] simplified expression, the results are unsafe for uniform load and conservative for the other two loading cases.

The difference between PEFSYS results for the four different support conditions at the fixed end varies from *10%* to *65%*.

Taking into account support conditions type *I* and *III*, the relation $M_{cr}/M_{0cr}$is practically independent of the parameter μ, but decreases with the increase of μgor support conditions *II* and *IV*. As support conditions significantly affect the values of the relation M_{cr}/M_{0cr}, more accurate analysis of this case is necessary, especially for different support conditions in the simply supported end. This study is not part of this work.

It is still observed that for an end moment loading case, the values of PEFSYS are similar to the presented ones in the case of simply supported beams under gradient moment with $\psi = 0.50$.

5.5 Cantilever with concentrated load at free end (Figure 10) and uniform load (Figure 11)

The values obtained from the Geometrically Exact Nonlinear Theory show that the technical literature and standard recommendations are conservative.

Figures 10 and 11 show that the difference between PEFSYS results and the values obtained by the authors that use only one effective length is as high as *60%*.

On the other hand, [15] and European Prestandard [4] values present low differences (5%), being also conservative when compared with those obtained from the Geometrically Exact Theory.

Nethercot; Rockey (1971) and **Nethercot; Rockey (1973)** results are inconsistent with PEFSYS ones, since the relation M_{cr}/M_{0cr} increases for lower values of μ, opposite to what is verified for the Geometrically Exact Theory. In turn, for higher values of μ, the relation M_{cr}/M_{0cr} for these two publications are practically independent of the parameter μ, although reveals itself very conservative.

On the other side, the resultant PEFSYS curve matches, in all extension, **Trahair (1993)** and European Prestandard **prEN1999-1-1:2004** ones.

It is observed for μglose to eighty that PEFSYS results ($C_b \cong 1.60$ for concentrated load and $C_b \cong 2.85$ for uniform load) are still higher than those commonly adopted in the literature for μg$\rightarrow \infty$ ($C_b = 1.28$ and $C_b = 2.05$, respectively). Moreover, PEFSYS values still decrease with the coefficient μ without presenting, apparently, a convergence point.

In sight of this, a special parametric analysis was performed for cantilever beams with high values of μ, showing that the relation M_{cr}/M_{0cr}tends to be 1.40 for concentrated load and 2.30 for uniform load.

6 Conclusions

The results presented in this paper are applicable to design only with a correct identification of boundary conditions, by means of a discerning analysis of connections and rigidity of elements where the beam is connected, since some conditions are of difficult accomplishment in practice.

The comparison of the results obtained from PEFSYS analysis with those of the literature and of standards allows to conclude that the majority of the recommendations for the attainment of the critical moment in elastic range is conservative; additionally, boundary conditions that consider restriction to warping and/or to lateral rotation are not dealt with adequately,.

The values suggested in [9] and [2] are estabilished only for support condition type I, in general, excessively conservative. The difficulty to establish boundary conditions in design justifies the use of these values for their simplicity. On the other hand, [4] recommendations are appropriate for practically all cases studied in this publication (including cantilever beams, for which the American Specifications [9] present excessively conservative results), however they are too complex for routine design.

Simplified expressions commonly used for the usual cases of loading and boundary conditions lead to poor results. The same occurs for the values obtained from expressions based only on one effective length for warping and lateral rotation.

On the other hand, the expression that uses two effective lengths (k_ω and k_y), associated respectively to the restrictions to warping and lateral rotation, is shown to be more adequate for lateral-torsional buckling analysis. In this case, the effect of each of the above-mentioned parameters on the value of the elastic critical moment is clear, as opposed to the expressions that use only one coefficient (C_b) to adjust the basic critical moment value.

A great part of the technical literature disregards the influence of the parameter μ on the critical moment, since emphasis on results of the boundary condition is given to type I (with no restriction to warping and lateral rotation, that is $k_y = k_\omega = 1.0$), for which this influence is negligible. Even in these cases, it is evidenced that the ratio M_{cr}/M_{0cr} obtained by means of the Geometrically Exact Nonlinear Theory is higher than that usually found in the literature, in particular for fully fixed beams and cantilevers.

In tables of Figures 12 and 13 the minimum PEFSYS values are indicated for boundary conditions I to IV. It is convenient to highlight that much higher values can be obtained if the influence of parameter μ is considered. For example, in the case of cantilever beams with a concentrated load applied at the free end, the C_b coefficient is higher than the suggested value 1.40, for which the C_b curve tends asymptotically. Moreover, according to [12], the usual range of the parameter μg in structure design is $4 \leq \mu \leq 40$, and for this case, $C_b > 1.68$. When boundary conditions *II*, *III* and *IV* are considered, it is verified that differences in the values used in structure design are even greater.

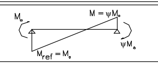
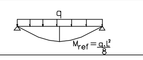
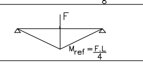
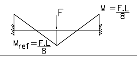
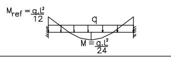
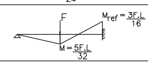
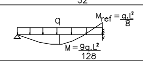
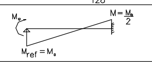
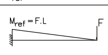
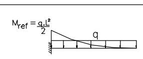
LOADING AND SUPPORT CONDITIONS	$C_b = {M_{cr}}/{M_{0cr}}$			
	SUPP. TYPE I	SUPP. TYPE II	SUPP. TYPE III	SUPP. TYPE IV
	(FIGURE 13)			
	1.14	1.44	1.85	2.20
	1.38	1.72	2.08	2.43
	1.74	2.13	1.90	2.45
	2.62	3.29	2.91	4.01
	1.81	2.03	1.94	2.25
	2.22	2.48	2.41	2.77
	2.47	2.61	2.58	2.73
	1.40			
	2.30			

Figure 12: Values of C_b coefficient for usual cases of loading and support condition

For simply supported beams, the influence of lateral rotation end restraints is more significant than that for warping, except for low values of μ; however, for fully fixed beams, the restriction to warping is shown to be more important.

The relation M_{cr}/M_{0cr} for support conditions type *I* and *III* with no warping restriction is, in general, practically independent of the parameter μ; however, for types *II* and *IV* with warping restriction, this relation starts to be strongly dependent on μ, decreasing with the increase of μ.

The results for simply supported beams under moment gradient present a similar behaviour, indicating that, for support conditions type *II* and *IV*, the ratio between the critical moment and basic critical moment also depends on the parameter μ.

It is interesting to emphasize that the shear force effect reduces the value of the elastic critical moment. This effect is not considered in most papers and is important to very low values of μ, although this condition is not usual in design. The results in this paper indicate that values applied to usual cases of loading and boundary conditions for lateral torsional buckling can be readily reviewed, implying a more economic design.

Considering the reasons shown, the values obtained here represent an orientation for a possible revision of the design codes, especially [2].

END MOMENTS	$C_b = M_{cr}/M_{0cr}$			
	SUPP. TYPE I	SUPP. TYPE II	SUPP. TYPE III	SUPP. TYPE IV
$\psi = -1.00$	1.01	1.31	1.93	2.23
$\psi = -0.75$	1.16	1.50	2.27	2.54
$\psi = -0.50$	1.34	1.74	2.62	2.94
$\psi = -0.25$	1.57	2.04	3.06	3.44
$\psi = 0$	1.84	2.42	3.59	4.06
$\psi = 0.25$	2.17	2.90	4.16	4.79
$\psi = 0.50$	2.52	3.45	4.67	5.51
$\psi = 0.75$	2.79	3.97	4.87	5.94
$\psi = 1.00$	2.73	4.05	4.55	5.65

Figure 13: Values of C_b coefficient for simply supported beams under moment gradient

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